Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

APPROXIMATION BY LINEAR MEANS OF FOURIER SERIES IN WEIGHTED ORLICZ SPACES

SADULLA Z. JAFAROV

Abstract. In this work the approximation of functions by linear means of Fourier series in reflexive weighted Orlicz spaces with Muckenhoupt weights is studied. This result is applied to the approximation of functions by linear means of Faber series in weighted Smirnov-Orlicz classes defined on simply connected domain of the complex plane.

1. Introduction and main results

Let $M(u)$ be a continuous increasing convex function on $[0,\infty)$ such that $M(u)/u \to 0$ if $u \to 0$, and $M(u)/u \to \infty$ if $u \to \infty$. We denote by N *the complementary function of* M *in Young's sense,* i.e. $N(u) = \max\{uv - M(v) : v \geq 0\}$ if $u \geq 0$. We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0$ with some constant c independent of u.

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$.

For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int\limits_{\mathbb{T}} M\left(|f(x)|\right) dx < \infty.$$

Let N be the complementary Young function of M. It is well-known [25, p. 69], [31, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx : g \in \widetilde{L}_N(\mathbb{T}), \int\limits_{\mathbb{T}} N\left(|g(x)|\right) dx \leq 1\right\},$$

or with the *Luxembourg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{k > 0 : \int\limits_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \leq 1\right\}$$

2010 *Mathematics Subject Classification.* 41A25, 41A27, 42A10, 46E30.

Key words and phrases. Young function, weighted Orlicz space, Boyd indices, modulus of smoothness, best approximation.

becomes a Banach space. The space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [25, p. 26]. The Luxembourg norm is equivalent to the Orlicz norm as

$$\|f\|^*_{L_M(\mathbb{T})} \leq \|f\|_{L_M(\mathbb{T})} \leq 2\,\|f\|_{L_M(\mathbb{T})}, \qquad f \in L_M(\mathbb{T})$$

holds true [25, p. 80].

If we choose $M(u) = u^p/p \quad (1 < p < \infty)$ then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \leq \|u\|_{L_M(\mathbb{T})} \leq q^{1/q}\,\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ denotes the usual norm of the $L_p(\mathbb{T})-$space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [25, p.80]

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \leq \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})},$$

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \leq \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}.$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [31, p. 113].

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* α_M, β_M [3, p. 350]

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t}$$

of the function

$$h : (0,\infty) \to (0,\infty], \quad h(t) := \lim_{y\to\infty} \sup \frac{M^{-1}(y)}{M^{-1}(ty)}, \ \ t > 0$$

first considered by Matuszewska and Orlicz [28], are called the *Boyd indices* of the Orlicz spaces $L_M(\mathbb{T})$.

It is known that the indices α_M and β_M satisfy $0 \leq \alpha_M \leq \beta_M \leq 1, \alpha_N + \beta_M = 1$, $\alpha_M + \beta_N = 1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0 < \alpha_M \leq \beta_M < 1$. The detailed information about the Boyd indices can be found in [2], [3], [4], [21] and [27].

A function ω is called a weight on $\mathbb{T}$ if $\omega : \mathbb{T} \to [0,\infty]$ is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure). With any given weight ω we associate the ω*-weighted Orlicz space* $L_M(\mathbb{T},\omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\,\omega)} := \|f\omega\|_{L_M(\mathbb{T})}.$$

Let $1 < p < \infty,\ 1/p + 1/p' = 1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy Muckenhoupt's A_p-condition on $\mathbb{T}$ if

$$\sup_J \left(\frac{1}{|J|}\int\limits_J \omega^p(t)\,dt\right)^{1/p} \left(\frac{1}{|J|}\int\limits_J \omega^{-p'}(t)\,dt\right)^{1/p'} < \infty,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length.

Let us denote by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

Note that by [20, Lemma 3.3], [22, Theorem 4.5] and [19, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and ω weight function satisfying the condition $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T}, \omega)$ is also reflexive.

Let $L_M(\mathbb{T}, \omega)$ be a weighted Orlicz space, let α_M and β_M be nontrivial, and let $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f \in L_M(\mathbb{T}, \omega)$ we set

$$(\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\, dt,\ 0 < h < \pi, x \in \mathbb{T}.$$

By reference [15, Lemma 1] the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T}, \omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\omega)} \leq C \|f\|_{L_M(\mathbb{T},\omega)}.$$

The function

$$\Omega^r_{M,\omega}(\delta, f) := \sup_{\substack{0<h_i\leq\delta \\ 1\leq i\leq k}} \left\| \prod_{i=1}^{k} (I - \nu_{h_i}) f \right\|_{L_M(\mathbb{T},\omega)},\ \delta > 0,\ r = 1, 2, ...$$

is called *r-th modulus of smoothness* of $f \in L_M(\mathbb{T}, \omega)$, where I is the identity operator.

