Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

ELLIPTIC EQUATIONS WITH MEASURABLE COEFFICIENTS IN GENERALIZED WEIGHTED MORREY SPACES

AHMET EROGLU, MEHRIBAN N. OMAROVA, AND SHEMSIYYE A. MURADOVA

Abstract. We obtain a global generalized weighted Morrey $M^{p,\varphi}_w$ estimate for the gradient of the weak solutions to divergence form elliptic equations with measurable coefficients in a nonsmooth bounded domain. The coefficients are assumed to be merely measurable in one variable and to have small BMO semi-norms in the remaining variables, while the boundary of the domain is supposed to be Reifenberg flat, which goes beyond the category of domains with Lipschitz continuous boundaries. As consequence of the main result, we derive global gradient estimate for the weak solution in the framework of the generalized weighted Morrey spaces.

1. Introduction

This work is concerned with generalized weighted Morrey $M^{p,\varphi}_w$ - regularity of the gradient of weak solutions to the Dirichlet problems regarding elliptic equations with possibly measurable coefficients in nonsmooth domains. The problems in mind are related to some important variational problems arising in the mechanics of membranes and films of simple nonhomogeneous materials which form a linear laminated medium. In particular, a highly twinned elastic or ferroelectric crystal is a situation where a laminate appears. The equilibrium equations of such linear laminates usually have merely bounded measurable coefficients, see [3, 11, 14, 15, 23].

In [25] Morrey studied some integral inequalities in connection with Hölder regularity of solutions of nonlinear elliptic and parabolic operators. The classical *Morrey spaces* $L^{p,\lambda}$, usually attributed to him, were formulated in terms of function spaces by Campanato, Brudnyi and Peetre in the 1960s. They introduced notations similar to those used in the definition below (cf. [34]).

A real valued function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ with $p \in (1,\infty)$, $\lambda \in (0,n)$ provided the following norm is finite

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \left(\sup_{(x,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{r^\lambda} \int_{C_r(x)} |f(y)|^p \, dy \right)^{1/p}$$

2010 *Mathematics Subject Classification.* Primary: 35R05; Secondary: 35J15, 35B45, 35B65, 46E30, 46E35.

Key words and phrases. Elliptic equation, measurable coefficients, gradient estimates, Muckenhoupt weight, generalized weighted Morrey spaces, Reifenberg flat domain.

where the supremo is taken over all balls $C_r(x) \subset \mathbb{R}^n$. The main result connected with these spaces is the following celebrated lemma: *Let* $|Df| \in L^{p,n-\lambda}$ *even locally, with* $n-\lambda < p$*, then* u *is Hölder continuous of exponent* $\alpha = 1 - \frac{n-\lambda}{p}$. In [12] Chiarenza and Frasca showed boundedness of *the Hardy-Littlewood maximal operator* $\mathcal{M}$ and the *Calderón-Zygmund integral operator* in $L^{p,\lambda}(\mathbb{R}^n)$. This allows them later to study the regularity of the solutions of the Dirichlet problem for linear elliptic PDE's with VMO coefficients.

In [24] Mizuhara extended the Morrey concept taking a weight function $\phi(x,r)$: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of r^λ in the definition of the norm while in [27], Nakai extended the results of Chiarenza and Frasca from $L^{p,\lambda}$ to $L^{p,\phi}$ imposing integral and doubling conditions on ϕ. These results allow to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $L^{p,\phi}$ (see [31] and the references therein). A further development of the generalized Morrey spaces can be found in the works of Guliyev at al. where he introduced the spaces $M^{p,\varphi}$ under different conditions on φ (see [1, 17] and the references therein). These results give the functional analysis tools to obtain regularity type results in $M^{p,\varphi}$ for various linear boundary value problems, see [6, 18, 19, 20, 29].

Recently, Komori and Shirai [21] defined the weighted Morrey spaces and studied the boundedness of some classical operators in $L^{p,k}(w)$, $p > 1$ where w satisfies the Muckenhoupt condition A_p. In the present work, we consider generalized weighted Morrey spaces $M_w^{p,\varphi}$ that can be seen as an extension of both $M^{p,\varphi}$ and $L_w^{p,\kappa}$ see [16].

Let us recall the definition of the Muckenhoupt classes A_p, $1 < p < \infty$, and the respective weighted Lebesgue spaces $L^p_\omega(\Omega)$. A positive locally integrable function ω on $\mathbb{R}^n$, $\omega \in L^1_{loc}(\mathbb{R}^n)$, is called to be a weight. Then, given $p \in (1,\infty)$ this weight belongs to the Muckenhoupt class A_p if

$$[\omega]_{A_p} = \sup_{C_r(x)} \Big(\frac{1}{|C_r(x)|}\int_{C_r(x)} w(y)dy\Big)\Big(\frac{1}{|C_r(x)|}\int_{C_r(x)} w(y)^{-\frac{1}{p-1}}dy\Big)^{p-1} < \infty, \tag{1.1}$$

where the supremum is taken over all balls $C_r(x)$. Then $w(C)$ means the weighted measure of C, that is,

$$w(C) = \int_C w(x)dx.$$

This measure satisfies strong and reverse doubling properties for each C and each measurable subset $A \subset C$, there exist constants $c_1 > 0$ and $\tau_1 \in (0,1)$ such that

$$\frac{1}{[w]_p}\left(\frac{|A|}{|C|}\right)^p \le \frac{w(A)}{w(C)} \le c_1 \left(\frac{|A|}{|C|}\right)^{\tau_1}, \tag{1.2}$$

where c_1 and τ_1 depend on n and p but not on C and A.

