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ON MULTIPLE COMPLETENESS OF EIGEN AND ASSOCIATED VECTORS OF A CLASS OF OPERATOR PENCILS

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Abstract. In the paper we obtain conditions providing multiple completeness of eigen and associated elements of higher order operator pencils on a finite interval. These conditions are directly expressed by properties of coefficients of the operator pencil.

1. Introduction

On separable Hilbert space H we consider a polynomial operator pencil of n-th order

$$P(\lambda) = (-1)^k \lambda^n E + \lambda^{n-1} A_1 + ... + A_n + A^n, \tag{1.1}$$

where $n = 2k$, $(k = 1, 2, ...)$, λ is a spectral parameter, E is a unit operator in H, the remaining coefficients of the operator pencil (1.1) satisfy the conditions:

1) A is a positive-definite operator with completely continuous inverse A^{-1};
2) The operators $B_j = A_j A^{-j}$ $(j = 1, ..., n)$ are bounded in H;
3) The operator $E + B_n$ is invertible in H.

Note that subject to conditions 1)-3), the operator pencil has a discrete spectrum with a unique limit point at infinity. Indeed,

$$P(\lambda) = (E + B_n) \left(\sum_{j=1}^{n-1} \lambda^{n-j} (E + B_n)^{-1} (A_j A^{-j}) \cdot A^{-n+j} \right.$$

$$\left. + (-1)^k \lambda^n (E + B_n)^{-1} A^{-n} + E \right) A^n = (E + B_n)(E + L(\lambda))A^n,$$

where

$$L(\lambda) = (-1)^k \lambda^n (E + B_n)^{-1} A^{-n} + \sum_{j=1}^{n-1} \lambda^{n-j} (E + B_n)^{-1} B_j A^{-n+j}.$$

As $L(0) = 0$, the coefficients $(E + B_n)^{-1} A^{-n}$ and $(E + B_n)^{-1} B_n A^{-n+j}$ $(j = \overline{1,\ n-1})$ are completely continuous in H, then $E + L(\lambda)$ by the Keldysh lemma [4] has only a discrete spectrum with a unique limit point at infinity. Then the operator pencil $P(\lambda) = (E + B_n)^{-1}(E + L(\lambda))A^n$ also possesses this property.

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Obviously, the domain of definition of the operator A^γ $(\gamma \geq 0)$ is Hilbert space H_γ with respect to the scalar product $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in H_\gamma$. For $\gamma = 0$ we assume $H_0 = H$.

Denote by $L_2(R_+ : H)$ Hilbert space of all vector-functions $f(t)$ determined almost everywhere in the interval $(0,\ 1)$ with values in H, for which

$$\|f\|_{L_2((0,1)\,:H)} = \left(\int_0^1 \|f(t)\|^2\, dt\right)^{1/2} < +\infty.$$

Further, following the monograph [5] we introduce Hilbert space

$$W_2^n((0,1):H) = \Big\{ u :\ u^{(n)} \in L_2((0,1):H),\quad A^n u \in L_2((0,1):H)\Big\}$$

with the norm

$$\|u\|_{W_2^n((0,\,1):H)} = \left(\|A^n u\|^2_{L_2((0,1):H)} + \|u^n\|^2_{L_2((0,1):H)}\right)^{1/2}.$$

Here and in the sequel, the derivatives are understood in the sense of theory of distributions in abstract Hilbert spaces [5].

Associate the pencil (1.1) with the boundary value problem

$$P(d/dt)u(t) = (-1)^k u^{(n)} + A^n u + \sum_{j=1}^{n-1} A_{n-j} u^{(j)} = 0, \quad t \in (0,\ 1) \tag{1.2}$$

$$u^{(2\,\nu)}(0) = \varphi_\nu, u^{(2\,\nu)}(1) = \psi_\nu, \nu = \overline{0,\ k-1}. \tag{1.3}$$

Definition 1.1. If $u(t) \in W_2^n((0,1) : H)$ satisfies the equation (1.2) almost everywhere in $(0,1)$, then $u(t)$ it said to be a regular solution of equation (1.2).