It can easily be shown that $\Omega^r_{M,\omega}(\cdot, f)$ is a continuous, nonnegative and non-decreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega^r_{M,\omega}(\delta, f) = 0,\ \Omega^r_{M,\omega}(\delta, f+g) \leq \Omega^r_{M,\omega}(\delta, f) + \Omega^r_{M,\omega}(\delta, g)$$

for $f, g \in L_M(\mathbb{T}, \omega)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $\alpha_k(f)$ are $b_k(f)$ the Fourier coefficients of the function f.

Let (1.1) be the Fourier series of the function f . For $f \in L_M(\mathbb{T}, \omega)$ we define the summability method by the triangular matrix $\Lambda = \{\lambda_{ij}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(x, f) = \lambda_{0n} \frac{a_0}{2} + \sum_{i=1}^{n} \lambda_{in}(a_i(f) \cos ix + b_i(f) \sin ix)$$

If the Fourier series of f is given by (1.1), then *Zygmund-Riesz means of order* k is defined as

$$Z^k_n(x,\ f) = \frac{a_0}{2} + \sum_{i=1}^{n} \left(1 - \frac{i^k}{(n+1)^k}\right) (a_i(f) \cos ix + b_i(f) \sin ix).$$

We denote by $E_n(f)_M$ the best approximation of $f \in L_M(\mathbb{T},\omega)$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{M,\omega} = \inf\left\{ \|f - T_n\|_{L_M(\mathbb{T},\omega)} : T_n \in \Pi_n \right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

Let $T_n \in \Pi_n$

$$T_n = \frac{c_0}{2} + \sum_{i=1}^{n} (c_i \cos ix + d_i \sin ix).$$

The conjugate polynomial $\widetilde{T_n}$ is defined by

$$\widetilde{T_n} = \sum_{i=1}^{n} (c_i \sin ix - d_i \cos ix).$$

We will say that the method of summability by the matrix Λ satisfies condition $b_{k,M}$ (respectively $b^*_{k,M}$) if for $T_n \in \Pi_n$ the inequality

$$\|T_n - U_n(T_n)\|_{L_M(\mathbb{T},\omega)} \le c(n+1)^{-k} \left\| T_n^{(k)} \right\|_{L_M(\mathbb{T},\omega)}$$

$$\left(\|T_n - U_n(T_n)\|_{L_M(\mathbb{T},\omega)} \le c(n+1)^{-k} \left\| \widetilde{T_n}^{(k)} \right\|_{L_M(\mathbb{T},\omega)} \right)$$

holds and the norms

$$\|\Lambda\|_1 := \int_0^{2\pi} \left| \frac{\lambda_{0n}}{2} + \sum_{i=1}^{n} \lambda_{in} \cos it \right| dt$$

are bounded.

We use the constants $c, c_1, c_2, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

The problems of approximation theory in the weighted and non-weighted Orlicz spaces have been investigated by several authors (see, for example, [1], [10], [11], [14]-[18], [23] and [30]).

In the present paper necessary and sufficient condition about the relationship between the approximation of functions by linear means of Fourier series and by Zygmund-Riesz means of order k was investigated in reflexive weighted Orlicz spaces with Muckenhoupt weights Also, we investigate the approximation of functions by linear means of Fourier series in terms of the modulus of smoothness of these functions in reflexive weighted Orlicz spaces with Muckenhoupt weights. This result was applied to the approximation of the functions by linear means of Faber series in weighted Smirnov-Orlicz classes defined on simply connected domain of the complex plane. The similar problems in different spaces were investigated in [5] ,[6], [12], [13], [24], [32], and [34]-[36].

Main results in the present work are the following theorems:

Theorem 1.1. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$. In order that for $f \in L_M(\mathbb{T},\omega)$*

$$\|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T},\omega)} \le c_1 \left\| f - Z_n^k(\cdot,\ f) \right\|_{L_M(\mathbb{T},\omega)} \tag{1.2}$$

it is sufficient and necessary that for $f \in L_M(\mathbb{T}, \omega)$

$$\|T_n - U_n(\cdot,\ T_n)\|_{L_M(\mathbb{T},\omega)} \leq c_2 \left\|T_n - Z_n^k(\cdot,\ T_n)\right\|_{L_M(\mathbb{T},\omega)}. \tag{1.3}$$

Theorem 1.2. *Let* $L_M(\mathbb{T})$ *be a reflexive Orlicz space and* $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$. *If the summability method with the matrix* Λ *satisfies the condition* $(b_{k,M})$ *or* $\left(b^*_{k,M}\right)$, *then for* $f \in L_M(\mathbb{T}, \omega)$ *the inequality*

$$\|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T},\omega)} \leq c_3 \Omega^r_{M,\omega}\left(\frac{1}{n+1}, f\right) \tag{1.4}$$

holds with a constant $c_3 > 0$ *independent of* n.