We note that the A_p classes are nested, that is, $A_{p_1} \subset A_{p_2}$ if $1 < p_1 \le p_2 < \infty$. To give an example, consider the function

$$\omega_\alpha(x) = |x|^\alpha, \ x \in \mathbb{R}^n.$$

Then $w_\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$. Thus, w_α is a typical weight which can be considered in the present paper.

Lemma 1.1. [32] *Let $\omega \in A_s$ for some $1 < s < \infty$, and let $C_r(y)$ be the balls $C_r(y)$ centered at $y = (y', y_n) \in \Omega$ and of size $r > 0$. Then we have*

$$\frac{1}{\gamma_1}\Big[\frac{|C_r(y)\cap\Omega|}{(C_r(y))}\Big]^s \le \frac{\omega(C_r(y)\cap\Omega)}{\omega(C_r(y))} \le \gamma_1\Big[\frac{|C_r(y)\cap\Omega|}{(C_r(y))}\Big]^\beta,$$

where γ_1 and $\beta > 0$ are constants depending only on $[\omega]_s$ and n.

The function $f \in L^1_{loc}$ belongs to the weighted Lebesgue space L^p_w, $p \in [1,\infty)$ if

$$\|f\|_{L^p_w} = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\Big)^{\frac{1}{p}} < \infty.$$

The notorious results of Muckenhoupt [26] states that (1.1) is a necessary and sufficient condition in order that the *maximal inequality* in L^p_w, $p \in (1,\infty)$ to hold

$$\|f\|_{L^p_w} \le \|\mathcal{M}f\|_{L^p_w} \le c(n,p)\|f\|_{L^p_w}, \tag{1.3}$$

where $\mathcal{M}f(x) = \sup\limits_{r>0} |C_r(x)|^{-1}\int_{C_r(x)} |f(y)|dy$ is the *maximal function* of f and the supremum is taken over all balls that contain x.

Consider the Dirichlet problem

$$\begin{cases} D_i(a^{ij}(x)D_ju) = D_if^i(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω be a bounded open domain in $\mathbb{R}^n$, $\partial\Omega$ stands the boundary of Ω and the summation convention over the repeated lower and upper indexes, running from 1 to n, is adopted. We are going to obtain *Calderón-Zygmund type* estimate for Du in the *generalized weighted Morrey spaces*: denote by $A = \{a^{\alpha\beta}\}^n_{\alpha,\beta=1}$ the coefficient matrix and by $\mathbf{F} = \big(f^1,\dots,f^n\big)$ the right-hand side. We are going to prove that

$$\mathbf{F} \in M^{p,\varphi}_w(\Omega) \qquad \text{implies} \qquad |Du| \in M^{p,\varphi}_w(\Omega),\ p > 2,$$

with $w \in A_{\frac{p}{2}}$ and φ as in (2.8).

According to the recent works in [9, 10] one can allow the coefficients a^{ij} to be merely measurable in one of the variables for the estimate

$$\int_\Omega |Du(x)|^p dx \le c\int_\Omega |\mathbf{F}(x)|^p dx, \tag{1.5}$$

for some $c = c(n,\nu,\Lambda,p,|\Omega|) > 0$, to be true, while the boundary of the domain Ω belongs to the class of Reifenberg flat domains which goes beyond the category of sets with Lipschitz continuous boundaries. More precisely, the global L^p-estimate holds true for all $p \in (1,\infty)$ if for each point and for each scale the coefficients are only measurable in one variable and are averaged in the sense of small BMO with respect to the remaining $n-1$ variables, while the boundary $\partial\Omega$ can be trapped into two hyperplanes depending on the scale chosen. Let us emphasize on the fact that there is no any regularity assumption with respect to one of the variables and the boundary of the domain, being Reifenberg flat, can have rough enough fractal structure (see the excellent survey by Toro [33]). The class of operators with coefficients which are only partially BMO was first considered by Krylov in [22] who developed a technique based on apriori pointwise estimates for the sharp maximal function of the gradient, and which gives a unified approach for both divergence and nondivergence form equations.

The present article is a natural outgrowth of [5] and deals with weighted Sobolev-Morrey $W^1 M_w^{p,\varphi}$- theory for the Dirichlet problem (1.4). In particular, the authors derive an extended version of the weighted L^p-estimate (1.5) in the settings of the generalized weighted Morrey spaces $M_w^{p,\varphi}$, generalizing this way the $W_w^{1,p}$-regularity theory recently elaborated in [5, 9, 10]. More precisely, were proved that under the same regularity assumptions on A and Ω as these in [9, 10], the following global weighted Sobolev-Morrey $W^1 M_w^{p,\varphi}$-regularity

$$|F|^2 \in M_w^{p/2,\varphi}(\Omega) \Rightarrow |Du|^2 \in M_w^{p/2,\varphi}(\Omega) \tag{1.6}$$

holds for any $p \in (2,\infty)$, $w \in A_{\frac{p}{2}}$ and φ as in (2.8).