Definition 1.2. If for any collection of n vectors $\varphi_\nu \in H_{n-2\,\nu-1/2}$, $\psi_\nu \in H_{n-2\,\nu-1/2}$, $(\nu = \overline{0,\ k-1})$ there exists the regular solution $u(t)$ of equation (1.2) satisfying boundary conditions (1.3) in the sense of convergence

$$\lim_{t\to+0}\left\|u^{(2\,\nu)}(t) - \psi_\nu\right\|_{n-2\,\nu-1/2} = 0, \lim_{t\to 1-0}\left\|u^{(2\,\nu)}(t) - \psi_\nu\right\|_{n-2\,\nu-1/2} = 0, \nu = \overline{0, k-1}$$

and the estimation

$$\|u(t)\|_{W_2^n((0,\,1):H)} \leq const \sum_{\nu=0}^{k-1}\left(\|\varphi_\nu\|_{n-2\,\nu-1/2} + \|\psi_\nu\|_{n-2\,\nu-1/2}\right),$$

then problem (1.2), (1.3) is said to be regularly solvable.

Definition 1.3. If the equation $P(\lambda_i)\, x_{0,i,j} = 0$ has nonzero solution $x_{0,i,j}$, then λ_i is called a characteristic number of $P(\lambda)$, and $x_{0,i,j}$ an eigen-vector of the operator pencil $P(\lambda)$, corresponding to λ_i. If the vectors $x_{0,i,j}$, $x_{1,i,j}$, ..., $x_{h,i,j}$, $h = \overline{0,\ m_{i,j}}$, $j = \overline{1,\ q_i}$ satisfy the equations

$$\sum_{p=0}^{h} \left.\frac{\partial^p P(\lambda)}{\partial \lambda^p}\right|_{\lambda=\lambda_i} x_{h-p,i,j} = 0, h = \overline{0,\ m_{i,j}}, j = \overline{1,\ q_i},$$

then $x_{0,i,j}$, ..., $x_{h,i,j}$ are said to be eigen and associated vectors of the pencil $P(\lambda)$, corresponding to the characteristic number λ_i.

The vector-functions

$$u_{h,i,j}(t)=e^{\lambda_i t}\left(\frac{t^h}{h!}x_{0,i,j}+\frac{t^{h-1}}{(h-1)!}x_{1,i,j}+...+x_{h,i,j}\right),h=\overline{0,\ m_{i,j}},j=\overline{1,\ q_i}$$

belong to the space $W_2^n((0,\ 1):H)$, satisfy equation (1.2) and are called elementary solutions of equation (1.2).

In the space $\overset{k-1}{\underset{\nu=0}{\oplus}} H_{n-2\nu-1/2}\times\overset{n-1}{\underset{\nu=0}{\oplus}} H_{n-2\nu-1/2}$ let us construct the system

$$\{\tilde{x}_{h,i,j}\}_{i=1,h=\overline{0,m_{i,j}},j=\overline{1,\ q_m}}^{\infty}\equiv\left\{u_{h,i,j}^{(2\nu)}(0),u_{h,i,j}^{(2\nu)}(1)\right\}_{i=1,h=\overline{0,m_{i,j}},j=\overline{1,\ q_i}}^{\infty}.$$

Definition 1.4. If the system $\{\tilde{x}_{h,i,j}\}_{i=1,h=\overline{0,m_{i,j}},j=\overline{1,\ q_i}}^{\infty}$ is complete in the space $\overset{k-1}{\underset{\nu=0}{\oplus}} H_{n-2\nu-1/2}\times\overset{n-1}{\underset{\nu=0}{\oplus}}\bar{H}_{n-2\nu-1/2}$, we say that the system of eigen and associated vectors $P(\lambda)$ is n-fold complete in the space of traces of regular solutions of equation (1.2).

In the present paper we find conditions on the coefficients of the operator pencil $P(\lambda)$, that provide n-fold completeness of the system of eigen and associated vectors of the pencil $P(\lambda)$ in the space of traces of solutions.

To this end we study regular solvability of problem (1.2), (1.3) and estimate the norms of the resolvent $P^{-1}(\lambda)$ on some rays.

Similar problems were studied for example in the papers [1]-[4], [6]-[8], [10]-[12].