Theorem 1.3. *Let* $L_M(\mathbb{T})$ *be a reflexive Orlicz space and* $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$. *If the summability method with the matrix* Λ *satisfies the condition* $(b_{k,M})$ *or* $\left(b^*_{k,M}\right)$, *then for* $f \in L_M(\mathbb{T}, \omega)$ *the estimate*

$$\Omega^r_{M,\omega}(\delta, U_n(\cdot,\ f)) \leq c_4 \Omega^r_{M,\omega}(\delta, f), \tag{1.5}$$

holds with a constant $c_4 > 0$ not depend on n, f and δ.

Corollary 1.4. This results obtained in Theorems 1.1 and 1.2 are valid for the Zugmund-Riesz means of order k.

Note that Theorem 1.1 was proved in[11], for modulus of continuity $\Omega_{M,\omega}(\delta, f)$ $(r = 1)$ and Zygmund-Riesz means of order 2.

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^- := ext\Gamma$. Further let

$$\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}, \mathbb{D} := int\,\mathbb{T} \text{ and } \mathbb{D}^- := ext\,\mathbb{T}.$$

Let $w = \phi(z)$ be the conformal mapping of G^- onto D^- normalized by

$$\phi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\phi(z)}{z} > 0,$$

and let ψ denote the inverse of ϕ.

Let $w = \phi_1(z)$ denote a function that maps the domain G conformally onto the disk $|w| < 1$. The inverse mapping of ϕ_1 will be denoted by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\phi_1(z)| = r$ under the mapping $z = \psi_1(w)$.

Let us denote by E_p, where $p > 0$, the class of all functions $f(z) \neq 0$ which are analytic in G and have the property that the integral

$$\int_{\Gamma_r} |f(z)|^p\, |dz|$$

is bounded for $0 < r < 1$. We shall call the E_p-class the *Smirnov class*. If the function $f(z)$ belongs to E_p, then $f(z)$ has definite limiting values $f(z')$ almost every where on Γ, over all nontangential paths; $|f(z')|$ is summable on Γ; and

$$\lim_{r\to 1} \int_{\Gamma_r} |f(z)|^p\, |dz| = \int_{\Gamma} \left|f(z')\right|^p |dz|.$$

It is known that $\varphi' = E_1(G^-)$ and $\psi' \in E_1(D^-)$. Note that the general information about Smirnov classes can be found in the books [9, pp. 438-453] and [7, pp. 168-185].

Let $L_M(\mathbb{T}, \omega)$ is a weighted Orlicz space defined on Γ. We define also the *ω-weighted Smirnov-Orlicz class* $E_M(G,\omega)$ as

$$E_M(G,\omega) := \{f \in E_1(G) : f \in L_M(\Gamma,\omega)\}.$$

With every weight function ω on Γ, we associate another weight ω_0 on $\mathbb{T}$ defined by

$$\omega_0(t) := \omega(\psi(t)), \ t \in \mathbb{T}.$$

For $f \in L_M(\Gamma,\omega)$ we define the function

$$f_0(t) := f(\psi(t)), \ t \in \mathbb{T}.$$

Let h be a continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0,2\pi], |t_1 - t_2| \leq t\}, t \geq 0.$$

The curve Γ is called *Dini-smooth* if it has a parametrization

$$\Gamma : \varphi_0(s), \ 0 \leq s \leq 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.

$$\int_0^\pi \frac{\omega(t,\varphi_0')}{t} dt < \infty$$

and $\varphi_0'(s) \neq 0$ [29, p. 48].