The paper is organized as follows. In Section 2 we define the space $M_w^{p,\varphi}$ and prove the maximal inequality in it. The Section 3 presents the problem and the main result. In the following, we obtain a gradient estimate. The technical approach is based on the Vitali-type covering lemma and estimates of the upper level sets of the maximal function of the gradient, see [5, 10] for more details.

Throughout the paper, the letter c will denote a universal constant that can be explicitly computed in terms of known quantities such as n, L, ν, p, φ, w and the geometric structure of Ω. The exact value of c may vary from one occurrence to another.

2. Generalized weighted Morrey spaces

In the following we use the next domains:

- *balls* centered in a point $y \in \mathbb{R}^n$ and of radius $r > 0$:

$$C \equiv C_r(x) = \{y \in \mathbb{R}^n : \ |x-y| < r\}$$

with Lebesgue measure $|C_r| = c(n)r^n$. For each $x \in \Omega$ we write

$$\Omega_r = C_r(x) \cap \Omega, \ 2C_r = C_{2r}(x) \text{ and } \complement(2C_r) = \mathbb{R}^n \setminus 2C_r.$$

- *balls* centered in $y = (y', y_n)$:

$$\mathcal{C} \equiv \mathcal{C}_r(y) = \{(x', x_n) \in \mathbb{R}^n : |x'-y'| < r, |x_n - y_n| < r\}$$

with $|\mathcal{C}_r| = c(n)r^n$.

- *open ball in* $\mathbb{R}^{n-1}$ with centered in $y' = (y_1, \dots, y_{n-1})$ and radius $r > 0$:

$$B'_r(y) = \{x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : |x'-y'| < r\}.$$

with $|B'_r| = c(n)r^{n-1}$.

Definition 2.1. Let Ω be a open domain in $\mathbb{R}^n$ and $p \in (1,\infty)$. A function $f \in L_w^p(\Omega)$, $w \in A_p$, belongs to the generalized weighted Morrey space $M_w^{p,\varphi}(\Omega)$ if the following norm is finite

$$\|f\|_{M_w^{p,\varphi}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)} \Big(\frac{1}{w(C_r(x))} \int_{C_r(x)\cap\Omega} |f(x)|^p \omega(x)dx \Big)^{\frac{1}{p}} < \infty. \tag{2.1}$$

where φ is a measurable non-negative function defined on $\Omega \times (0, +\infty)$ (see [16]).

If $w \equiv 1$, then $M^{p,\varphi}_w(\Omega) \equiv M^{p,\phi}(\Omega)$ with $\phi(C_r(x)) = \varphi(C_r(x))^p r^n$.
If $\varphi \equiv r^{(\lambda-n)/p}$ and $w \equiv 1$, then $M^{p,\varphi}_w(\Omega) \equiv L^{p,\lambda}(\Omega)$, $\lambda \in (0, n)$.
If $\varphi \equiv w^{-1/p}$, then $M^{p,\varphi}_w(\Omega) \equiv L^p_w(\Omega)$.

In [1] we proved maximal inequality in generalized Morrey spaces under quite general condition on the weight φ. In fact, we consider a couple of measurable non-negative functions (φ_1, φ_2) defined on $\mathbb{R}^n \times \mathbb{R}_+$, then the next result holds.

Theorem 2.1. [1] *Let $q \in (1, \infty)$. Suppose that for any fixed $x \in \mathbb{R}^n$ and any $r > 0$ there is a constant $\kappa > 0$ independent on x and r such that*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(C_\sigma(x))\sigma^{\frac{n}{q}}}{s^{\frac{n}{q}}} \le \kappa\, \varphi_2(C_r(x))\,. \tag{2.2}$$

Then the operator $\mathcal{M}$ is bounded from M^{q,φ_1} to M^{q,φ_2} and

$$\|\mathcal{M}f\|_{M^{q,\varphi_2}(\mathbb{R}^n)} \le c\|f\|_{M^{q,\varphi_1}(\mathbb{R}^n)}$$

with a constant independent of f, r and the point x.

We need to prove an analogous result in $M^{p,\varphi}_w$. Let $L^{\infty,v}(0,\infty)$ be the space of all functions $g(\eta)$, $\eta > 0$ with finite norm

$$\|g\|_{L^{\infty,v}(0,\infty)} = \sup_{\eta>0} v(\eta)g(\eta)$$

and $L^\infty(0,\infty) \equiv L^{\infty,1}(0,\infty)$. Denote by $\mathfrak{M}(0,\infty)$ the set of all Lebesgue-measurable functions on $(0,\infty)$, by $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$, by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all non-decreasing functions in $\mathfrak{M}^+(0,\infty)$, and by $\mathbb{A}$ the subset

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{\eta\to 0+} \varphi(\eta) = 0 \right\}.$$

Let ν be a continuous and non-negative function on $(0,\infty)$. We define the *sup-operator* $\overline{S}_\nu$ acting on $\mathfrak{M}(0,\infty)$ by

$$(\overline{S}_\nu g)(\eta) := \|\nu\, g\|_{L^\infty(\eta,\infty)}, \quad \eta \in (0,\infty)\,.$$

This operator turns to be bounded on $\mathbb{A}$, as it is proved in [2].