2. On regular solvability of problem (1.2), (1.3).

At first we consider the problem

$$(-1)^k u^{(n)}(t)+A^n u(t)=0,t\in(0,\ 1)\tag{2.1}$$

$$u^{(2\nu)}(0)=\varphi_\nu,u^{(2\nu)}(1)=\psi_\nu,\nu=\overline{0,\ k-1}\tag{2.2}$$

It holds

Theorem 2.1. *Problem (2.1), (2.2) is regularly solvable.*

Proof. Let $\omega_0,\omega_1,...,\omega_{n-1}$ be the solution of the equation $(-1)^n\omega^n+1=0$, moreover $Re\omega_l<0$ $(l=0,...,k-1)$, $Re\omega_l>0$ $(l=k,...,n-1)$. Then the general solution of equation (2.1) from the space $W_2^n((0,\ 1):H)$ has the form

$$u_0(t)=\sum_{l=0}^{k-1}e^{\omega_l tA}c_l+\sum_{l=k}^{n-1}e^{\omega_l(t-1)A}c_l,\tag{2.3}$$

where the vectors $c_l\in H_{n-1/2}$. From condition (2.2) it follows that

$$\sum_{l=0}^{k-1}\omega_l^{2\nu}A^{2\nu}c_l+\sum_{l=k}^{n-1}(\omega_e)^{2\nu}A^{2\nu}e^{-\omega_l A}c_l=\varphi_\nu,\nu=\overline{0,\ k-1},$$

$$\sum_{l=0}^{k-1}\omega_l^{2\nu}A^{2\nu}l^{\omega_l A}c_l+\sum_{l=k}^{n-1}\omega_l^{2\nu}A^{2\nu}c_l=\psi_\nu,\nu=\overline{0,\ k-1},$$

or

$$\sum_{l=0}^{k-1}\omega_l^{2\,\nu}c_l+\sum_{l=k}^{n-1}\omega_l^{2\,\nu}A^{-\omega_l A}c_l=A^{-2\nu}\varphi_\nu,\nu=\overline{0,\ k-1},$$

$$\sum_{l=0}^{k-1}\omega_l^{2\,\nu}e^{\omega_l A}c_l+\sum_{l=k}^{n-1}\omega_l^{2\,\nu}c_l=A^{-2\nu}\psi_\nu,\nu=\overline{0,\ k-1}.$$

Thus, we get the equation $\Delta(A)\,\tilde{c}=\bar{\theta}$, $\tilde{\theta}=\left(\varphi_0,...,\ A^{-2\,\nu}\varphi_{k-1},\ \psi_0,...,A^{-2\,\nu}\psi_{n-1}...\right)$, $\tilde{c}=(c_0,\ c_1,\ ...,\ c_{n-1})$, where

$$\Delta\left(A\right)=\begin{pmatrix} E & \cdots & E & e^{-\omega_k A} & \cdots & e^{-\omega_{n-1}A}\\ \omega_0^2E & & \omega_{k-1}^2E & \omega_k^2e^{-\omega_k A} & & \omega_{n-1}^2e^{\omega_{n-1}A}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \omega_0^{2(k-1)}E & & \omega_{k-1}^{2(k-1)}E & \omega_k^{2(k-1)}e^{-\omega_k A} & & \omega_{n-1}^{2(k-1)}e^{-\omega_{n-1}A}\\ e^{\omega_0 A} & & e^{\omega_{k-1}A} & E & & E\\ \omega_0^2e^{\omega_0 A} & & \omega_{k-1}^2e^{\omega_{k-1}A} & \omega_k^2E & & \omega_{n-1}^2E\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \omega_0^{2(k-1)}e^{\omega_0 A} & \cdots & \omega_{k-1}^{2(k-1)}e^{\omega_{k-1}A} & \omega_k^{2(k-1)}E & \cdots & \omega_{n-1}^2E \end{pmatrix} \tag{2.4}$$