If Γ is Dini-smooth curve, then there exist [37] the constants c_5 and c_6 such that

$$0 \leq c_5 \leq |\psi'(t)| \leq c_6 < \infty, \ \ |t| > 1. \tag{1.6}$$

Note that if Γ is a Dini-smooth curve, then by (1.6) we have $f_0 \in L_M(\mathbb{T}, \omega_0)$ for $f \in L_M(\Gamma,\omega)$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'}$ and let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p -condition on Γ if

$$\sup_{z\in\Gamma} \sup_{r>0} \left(\frac{1}{r} \int_{\Gamma\cap D(z,r)} |\omega(\tau)|^p |d\tau| \right)^{1/p} \left(\frac{1}{r} \int_{\Gamma\cap D(z,r)} [\omega(\tau)]^{-p'} |d\tau| \right)^{1/p'} < \infty,$$

where $D(z,r)$ is an open disk with radius r and centered z.

Let us denote by $A_p(\Gamma)$ the set of all weight functions satisfying Muckenhoupt's A_p -condition on Γ. For a detailed discussion of Muckenhoupt weights on curves, see, e.g. [8].

Let Γ be a rectifiable Jordan curve and $f \in L_1(\Gamma)$. Then the function f^+ defined by

$$f^+(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(s)ds}{s-z}, \ z \in G$$

is analytic in G. Note that if $0 < \alpha_M \leq \beta_M < 1$, $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$ and $f \in L_M(\Gamma,\omega)$, then by Lemma 1 in [14] $f^+ \in E_M(G,\omega)$.

Let $\phi_k(z),\ k=0,1,2,...$ be the Faber polynomials for G. The Faber polynomials $\phi_k(z)$, associated with $G\cup\Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{\phi_k(z)}{w^{k+1}},\ \ z\in G,\ w\in D^- \tag{1.7}$$

and the equalities

$$\begin{aligned}\phi_k(z) &= \frac{1}{2\pi i}\int\limits_T \frac{t^k\psi'(t)}{\psi(t)-z}dt,\ \ z\in G, \\ \phi_k(z) &= \phi^k(z)+\frac{1}{2\pi i}\int\limits_T \frac{\phi^k(s)}{s-z}ds,\ \ z\in G^-\end{aligned} \tag{1.8}$$

hold [33, p. 33-48].

Let $f\in E_M(G,\omega)$. Since $f\in E_1(G)$. we obtain

$$f(z):=\frac{1}{2\pi i}\int\limits_\Gamma \frac{f(s)ds}{s-z}=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(t))\psi'(t)}{\psi(t)-z}dt,$$

for every $z\in G$. Considering this formula and expansion (1.7), we can associate with f the formal series

$$f(z)\sim\sum_{i=0}^{\infty}a_i(f)\phi_i(z),\ \ z\in G, \tag{1.9}$$

where

$$a_i(f):=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(t))}{t^{i+1}}dt.\ i=0,\ 1,\ 2,...$$

This series is called the *Faber series* expansion of f, and the coefficients $a_i(f)$ are said to be the *Faber coefficients* of f.

Let (1.9) be the Faber series of the function $f\in E_M(G,\omega)$. For the function f we define the summability method by the triangular matrix $\Lambda=\{\lambda_{ij}\}_{i,j=0}^{j,\infty}$ by the linear means

$$U_n(z,\ f)=\sum_{i=0}^{n}\lambda_{in}a_i(f)\phi_i(z),$$

The n-the *partial sums* and *Zygmund means of order* k of the series (1.9) are defined, respectively, as

$$S_n(z,\ f)=\sum_{k=0}^{n}a_k(f)\phi_k(z),$$

$$Z_n^k(z,\ f)=\sum_{i=0}^{n}\left(1-\frac{i^k}{(n+1)^k}\right)a_i(f)\phi_i(z).$$

Let Γ be a Dini-smooth curve. Using the nontangential boundary values of f_0^+ on $\mathbb{T}$ we define the *$r-th$ modulus of smoothness*

of $f\in L_M(\Gamma,\omega)$ as

$$\Omega^r_{\Gamma,.M,\omega}(\delta,f):=\Omega^r_{M,\omega_0}\left(\delta,f_0^+\right),\ \ \delta>0,$$

for $r=1,2,3,...$

The following theorem holds.

Theorem 1.5. *Let Γ be a Dini-smoth curve, and let $L_M(\Gamma)$ be a reflexive Orlicz space. If $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$ and the summability method with the matrix Λ satisfies the condition $(b_{k,M})$ or $\left(b^*_{k,M}\right)$, then for $f \in E_M(G,\omega)$ the estimate*

$$\|f - U_n(\cdot, f)\|_{L_M(\Gamma,\omega)} \le c_7 \Omega^r_{\Gamma,M,\omega}\left(\frac{1}{n+1}, f\right) \tag{1.10}$$

holds with a constant $c_7 > 0$, independent of n.