Theorem 2.2. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L^\infty(\eta,\infty)} < \infty$ for any $\eta > 0$ and let ν be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_\nu$ is bounded from $L^{\infty,v_1}(0,\infty)$ to $L^{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_\nu \left(\|v_1\|^{-1}_{L^\infty(\cdot,\infty)} \right) \right\|_{L^\infty(0,\infty)} < \infty. \tag{2.3}$$

The following lemmata give some estimates of the maximal function.

Lemma 2.1. *Let $w \in A_p$, $p \in (1,\infty)$. Then the inequality*

$$\|\mathcal{M}f\|_{L^p_w(C_r)} \le c \left(\|f\|_{L^p_w(C_{2r})} + w(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} s^{-n}\|f\|_{L^1(C_s)} \right) \tag{2.4}$$

holds for all $f \in L^p_{w,\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Fix a point $y \in \mathbb{R}^n$ and write $C_r \equiv C_r(y)$ and $C_s \equiv C_s(y)$ to denote balls centered in y. Consider the decomposition

$$f = f_1 + f_2 = f\chi_{2C_r} + f\chi_{\complement(2C_r)}\,.$$

Then

$$\|\mathcal{M}f\|_{L^p_w(C_r)} \le \|\mathcal{M}f_1\|_{L^p_w(C_r)} + \|\mathcal{M}f_2\|_{L^p_w(C_r)}.$$

Because of (1.3) we have

$$\|\mathcal{M}f_1\|_{L^p_w(C_r)} \le \|\mathcal{M}f_1\|_{L^p_w(\mathbb{R}^n)} \le c\|f_1\|_{L^p_w(\mathbb{R}^n)} = c\|f\|_{L^p_w(C_{2r})}.$$

As in [18, Lemma 3.2] we get that for all $x \in C_r$ holds

$$\mathcal{M}f_2(x) \le c \sup_{s>2r} \frac{1}{|C_s|}\int_{C_s} |f(z)|\,dz \tag{2.5}$$

where the right hand side does not depend on x anymore. Hence

$$\|\mathcal{M}f_2\|_{L^p_w(C_r)} \le c\sup_{s>2r} \frac{1}{|C_s|}\int_{C_s} |f(z)|\,dz \left(\int_{C_r} w(x)\,dx\right)^{\frac{1}{p}}.$$

Unifying the both estimates we get

$$\|\mathcal{M}f\|_{L^p_w(C_r)} \le c\left(\|f\|_{L^p_w(2C_r)} + w(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{|C_s|}\int_{C_s} |f(z)|\,dz\right).$$

□

Lemma 2.2. *Let $w \in A_p$, $p \in (1,\infty)$, then the inequality*

$$\|\mathcal{M}f\|_{L^p_w(C_r)} \le cw(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} w(C_s)^{-\frac{1}{p}}\,\|f\|_{L^p_w(C_s)} \tag{2.6}$$

holds for all $f \in L^p_{w,\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Denote by

$$\mathcal{A}_1 := w(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{|C_s|}\int_{C_s} |f(z)|\,dz\,,$$
$$\mathcal{A}_2 := \|f\|_{L^p_w(2C_r)}\,.$$

Applying Hölder's inequality and (1.1) we get

$$\mathcal{A}_1 \le [w]_p^{\frac{1}{p}} w(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{w(C_s)^{\frac{1}{p}}}\left(\int_{C_s} |f(z)|^p w(z)\,dz\right)^{\frac{1}{p}}.$$

On the other hand,

$$\begin{aligned} w(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{w(C_s)^{\frac{1}{p}}}\left(\int_{C_s} |f(z)|^p w(z)\,dz\right)^{\frac{1}{p}} \\ \ge\ [w]_p^{-\frac{1}{p}} w(C_r)^{\frac{1}{p}}\left(\int_{C_{2r}} |f(z)|^p w(z)\,dz\right)^{\frac{1}{p}} \cdot \sup_{s>2r} \frac{1}{w(C_s)^{\frac{1}{p}}} = c\mathcal{A}_2\,. \end{aligned}$$

Hence by Lemma 2.1 we get (2.6) (see [18]).