Let $\sigma\geq\mu_0$, i.e. $\sigma\in[\mu_0,\infty)$, where μ_0 is the lower bound of spectrum A and consider the scalar matrix $\Delta(\sigma)$ in the domain $[\mu_0,\infty)$. Obviously, as $\sigma\to\infty$ $|\det\Delta(\sigma)|=\left|wronskian(\omega_0^2,...,\omega_{k-1}^2)\right|\times\left|wronskian(\omega_k^2,...,\omega_{n-1}^2)\right|\neq0$. Therefore, there exists $R_0>0$ such that for $\sigma\in[R_0,\infty)$ $|\det\Delta(\sigma)|>c>0$. Now show that for any $\sigma\in[\mu_0,R_0]$ $\det\Delta(\sigma)\neq0$. If it is not so, then $\det\Delta(\sigma_0)=0$ for some $\sigma_0\in[\mu_0,R_0]$.

Then the equation $\Delta(\sigma_0)\bar{\xi}=0$ has the nonzero solution $\bar{\xi}=(\xi_0,\xi_1,...,\xi_{n-1})$. This means that the scalar function

$$\xi(t)=\sum_{l=0}^{k-1}e^{\omega_l t\sigma_0}\xi_l+\sum_{l=k}^{n-1}e^{\omega_l(t-1)\sigma_0}\xi_l$$

satisfies the equation

$$(-1)^k\xi^{(n)}(t)+\sigma_0^n\xi(t)=0,t\in(0,\ 1) \tag{2.5}$$

and the boundary condition

$$\xi^{(2\,\nu)}(0)=0,\xi^{(2\,\nu)}(1)=0,\ \ \nu=\overline{0,k-1}. \tag{2.6}$$

Multiplying the equation (2.5) scalarly by the function $\xi(t)$ in $L_2(0,\ 1)$ and taking into account condition (2.6), after integrating by parts, we get

$$\left\|\xi^{(k)}\right\|^2_{L_2(0,\,1)}+\sigma_0^n\left\|\xi\right\|^2_{L_2(0,\,1)}=0.$$

Hence we get $\xi(t)=0$, i.e. $\bar{\xi}=(\xi_0,...,\xi_{n-1})=0$ and this contradicts the condition $\bar{\xi}\neq0$. As $|\det\Delta(\sigma)|$ is a continuous function on $[\mu_0,\ R_0]$, then $\inf_\sigma|\det\Delta(\sigma)|\geq d_0>0$. Then we get that for all $\sigma\in[\mu_0,\infty)$ the inequality $|\det\Delta(\sigma)|>d>0$ is valid.

Using spectral expansion of A we get that $\Delta^{-1}(A)$ exists and takes the space $\oplus H_{n-1/2}$ to $\oplus H_{n-1/2}$ and $\left\|\Delta^{-1}(A)\right\|\leq const$. Here the n copies of direct sum of spaces $H_{n-1/2}$ is taken. Then, obviously

$$\sum_{l=0}^{n-1} \|c_l\|_{n-1/2} \leq const \left|\left|\Delta^{-1}(A)\right|\right| \left(\sum_{l=0}^{n-1} \left\|A^{-2\nu}\varphi_\nu\right\| + \sum_{l=0}^{n-1} \left\|A^{-2\nu}\psi_\nu\right\|_{n-1/2}^2 \right)$$

$$\leq const \left(\sum_{l=0}^{n-1} \|\varphi_\nu\|_{n-2\,\nu-1/2} + \sum_{l=k}^{k+1} \|\psi_\nu\|_{n-2\,\nu-1/2} \right).$$

Hence we get

$$\|u_0(t)\|_{W_2^n((0,1):H)} \leq const \left(\sum_{l=0}^{k} \|\varphi_\nu\|_{n-2\,\nu-1/2} + \sum_{l=0}^{n-1} \|\psi_\nu\|_{n-2\,\nu-1/2} \right). \qquad (2.7)$$

The theorem is proved. □

Now solve the problem (1.2), (1.3).
Denote by

$$\overset{\circ}{W}{}_2^n = \left\{ u:\ u \in W_2^n((0,\ 1)H),\ u^{(2\,\nu)}(0) = u^{(2\,\nu)}(1) = 0 \right\}$$

and consider the operators

$$P_0 u = P_0(d/dt)u = (-1)^{(k)} u^{(n)}(t) + A^n u(t),$$

$$P_1 u = P_1(d/dt)u = \sum_{j=0}^{n-1} A_{n-j} u^{(j)}(t), u \in \overset{\circ}{W}{}_2^n((0,\ 1) : H).$$