Let $\mathcal{P}$ be the set of al algebraic polynomials (with no restriction on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$. We define the operator

$$T : \mathcal{P}(\mathbb{D}) \longrightarrow E_M(G,\omega)$$

as

$$T(P)(z) := \frac{1}{2\pi i}\int\limits_{T} \frac{P(w)\psi'(w)}{\psi(w) - z} dw, \ z \in G.$$

Then from (1.8) we have

$$T\left(\sum_{k=0}^{n} \beta_k w^k\right) = \sum_{k=0}^{n} \beta_k \phi_k(z).$$

The following result hold for the linear operator T [14].

Theorem 1.6. *Let Γ be a Dini- smooth curve and $L_M(\Gamma)$ be a reflexive Orlicz space. If $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$,then linear operator $T : P(D) \longrightarrow E_M(G,\omega)$ is bounded.*

Theorem 1.7. I*f Γ is a Dini- smooth curve, $0 < \alpha_M \le \beta_M < 1$ and $\omega \in A_{1/\alpha_M}(\Gamma) \cap A_{1/\beta_M}(\Gamma)$, then the operator*

$$T : E_M(\mathbb{D},\omega_0) \longrightarrow E_M(G,\omega)$$

is one-to-one and onto.

2. Proof of main results

Proof of Theorem 1.1. Necessity. It is clear that the inequality (1.3) follows from the inequality (1.2).

Sufficiency. Let $f \in L_M(\mathbb{T},\omega)$ and let $T_n \in \Pi_n$ $(n = 0, 1, 2, ...)$ be the polynomial of best approximation to f . We obtain

$$\begin{aligned}
& \|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T},\omega)} \\
\leq\ & \|f - T_n\|_{L_M(\mathbb{T},\omega)} - \|T_n - U_n(\cdot, T_n)\|_{L_M(\mathbb{T},\omega)} \\
& + \|U_n(\cdot,\ T_n) - T_n\|_{L_M(\mathbb{T},\omega)} \\
\leq\ & E_n(f)_{M,\omega} + c_2 \left\| T_n - Z_n^k(\cdot, T_n) \right\|_{L_M(\mathbb{T},\omega)} + c_8 E_n(f)_{M,\omega} \\
\leq\ & c_9 E_n(f)_{M,\omega} + c_2 \left(\|T_n - f\|_{L_M(\mathbb{T},\omega)} + \left\| f - Z_n^k(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \right) \\
& + c_2 \left\| Z_n^k(\cdot, f - T_n) \right\|_{L_M(\mathbb{T},\omega)} \leq c_9 E_n(f)_{M,\omega} + c_2 E_n(f)_{M,\omega} \\
& + c_2 \left\| f - Z_n^k(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} + c_2 c_{10} E_n(f)_{M,\omega} \\
\leq\ & c_{11} E_n(f)_{M,\omega} + c_2 \left\| f - Z_n^k(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} \\
\leq\ & c_{12} \left\| f - Z_n^k(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)}
\end{aligned}$$

and Theorem 1.1 is proved.

Proof of Theorem 1.2. We suppose that the condition $b_{k,M}^*$ is satisfied. Let $f \in L_M(\mathbb{T},\omega)$ and $T_n \in \Pi_n$ be the polynomial of best approximation to f. Note that $U_n(f) = \Lambda_n * f$. The operator $U_n(f)$ is bounded in $L_p(\mathbb{T},\omega)$ and $L_q(\mathbb{T},\omega)$ (see [8] and[26]). Using the method of proof of Lemma 1 in [15] we can show that the operator $U_n(f)$ is bounded in $L_M(\mathbb{T},\omega)$, i.e. $\|U_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} \leq c_5 \|f\|_{L_M(\mathbb{T},\omega)}$. Then we get

$$\begin{aligned}
& \|f - U_n(\cdot,\ f)\|_{L_M(\mathbb{T},\omega)} \\
\leq\ & \|f - T_n\|_{L_M(\mathbb{T},\omega)} + \|T_n - U_n(\cdot, T_n)\|_{L_M(\mathbb{T},\omega)} \\
& + \|U_n(\cdot, T_n) - U_n(\cdot, f)\|_{L_M(\mathbb{T},\omega)} \\
\leq\ & c_{13} E_n(f)_{M,\omega} + c_7 E_n(f)_{M,\omega} + c_{14}(n+1)^{-2r} \left\| \widetilde{T_n}^{(2r)} \right\|_{L_M(\mathbb{T},\omega)} \\
\leq\ & c_{15} E_n(f)_{M,\omega} + c_{16} n^{-2r} \left\| \widetilde{T_n}^{(2r)} \right\|_{L(\mathbb{T},\omega)}. \qquad (2.1)
\end{aligned}$$

Using boundedness of the linear operator $f \to \widetilde{f}$ in $L_M(\mathbb{T},\omega)$ into account [15, (15)] we have

$$\left\| \widetilde{T_n}^{(2r)} \right\|_{L_M(\mathbb{T},\omega)} \leq c_{17} \left\| T_n^{(2r)} \right\|_{L_M(\mathbb{T},\omega)} \qquad (2.2)$$

where $\widetilde{f}$ is the conjugate function of $f \in L_M(\mathbb{T},\omega)$.