$$\|\mathcal{M}f\|_{L^p_w(C_r)} \le c\big(\mathcal{A}_1 + \mathcal{A}_2\big) \le cw(C_r)^{\frac{1}{p}} \cdot \sup_{s>2r} w(C_s)^{-\frac{1}{p}}\|f\|_{L^p_w(C_s)}$$

we arrive at (2.6). □

Theorem 2.3. *Let $w \in A_p$, $p \in (1,\infty)$ and φ_1, φ_2 be a non-negative measurable functions. Assume that there is a positive constant κ independent on y and r such that*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi_1(C_\sigma(y)) w\big(C_\sigma(y)\big)^{\frac{1}{p}}}{w\big(C_s(y)\big)^{\frac{1}{p}}} \le \kappa\, \varphi_2(C_r(y))\,. \tag{2.7}$$

Then the operator $\mathcal{M}$ is bounded from M_w^{p,φ_1} to M_w^{p,φ_2} and

$$\|\mathcal{M}f\|_{M_w^{p,\varphi_2}(\mathbb{R}^n)} \le c\|f\|_{M_w^{p,\varphi_1}(\mathbb{R}^n)}\,.$$

The proof follows by Lemma 2.2 as in [18, Theorem 3.4].

Corollary 2.1. *(Maximal inequality) Let $w \in A_p$, $p \in (1,\infty)$ and φ satisfies*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\sigma<\infty} \varphi(C_\sigma(y)) w\big(C_\sigma(y)\big)^{\frac{1}{p}}}{w\big(C_s(y)\big)^{\frac{1}{p}}} \le \kappa\, \varphi(C_r(y)) \tag{2.8}$$

with κ independent of r and y. Then there is a constant $c_p > 0$ such that

$$\|f\|_{M_w^{p,\varphi}(\mathbb{R}^n)} \le \|\mathcal{M}f\|_{M_w^{p,\varphi}(\mathbb{R}^n)} \le c_p\|f\|_{M_w^{p,\varphi}(\mathbb{R}^n)}, \quad \forall\, f \in M_w^{p,\varphi}(\mathbb{R}^n)\,.$$

Impose in addition a kind of monotonicity condition on φ, precisely

$$\varphi(C_r(y))^p w\big(C_r(y)\big) \le \varphi(C_s(z))^p w\big(C_s(z)\big) \text{ for all } C_r(y) \subset C_s(z)\,. \tag{2.9}$$

This implies that for a given $\Omega \subset \mathbb{R}^n$, there holds

$$\sup_{\substack{y\in\Omega\\ r>0}} \frac{w\big(C_r(y)\cap\Omega\big)}{\varphi\big(C_r(y)\big)^p\; w\big(C_r(y)\big)} \le \varkappa, \tag{2.10}$$

with $\varkappa = \varkappa(n, q, \kappa, \varphi, w, \Omega)$ (see [18]).

3. Statement of the Problem and Main Result

Consider the Dirichlet problem

$$\begin{cases} D_i(a^{ij}(x)D_j u) = D_i f^i(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Suppose that the coefficients are uniformly bounded and uniformly elliptic, that is, there exist positive constants Λ and ν such that

$$\begin{cases} \|\mathbf{a}\|_{L^\infty(\Omega)} \le \Lambda \\ a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \ge \nu|\xi|^2, \quad \forall\xi \in \mathbb{R}^n, \text{for a.a. } x \in \Omega. \end{cases} \tag{3.2}$$

The balls in $\mathbb{R}^{n-1}\times\mathbb{R}$ with center $y = (y', y_n)$ and size $r > 0$ in the x_n-axis is denoted by

$$C_r(y) = B'_r(y') \times (y_n - r, y_n + r).$$

If the center is the origin, we do not specify it and write just C_r for the sake of simplicity.

For each fixed $x_n \in \mathbb{R}$ and for each bounded subset U' of $\mathbb{R}^{n-1}$, the integral average of a function $g(\cdot, x_n)$ with respect to x'-variables in U' is denoted by

$$\bar{g}_{U'}(x_n) \equiv \frac{1}{|U'|}\int_{U'} g(x', x_n)dx',$$

and $|U'|$ stands for the $(n-1)$-dimensional Lebesgue measure of U'.

We now state the main assumptions on the data of problem (1.4) regarding the coefficients matrix $A(x)$ and the domain Ω.

Definition 3.1. We say that (A,Ω) is (δ,R)-vanishing of codimension 1 if for every point $y\in\Omega$ and for every number $r\in(0,R]$ such that

$$dist(y,\partial\Omega)=\min_{x\in\partial\Omega} dist(y,x)>\sqrt{2}r, \tag{3.3}$$

there exists a coordinate system depending on y and r, whose variables we still denote by $x=(x',x_n)$, such that in this new coordinate system y is the origin and

$$\frac{1}{|C_{\sqrt{2}r}|}\int_{C_{\sqrt{2}r}} |A(x',x_n)-\overline{A}_{B'_{\sqrt{2}r}}(x_n)|^2dx\le\delta^2, \tag{3.4}$$

while, for every point $y\in\Omega$ and for every number $r\in(0,R]$ with

$$dist(y,\partial\Omega)=\min_{x\in\partial\Omega} dist(y,x)=dist(y,x_0)\le\sqrt{2}r$$

for some $x_0\in\partial\Omega$, there exists a coordinate system depending on y and r, whose variables we still denote by $x=(x',x_n)$, such that in this new coordinate system x_0 is the origin,

$$C_{3r}\cap\{x:x_n>3r\delta\}\subset C_{3r}\cap\Omega\subset C_{3r}\cap\{x:x_n>-3r\delta\} \tag{3.5}$$

and

$$\frac{1}{|C_{3r}|}\int_{C_{3r}} |A(x',x_n)-\overline{A}_{B'_{3r}}(x_n)|^2dx\le\delta^2. \tag{3.6}$$