After substitution $u(t) = \omega(t) + u_0(t)$, where $\omega(t)$ is an unknown function from $\overset{\circ}{W}{}_2^n((0,\ 1) : H)$, and $u_0(t)$ is determined from (2.3) as the solution of problem (2.1), (2.2). From boundary condition (1.2), (1.3) we get

$$(P_0(d/dt) + P_1(d/dt))\,\omega(t) = g(t), t \in (0,\ 1) \qquad (2.8)$$

$$\omega^{(2\,\nu)}(0) = 0, \omega^{(2\,\nu)}(1) = 0, \qquad (2.9)$$

where $\omega \in \overset{\circ}{W}{}_2^n((0,\ 1) : H)$, while $g(t) \in L_2((0,\ 1) : H)$.
Indeed, after substitution $\omega(t) = u(t) - u_0(t)$ we have:

$$(P_0(d/dt) + P_1(d/dt))\,\omega(t) = -P_1(d/dt)u_0(t),$$

$$\omega^{(2\,\nu)}(0) = 0, \omega^{(2\,\nu)}(1) = 0$$

moreover

$$\|g(t)\|_{L_2((0,\,1):H)} = \|P_1(d/dt)u_0(t)\|_{L_2((0,\,1):H)}$$

$$\leq \sum_{j=0}^{n-1} \|B_j\| \cdot \left\|A^{n-j} u_0^{(j)}(t)\right\| \leq const\, \|u_0(t)\|_{W_2^n((0,\,1):H)}$$

$$\leq const \left(\sum_{\nu=0}^{k-1} \|\varphi_\nu\|_{n-2\,\nu-1/2} + \sum_{\nu=k}^{n-1} \|\psi_\nu\|_{n-2\,\nu-1/2} \right).$$

Here we used the theorem on intermediate derivatives [5] and inequality (2.7). Thus, we obtained problem (2.8), (2.9).

Now we will use the following result from the paper [9].

Theorem 2.2. [9] *Let conditions 1), 2) be fulfilled, and the inequality*

$$q=\sum_{j=0}^{n-1} d_{n,j}\left\|B_j\right\|<1, \tag{2.10}$$

where

$$d_{n,j}=\begin{cases}\left(\frac{n-j}{n}\right)^{\frac{n-j}{n}}\left(\frac{j}{n}\right)^{j/n}, & j=1,\ldots n-1\\ 1, & j=0\end{cases}, \tag{2.11}$$

hold. Then for any $g(t)\in L_2((0,1):H)$ *there exists* $\omega(t)\in \overset{\circ}{W}{}_2^n((0,\,1):H)$ *that satisfies equation (2.8) almost everywhere and the estimation*

$$\|\omega\|_{\dot{W}_2^n((0,1):H)}\leq const\,\|g(t)\|_{L_2((0,1):H)}\,.$$

Thus, we obtain

Theorem 2.3. *Let all conditions of theorem 2.2 be fulfilled. Then problem (1.2), (1.3) is regularly solvable.*

3. On completeness of eigen and associated vectors of the pencil $P(\lambda)$

At first we prove a theorem on estimation of the resolvent $P^{-1}(\lambda)$.

Theorem 3.1. *Let conditions 1), 2) be fulfilled,* $\omega_0,\ldots,\omega_{n-1}$ *be the roots of the equation* $(-1)^k\lambda^n+1=0$ *(*$n=2k,\ k=1,2,\ldots$*) and the number* $0<\alpha\leq\pi/n$*. Then subject to the condition*

$$q=\sum_{j=0}^{n-1} d_{n,j}\left\|B_j\right\|<\sin\frac{n\alpha}{2}, \tag{3.1}$$

on the rays $\Gamma_{s,\alpha}^{\pm}=\{\lambda:\arg\lambda=\arg\omega_s\pm\alpha\}$, $s=\overline{0,\ n-1}$ *there exists the resolvent* $P^{-1}(\lambda)$*, and on these rays it holds the estimation*

$$\left\|A^{\beta}P^{-1}(\lambda)\right\|\leq const\,(|\lambda|+1)^{-n+\beta}\,,0\leq\beta\leq n.$$

Here the numbers $d_{n,j}$ *(*$j=\overline{0,\ n-1}$*) are determined from equality (2.11).*

Proof. Obviously $\|B_n\|<1$, i.e. $E+B_n$ has a bounded inverse in H. Then $P(\lambda)$ has a discrete spectrum.