Note that according to the direct theorem of approximation in $L_M(\mathbb{T},\omega)$ given in [15] following inequality holds:

$$E_n(f)_{M,\omega} \leq c_{18} \Omega_{M,\omega}^r \left(\frac{1}{n+1}, f \right).$$

Using (2.2) and [15] we get

$$\begin{aligned} n^{-2r}\left\Vert \widetilde{T_n}^{(2r)}\right\Vert_{L_M(\mathbb{T},\omega)} &\leq c_{19}n^{-2r}\left\Vert T_n^{(2r)}\right\Vert_{L_M(\mathbb{T},\omega)} \\ &\leq c_{20}\Omega^r_{M,\omega}\left(\frac{1}{n+1},f\right). \end{aligned} \tag{2.3}$$

Note that according to the direct theorem of approximation in $L_M(\mathbb{T},\omega)$ given in [15] following inequality holds:

$$E_n(f)_{M,\omega} \leq c_{21}\Omega^r_{M,\omega}\left(\frac{1}{n+1},f\right). \tag{2.4}$$

Taking into account the relations (2.1), (2.3), and (2.4) we have

$$\left\Vert f-U_n(\cdot,f)\right\Vert_{L_M(\mathbb{T},\omega)} \leq c_{22}\Omega^r_{M,\omega}\left(\frac{1}{n+1},f\right).$$

If the summability method with the matrix Λ satisfies condition $(b^*_{k,M})$, the proof is made anologously to the above.

The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. By [15] the following inequality holds:

$$\Omega^r_{M,\omega}\left(\delta,U_n(f)-f\right) \leq c_{23}\left\Vert U_n(\cdot,f)-f\right\Vert_{L_M(\mathbb{T},\omega)}. \tag{2.5}$$

Let $\delta \geq (n+1)^{-1}$. Using Theorem 1.2 and (2.5) we have

$$\begin{aligned} \Omega^r_{M,\omega}\left(\delta,U_n(f)\right) &\leq \Omega^r_{M,\omega}\left(\delta,f\right)+\Omega^r_{M,\omega}\left(\delta,U_n(\cdot,f)-f\right) \\ &\leq \Omega^r_{M,\omega}\left(\delta,f\right)+c_{24}\left\Vert U_n(\cdot,f)-f\right\Vert_{L_M(\mathbb{T},\omega)} \\ &\leq \Omega^r_{M,\omega}\left(\delta,f\right)+c_{25}\Omega^r_{M,\omega}\left(\frac{1}{n+1},f\right) \\ &\leq c_{26}\Omega^r_{M,\omega}\left(\delta,f\right). \end{aligned} \tag{2.6}$$

Now we suppose that $\delta < (n+1)^{-1}$. Then by virtue of Corollary 5 and Theorem 8 in [15] we obtain

$$\begin{aligned} &\Omega^r_{M,\omega}\left(\delta,U_n(\cdot,f)\right) \\ \leq\ & c_{27}\delta^{2r}\left\Vert U_n^{(2r)}(\cdot,f)\right\Vert_{L_M(\mathbb{T},\omega)} \leq c_{28}\Omega^r_{M,\omega}(\delta,f). \end{aligned} \tag{2.7}$$

Now combining (2.6) and (2.7) we obtain the inequality (1.3) of Theorem 1.3.