Because of the scaling invariance property of the Reifenberg domains (see [8, 9]). The condition (3.3) means that away from the boundary the coefficients are *small BMO* in all variables except x_1. In that variable thay are *only measurable* and may have arbitrary jumps in the direction x_1. In addition, the domain Ω is (δ,R)-*Reifenberg flat* satisfying (3.5) (see Reifenberg [28]) and the coefficients have a small oscillation along the flat direction x' of the boundary and are only measurable along the normal direction x_1. The number $\sqrt{2}r$ in (3.3) is selected for convenience since we need to take the size of the balls in (3.4) such that there is enough room to have the rotation of $C_r(y)$ in any spatial direction.

Recall that under a weak solution of (3.1), we mean a function $u\in H^1_0(\Omega)$ that satisfies

$$\int_\Omega a^{ij}D_juD_i\phi dx=\int_\Omega f^iD_i\phi dx$$

for all $\phi\in H^1_0(\Omega)$. Moreover, the following L^2-estimate holds

$$\int_\Omega|Du(x)|^2dx\le c\int_\Omega|\mathbf{F}(x)|^2dx, \tag{3.7}$$

where the constant c depends only on n, ν, Λ and the Lebesgue measure $|\Omega|$ of the domain Ω.

We suppose that $\mathbf{F}\in M^{p,\varphi}_w(\Omega)$, $w\in A_{\frac{p}{2}}$, $p\in(2,\infty)$ with a weight φ satisfying (2.8). This implies $\mathbf{F}\in L^p_w(\Omega)$. Precisely, choose $y\in\Omega$, then

$$\sup_{z\in\Omega}\{|y-z|\}<\operatorname{diam}\Omega\,.$$

Hence there exists $r^* < \operatorname{diam}\Omega$ and such that $\Omega \subset C_r(y) \subset C_{2d}$ and

$$\|\mathbf{F}\|^2_{L^p_w(\Omega)} = \||\mathbf{F}|^2\|_{L^{\frac{p}{2}}_w(\Omega)} \leq \varphi(C_{r^*}(y)) w(C_{r^*}(y))^{\frac{2}{p}} \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}_w(\Omega)}.$$

By the Hölder inequality and (1.1), we get

$$\begin{aligned}
\|\mathbf{F}\|^2_{L^2(\Omega)} &= \int_\Omega |\mathbf{F}(x)|^2 \, w(x)^{\frac{2}{p}} \, w(x)^{-\frac{2}{p}} \, dx \\
&\leq \Big(\int_\Omega \big(|\mathbf{F}(x)|^2\big)^{\frac{p}{2}} \, w(x)\, dx \Big)^{\frac{2}{p}} \Big(\int_\Omega w(x)^{-\frac{2}{p-2}} \, dx \Big)^{\frac{p-2}{p}} \\
&\leq |\Omega| [w]^{\frac{2}{p}}_{\frac{p}{2}} \Big(\frac{w(C_{r^*}(y))}{w(\Omega)} \Big)^{\frac{2}{p}} \varphi(C_{r^*}(y)) \\
&\times \frac{1}{\varphi(C_{r^*}(y))} \Big(\frac{1}{w(C_{r^*}(y))} \int_{C_{r^*}(y)} (|\mathbf{F}(x)|^2)^{\frac{p}{2}} w(x) dx \Big)^{\frac{2}{p}}.
\end{aligned}$$

Because of the doubling property (1.2) of w, we get

$$\Big(\frac{w(C_{r^*}(y))}{w(\Omega)} \Big)^{\frac{2}{p}} \leq [w]^{\frac{2}{p}}_{\frac{p}{2}} \frac{|C_{r^*}(y)|}{|\Omega|}.$$

Hence, applying first (1.2) to w and then (2.9) to φ, we get

$$\|\mathbf{F}\|^2_{L^2(\Omega)} \leq c \, \varphi(C_{2d}) w\big(C_{2d}\big)^{\frac{2}{p}} \||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}_w(\Omega)}. \tag{3.8}$$

Let $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^1_2(\Omega)$ norm.

We are in a position now to state the main result of the paper.

Theorem 3.1. *Let $p \in (2,\infty)$, $w \in A_{p/2}$ and φ be a satisfying (2.8), there exist a small positive constant $\delta = \delta(\nu, L, n, p, [w]_{\frac{p}{2}}, \Omega)$ and a positive constant $c(\nu, L, n, p, [w]_{\frac{p}{2}}, \Omega)$ such that if (A, Ω) is (δ, R)-vanishing of codimension 1 and $F \in M^{p,\varphi}_w(\Omega)$, then the unique weak solution $u \in H^1_0(\Omega)$ of (1.4) satisfies $Du \in M^{p,\varphi}_w(\Omega)$ and the following estimate holds*

$$\|Du\|_{M^{p,\varphi}_w(\Omega)} \leq c\|\mathbf{F}\|_{M^{p,\varphi}_w(\Omega)}. \tag{3.9}$$

with a constant c depending on known quantities.