At first we show that on the rays $\Gamma_{s,\alpha}^{\pm}=\{\lambda:\arg\lambda=\arg\omega_s\pm\alpha\}$ $s=\overline{0,\ n-1}$ the estimations

$$\left\|A^{n-j}\lambda^j\left((-1)^k\lambda^nE+A^n\right)^{-1}\right\|\leq d_{n,j}\left(\sin\frac{n\alpha}{2}\right)^{-1} \tag{3.2}$$

hold.

Indeed, for $\lambda\in\Gamma_{s,\alpha}^{\pm}$ we have:

$$\left\|A^{n-j}\lambda^j\left((-1)^k\lambda^nE+A^n\right)^{-1}\right\|=\left\|A^{n-j}r^j\left(-r^ne^{\pm in\alpha}E+A^n\right)^{-1}\right\|$$

$$=\sup_{\mu\in\sigma(A)}\left|\mu^{n-j}r^j(-r^ne^{\pm in\alpha}+\mu^n)^{-1}\right|=\sup_{\mu\in\sigma(A)}\left|\mu^{n-j}r^j(r^{2n}+\mu^{2n}-2r^n\mu^n\cos n\alpha)^{-1/2}\right|.$$

After substitution $\tau = r/\mu$ and using the Cauchy inequality, we get that for $\lambda \in \Gamma_{\alpha}^{\pm}$ the following inequalities

$$\left\| A^{n-j}\lambda^j \left((-1)^k \lambda^n E + A^n\right)^{-1} \right\| \le \sup_{\tau>0} \left| \tau^j (\tau^{2n} + 1 - 2\tau^n \cos n\alpha)^{-1/2} \right|$$

$$= \sup_{\tau>0} \left| \tau^j ((\tau^n + 1)^2 - 2\tau^n (1 + \cos n\alpha))^{-1/2} \right|$$

$$= \sup_{\tau>0} \left| \tau^j (\tau^n + 1)^{-1} \left(1 - \frac{4\tau^n}{(\tau^n + 1)^2} \cos^2 \frac{n\alpha}{2}\right)^{-1/2} \right|$$

$$\le \sup_{\tau>0} \left| \tau^j (\tau^n + 1)^{-1} \right| \cdot \left(1 - \cos^2 \frac{n\alpha}{2}\right)^{-1/2}$$

$$= \left(\sin \frac{n\alpha}{2}\right)^{-1} \sup_{\tau>0} \left| \tau^j (\tau^n + 1)^{-1} \right| = d_{n,j} \left(\sin \frac{n\alpha}{2}\right)^{-1}$$

hold.

Then from the equality

$$P(\lambda) = P_0(\lambda) + P_1(\lambda) = (E + P_1(\lambda) P_0^{-1}(\lambda)) P_0(\lambda) \tag{3.3}$$

we get that on the rays $\Gamma_{s,\alpha}^{\pm}$ there hold the estimations

$$\left\| P_1(\lambda) P_0^{-1}(\lambda) \right\| = \left\| \sum_{j=0}^{n-1} \lambda^j A_{n-j} ((-1)^k \lambda^n E + A^n)^{-1} \right\|$$

$$= \sum_{j=0}^{n-1} \|B_{n-j}\| \left\| A^{n-j} \lambda^j ((-1)^k \lambda^n E + A^n)^{-1} \right\| \le \sum_{j=0}^{n-1} d_{n,j} \left(\sin \frac{n\alpha}{2}\right)^{-1} \|B_{n-j}\|$$

$$= \sum_{j=0}^{n-1} d_{n,j} \left(\sin \frac{n\alpha}{2}\right)^{-1} \|B_j\| < 1.$$