Proof of Theorem 1.5. Let $f \in L_M(G,\omega)$. Then by virtue of Theorem 1.7 the operator $T: E_M(\mathbb{D},\omega_0) \longrightarrow E_M(G,\omega)$ is bounded one-to-one and onto and $T(f_0^+)=f$.The function f has the following Faber series

$$f(z) \backsim \sum_{m=0}^{\infty} a_m(f)\phi_m(z).$$

Since $\omega_0 \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\alpha_M}(\mathbb{T})$,using Lemma 1 in [14, p.760] we conclude that $f_0^+ \in E_M(\mathbb{D},\omega_0)$. For the function f_0^+ the following Taylor series holds:

$$f_0^+(w)=\sum_{m=0}^{\infty} a_m(f)w^m.$$

Note that $f_0^+ \in E_1(\mathbb{D})$ and boundary function $f_0^+ \in L_M(\mathbb{T},\omega)$. Then by [7, Theorem, 3.4] for the function f_0^+ we have the following Fourier expansion:

$$f_0^+(w) \backsim \sum_{m=0}^{\infty} a_m(f)e^{imt}.$$

Hence, if we consider boundedness of the operator $T : E_M(\mathbb{D},\omega_0) \longrightarrow E_M(G,\omega)$ and Theorem 1.2, we obtain

$$\begin{aligned} & \|f - U_n(.,f)\|_{L_M(\Gamma,\omega)} \\ = \;& \left\|T(f_0^+) - T(U_n(.,f_0^+))\right\|_{L_M(\Gamma,\omega)} \leq c_{29} \left\|f_0^+ - U_n(.,f_0^+)\right\|_{L_M(\mathbb{T},\omega)} \\ \leq \;& c_{30}\Omega^r_{M,\omega_0}\left(\frac{1}{n+1}, f_0^+\right) = c_{31}\Omega^r_{\Gamma,M,\omega}\left(\frac{1}{n+1}, f\right). \end{aligned}$$

and (1.8) is proved.

Remark 2.1. Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leq \beta_M < 1$, and $\omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T})$. Then by virtue of Theorem 4 in [15] for $f \in L_M(\mathbb{T},\omega)$ the inequality

$$\Omega^r_{M,\omega}\left(\frac{1}{n}, f\right) \quad \leq \quad \leq \frac{c_{32}}{n^{2r}}\left\{E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2r-1}E(f)_{M,\omega}\right\}, \tag{2.8}$$

holds with a constant c independent of n. If the summability method with the matrix Λ satisfy the condition $(b_{k,M})$ or $\left(b^*_{k,M}\right)$ then for $f \in L_M(\mathbb{T},\omega)$ relation (1.2) and inequality (2.8) immediately yield

$$\begin{aligned} & \|f - U_n(.,f)\|_{L_M(\mathbb{T},\omega)} \\ \leq \;& \frac{c_{39}}{n^{2r}}\left\{E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2r-1}E(f)_{M,\omega}\right\}. \end{aligned} \tag{2.9}$$

The inequality (2.9) holds for Zygmund-Riesz means of order k. Note that in the Lebesgue spaces $L_p(\mathbb{T})$, $1 < p \leq \infty$ the inequality (2.9) was proved in [34]

Acknowledgements

The author would like to thank referee for all precious advices and very helpful remarks.

References

[1] R. Akgun, D. M. Israfilov, Simultaneous and converse approximation theoremsin weighted Orlicz spaces, *Bull. Belg. Math. Soc. Simon Stevin* **17** (2010), 13-28.
[2] C.Bennet and R. Sharpley, *Interpolation of operators*, Academic Press, 1988.
[3] A. Bottcher and Yu. I. Karlovich, *Carleson curves, Muckenhoupt weights and Toeplitz operators*, Birkhauser-Verlag (1997).
[4] D.W. Boyd, Indices for the Orlicz spaces, *Pacific J. Math.* **38** (1971), 315-323.
[5] J.-D. Cao, Stechkin inequalities for summability methods, *Internat. J. Math. Sci.*, **20** (1997), 93-100.