4. Gradient estimates in generalized weighted Morrey spaces

In this section we will obtain the optimal weighted Sobolev-Morrey $W^1M^{p,\varphi}_w$-regularity for the Dirichlet problem (1.4). The main analytic tool of our approach is the maximal function, so let us recall first of all its definition and basic properties, see [13, 30].

In what follows we establish a suitable version of the Vitali covering lemma. We use it to derive a power decay estimate of the upper level sets for the Hardy-Littlewood maximal function of the spatial gradient of the weak solution. The regularity estimate in the main result then follows by the standard procedure of summation over the level sets.

Fix a point $y_0 \in \Omega$, take $C_r(y_0)$ and consider $\Omega_r = C_r(y_0) \cap \Omega$. Let u be a weak solution of (3.1), then we define the sets

$$\begin{aligned} \mathfrak{C} &= \{x \in \Omega_r : \mathcal{M}(|Du|^2) > \lambda_1^2\} \\ \mathfrak{D} &= \{x \in \Omega_r : \mathcal{M}(|Du|^2) > 1\} \cup \{\mathcal{M}(|\mathbf{F}|^2) > \delta^2\} \end{aligned} \tag{4.1}$$

with $\lambda_1 > 1$ and $\delta > 0$. It is easy to see that $\mathfrak{C} \subset \mathfrak{D} \subset \Omega_r$. For each $y \in \mathfrak{C}$ consider $C_\rho(y)$ and define the following auxiliary function

$$\Theta(\rho) = \frac{w(\mathfrak{C} \cap C_\rho(y))}{w(C_\rho(y))}, \qquad \rho > 0$$

with $w \in A_p$, $p \in (1, \infty)$. Because of (1.2) $\Theta(0) = \lim_{\rho \to 0_+} \Theta(\rho) = 1$ and $\lim_{\rho \to +\infty} \Theta(\rho) = 0$. We start with some preliminary lemma taking $R = 1$ because of the invariance property of the Reifenberg domain [9].

Lemma 4.1. *Let* $(\mathbf{a}, \Omega)$ *be* $(\delta, 1)$*-vanishing of codimension* 1 *and* $\mathfrak{C}$, $\mathfrak{D}$ *and* $\Theta(\rho)$ *be as above. Suppose that for any* $y \in \mathfrak{C}$ *there exists* $\varepsilon \in (0, 1)$ *such that* $\Theta(1) < \varepsilon$. *Then* $\Theta(\rho) \geq \varepsilon$ *implies* $\Omega_r \cap C_\rho(y) \subset \mathfrak{D}$ *and*

$$w(\mathfrak{C}) \leq \varepsilon [w]_p^2 \left(\frac{10\sqrt{3}}{1-\delta} \right)^{np} w(\mathfrak{D}) \,. \tag{4.2}$$

Proof. The implication follows by (1.2). Since $\Theta(1) < \varepsilon$, there exists $\rho_y \in (0, 1)$ such that $\Theta(\rho(y)) = \varepsilon$ and $\Theta(\rho) < \varepsilon$ for all $\rho > \rho_y$ and $y \in \mathfrak{C}$. Consider the family of balls $\{C_{\rho_y}(y)\}_{y \in \mathfrak{C}}$ which is an open covering of $\mathfrak{C}$. By the Vitali lemma (cf. [20, Lemma I.3.1]), there exists a disjoint sub-collection $\{C_{\rho_i}(y_i)\}_{i \geq 1}$ with $\rho_i = \rho_{y_i} \in (0, 1)$, $y_i \in \mathfrak{C}$ such that $\Theta(\rho_i) = \varepsilon$,

$$\sum_{i \geq 1} |C_{\rho_i}(y_i)| \geq c |\mathfrak{C}| \quad \text{ and } \quad \mathfrak{C} \subset \bigcup_{i \geq 1} C_{5\rho_i}(y_i)$$

with a positive constant $c = c(n)$. Since $\Theta(5\rho_i) < \varepsilon$, we have by (1.2)

$$w(\mathfrak{C} \cap C_{5\rho_i}(y_i)) < \varepsilon w(C_{5\rho_i}(y_i)) \leq \varepsilon [w]_p 5^{np} w(C_{\rho_i}(y_i)) \,.$$

Further, making use of the bound (see [7, 8])

$$\sup_{\substack{y \in \Omega_r \\ 0 < \rho < 1}} \frac{|C_\rho(y)|}{|\Omega_r \cap C_\rho(y)|} \leq \left(\frac{2\sqrt{2}}{1-\delta} \right)^n,$$

we get by (1.2)

$$w(C_{\rho_i}(y_i)) \leq [w]_p \left(\frac{2\sqrt{2}}{1-\delta} \right)^{np} w(\Omega_r \cap C_{\rho_i}(y_i)).$$