Then from equality (3.3) it follows that

$$\left\| A^\beta P^{-1}(\lambda) \right\| \le \left\| A^\beta P_0^{-1}(\lambda) \right\| \left\| (E + P_1(\lambda) P_0^{-1}(\lambda)) \right\| \le const \, (|\lambda|^n + 1)^{-n+\beta}.$$

Now using the method of the papers [10], [11] we prove a theorem on n -fold completeness of eigen and associated vectors in the space of traces. □

Theorem 3.2. *Let conditions 1), 2), $A^1 \in \sigma_\rho$ $(0 < \rho < \infty)$ be fulfilled and the inequality*

$$q(\rho) = \sum_{j=0}^{n-1} d_{n,j} \|B_j\| < \begin{cases} 1, & \text{for } 0 < \rho \le k \\ \sin \frac{\pi n}{4\rho}, & \text{for } k \le \rho < \infty \end{cases} \tag{3.4}$$

hold. Then the system of eigen and associated vectors of the pencil $P(\lambda)$ is n -fold complete in the space of traces.

Proof. As $A^{-1} \in \sigma_\rho$ $(0 < \rho < \infty)$, the operators $B_j = A_j A^{-j}$ $(j = 1,\ n)$ are completely continuous in H, then from the Keldysh theorem [4],[9] it follows that $A^n P^{-1}(\lambda)$ is represented in the form of ratio of two entire functions of order ρ and of minimal type of order ρ. Assume that the system of eigen and associated vectors of the operator-pencil $P(\lambda)$ is not n -fold complete in the space of traces. Then from the results of the paper of M.V. Keldysh [1], M.G. Gasymov and S.S. Mirzoyev [6] it follows that there exist the vectors $\chi_\nu \in H_{n-2\nu-1/2}$ $(\nu = \overline{0,\ n-1})$ such that $\sum_{\nu=0}^{n-1} \|\chi_\nu\|_{n-2\nu-1/2} \neq 0$, and the operator-function

$$g(\lambda) = \sum_{\nu=0}^{k-1} \lambda^{2\nu} \left(A^{n-2\nu-1/2} P_\nu^{-1}(\bar{\lambda})\right)^* A^{n-\nu-1/2} \chi_\nu$$

$$+ \sum_{\nu=k}^{n-1} \lambda^{2\nu} e^{\lambda} \left(A^{n-2\nu-1/2} P^{-1}(\bar{\lambda})\right)^* A^{n-2\nu-1/2} \chi_\nu$$

is an entire function. From the theorem [2] on estimation of the resolvent for $0 < \rho \le k$ it follows that on the rays $\Gamma^{\pm}_{s,\pi/2k} = \left\{\lambda : \arg\lambda = \arg\omega_k + \frac{\pi}{2k}\right\}$ the estimations

$$\|g(\lambda)\| \le const \cdot \lambda^{-1/2}(1 + e^{Re\lambda}) \tag{3.5}$$

hold.

Then taking into account that the angle between these rays equals $\frac{\pi}{n}$, then for $\frac{\pi}{n} \le \frac{\pi}{2p}$, i.e. when $0 < \rho \le \frac{n}{2} = k$ applying the Fragmen-Lindeloff theorem we get that estimation (3.5) holds on the whole of complex plane. If $k \le \rho < \infty$, then again using theorem 5 on estimation of the resolvent, and the Fragmen-Lindeloff theorem, we get that estimation (3.5) holds for all λ from the complex plane. Thus, in both cases, inequality (3.5) holds for all λ. Further it is obvious that for $\lambda = i\xi$, $\xi \in R$, $\lim\limits_{\xi\to\infty} \|g(i\xi)\| = 0$.

As the boundary value problem (1.2), (1.3) is regularly solvable, we denote its solution by $u(t)$ and assume

$$\hat{u}(\lambda) = \int_0^1 u(t) e^{-\lambda t} dt.$$

Obviously, $\hat{u}(\lambda)$ is an entire function. Further repeating all the reasonings from the paper [11, pp. 95-99] we get $\chi_\nu = 0$ $(\nu = \overline{0,\ n-1})$. And this contradicts the condition $\sum_{\nu=0}^{n-1} |\chi_\nu| \neq 0$. The theorem is proved. □

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