[6] T. S. Chikina, Approximation by Zymund- Riesz means in the $p-$ variation metric, *Anal. Math.* **39** (2013), no. 1, 29-44.
[7] P. L. Duren, *Theory of H^p spaces,* Academic Press, 1970.
[8] E. M. Dynkin and B. P. Osilenker, Weighted norm estimates for singular integrals and their applications, *Itogi Nauki i Tekhniki Ser. Mat. Anal.* 21 (1983). 42-129 (in Russian); English transl.: J. Soviet Math. **30** (1985), 2094-2154.
[9] G. M. Goluzin, *Geometric theory of functions of a complex variable,* Translation of Mathematical Monographs, vol. 26, Providence, RI: AMS (1968).
[10] A. Guven, D. M. Israfilov, Polynomial approximation in Smirnov-Orlicz classes, *Comput. Methods and Function Theory*, 2 (2002), no.:2, 509-517.
[11] A. Guven, D. M. Israfilov, Approximation by means of Fourier trigonometric series in weighted Orlicz spaces, *Adv. Stud. Contemp. Math (Kyungshang)*, 19 (2009), no. 2, 283-295.
[12] V. G. Gavrilyuk, Linear methods of summing Fourier series and best approximation, *Ukr. Math. J.* **15** (1963), no. 4, 412-417.
[13] N. A. Il'yasov, Approximation of periodic functions by Zygmund means, *Mat. Zametki* **39** (1986), no. 3, 367-382 (in Russian).
[14] D. M. Israfilov, R. Akgun, Approximation in weighted Smirnov-Orlicz classes, *J. Math. Kyoto Univ.* **46**-4 (2006), 755-770.
[15] D. M. Israfilov, A. Guven, Approximation by trigonometric polynomials in weighted Orlicz space, *Studia Math.* **174** (2) (2006) 147-168.
[16] S. Z. Jafarov, Approximation by Rational Functions in Smirnov-Orlicz Classes, *J. Math. Anal. Appl.* **379** (2011) 870-877.
[17] S. Z. Jafarov, Approximation by polynomials and rational functions in Orlicz spaces, *J. Comput Anal. Appl. (JoCAAA)*, 13 (2011) no 5, 953-962.
[18] S. Z. Jafarov, Approximation by Fejer sums of Fourier trigonometric series in weighted Orlicz spaces, *Hacet. J. of Math. Stat.* **42** (2013), no. 3, 259-268.
[19] A.Yu. Karlovich, Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces, *J. Integ. Eq. Appl.* **15** (2003), 263-320.
[20] A. Yu. Karlovich, Singular integral operators with PC coefficients in reflexive rearrangement invariant spaces, *Integ. Eq. and Oper. Th.*, **32** (1998), 436-481.
[21] A.Yu. Karlovich, Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces, *Math. Nachr.* **179** (1996), 187-222.
[22] A. Yu. Karlovich, Algebras of singular integral operators with PC coefficients in reflexive rearrangement invariant spaces with Muckenhoupt weights, *J. Operator Theory*, 47 (2002), 303-323.
[23] V. Kokilashvili, On analytic functions of Smirnov-Orlicz classes, *Studia Math.* **31** (1968), 43-59.
[24] V. Kokilashvili, Ts. Tsanava, Approximation by linear summability means in weighted variable exponent Lebesgue spaces, *Proc. A. Razmadze Math. Inst.* 154 (2010), 147-150.
[25] M. A. Krasnosel'skii and Ja. B. Rutickii, *Convex Functions and Orlicz Spaces,* Nordhoff Ltd., (1961) (translated from the first Russian edition): Russian orginal: Problems of Contemporary Mathematics Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958.
[26] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207-226.
[27] L. Maligranda, *Indices and interpolation,* Dissertationes Math. 234 (1985).
[28] W. Matuszewska and W. Orlicz, On certain properties of ϕ-functions, *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys.* **8** (1960), 439-443.

[29] Ch. Pommerenke, *Boundary Behavior of Conformal Maps, Berlin, Springer-Verlag,* 1992.

[30] A.-R. K. Ramazanov, On approximation by polynomials and rational functions in Orlicz spaces, *Anal. Math.* **10** (1984), 117-132.

[31] M. M. Rao and Z. D. Ren, *Applications of Orlicz spaces,* Marcel Dekker Inc. (2002).

[32] S. B. Stechkin, The approximation of periodic functions by Fejer sums, *Trudy Math Inst. Steklov*, **62** (1961), 48-60 (in Russian).

[33] P. K. Suetin, *Series of Faber Polynomials*, Gordon and Breach Science Publishers, 1998.

[34] M. F. Timan, Best approximation of a function and linear methods of summing Fourier series, *Izv. Akad. Nauk SSSR ser: Math.* **29** (1965), 587-604 (in Russian).

[35] M. F. Timan, The approximation of continuous periodic functions by linera operators which are construced on the basis of their Fourier series, *Dokl. Akad. Nauk SSSR* 181 (1968), 1339-1342 (in Russian).

[36] M. F. Timan, Some linear summation processes fr Fourier series and best approximation, *Dokl. Akad. Nauk SSSR*, **145** (1962), 741-743.

[37] S. E. Warschawskii, Über das Ranverhalten der Ableitung der Abbildungsfunctionen bei konformer Abbildung, *Math. Z.*, **35** (1932), 321-456.

Sadulla Z. Jafarov

Faculty of Education, Alparslan University, 49250, Turkey; Institute of Mathematics and Mechanics, ANAS, AZ1141, Baku, Azerbaijan

E-mail address: `s.jafarov@alparslan.edu.tr.`

Received: January 12, 2017; Revised: March 28, 2017; Accepted: April 17, 2017