Now we have

$$\begin{aligned} w(\mathfrak{C}) &= w\Big(\bigcup_{i \geq 1} (\mathfrak{C} \cap C_{5\rho_i}(y_i)) \Big) \leq \sum_{i \geq 1} w(\mathfrak{C} \cap C_{5\rho_i}(y_i)) \\ &< \varepsilon \sum_{i \geq 1} w(C_{5\rho_i}(y_i)) \leq \varepsilon [w]_p^2 5^{np} \sum_{i \geq 1} w(C_{\rho_i}(y_i)) \\ &\leq \varepsilon [w]_p^2 \left(\frac{10\sqrt{2}}{1-\delta} \right)^{np} \sum_{i \geq 1} w(\Omega_r \cap C_{\rho_i}(y_i)) \,. \end{aligned}$$

Having in mind that $\{C_{\rho_i}(y_i)\}_{i\geq 1}$ are mutually disjoint, $\Theta(\rho_i)=\varepsilon$ and condition (4.2) we get

$$\begin{aligned} w(\mathfrak{C}) \leq & \varepsilon[w]_p^2 \left(\frac{10\sqrt{2}}{1-\delta}\right)^{np} w\Big(\bigcup_{i\geq 1} \Omega_r \cap C_{\rho_i}(y_i)\Big) \\ \leq & \varepsilon[w]_p^2 \left(\frac{10\sqrt{2}}{1-\delta}\right)^{np} w(\mathfrak{D})\,. \end{aligned}$$

□

Lemma 4.2. *Let* $|F|^2 \in M_w^{\frac{p}{2},\varphi}(\Omega) \subset L^1(\Omega)$, $2<p<\infty$, *and let* $u \in H_0^1(\Omega)$ *be the weak solution of* (1.4). *Suppose* (A,Ω) *is* (δ,R)*-vanishing of codimension* 1 *and set* $\varepsilon_1=\gamma_1\varepsilon$. *Then we have*

$$\begin{aligned} w(\{x\in\Omega : \mathcal{M}(|Du|^2) > \lambda_2^{2k}\}) &\leq \varepsilon_1^k w(\{x\in\Omega : \mathcal{M}(|Du|^2)>1\}) \\ &+\sum_{i=1}^k \varepsilon_1^i w\Big(\{x\in\Omega : \mathcal{M}(|F|^2) > \delta^2\lambda_2^{2(k-i)}\}\Big)\ (k=1,2,\dots). \end{aligned} \tag{4.3}$$

Proof. By Lemma 4.1, we have

$$\begin{aligned} w\left(\{x\in\Omega_r : \mathcal{M}(|Du|^2) > \lambda_1^2\}\right) \leq \varepsilon_1 w\left(\{x\in\Omega_r : \mathcal{M}(|Du|^2)>1\}\right) \\ + \varepsilon_1 w\left(\{x\in\Omega_r : \mathcal{M}(|\mathbf{F}|^2)>\delta^2\}\right) \end{aligned}$$

where $\varepsilon_1=\varepsilon\,[w]_p^2\left(\frac{10\sqrt{2}}{1-\delta}\right)^{np}$.

The last inequality is exactly (4.3) with $k=1$. Further, we proceed with the proof by induction, as it is done in [4, Corollary 3.10]. Suppose that (4.3) holds true for some $k\geq 1$. Define the functions $u_1=\frac{u}{\lambda_1}$ and $\mathbf{F}_1=\frac{\mathbf{F}}{\lambda_1}$. It is easy to see that u_1 is a weak solution to the problem (3.1) with a right-hand side $\mathbf{F}_1$. Hence Lemma 4.1 holds with sets $\mathfrak{C}$ and $\mathfrak{D}$ corresponding to u_1 as defined in (4.1). According to (4.3), the inductive assumption holds true for u_1 with the same $k\geq 1$. The definition of u_1 ensures the inductive passage from k to $k+1$ for u. Namely,

$$\begin{aligned} w\left(\left\{x\in\Omega_r\colon \mathcal{M}(|Du|^2) > \lambda_1^{2(k+1)}\right\}\right) &= w\left(\{x\in\Omega_r\colon \mathcal{M}(|Du_1|^2)>\lambda_1^{2k}\}\right) \\ \leq \varepsilon_1^k\, w\left(\{x\in\Omega_r\colon \mathcal{M}(|Du_1|^2)>1\}\right) & \\ +\sum_{i=1}^k \varepsilon_1^i w\left(\left\{x\in\Omega_r\colon \mathcal{M}(|\mathbf{F}_1|^2)>\delta^2\lambda_1^{2(k-i)}\right\}\right) & \\ = \varepsilon_1^k\, w\left(\{x\in\Omega_r\colon \mathcal{M}(|Du|^2)>\lambda_1^2\}\right) & \\ +\sum_{i=1}^k \varepsilon_1^i\, w\left(\left\{x\in\Omega_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k-i)}\lambda_1^2\right\}\right) & \\ \leq \varepsilon_1^{k+1}\, w\left(\{x\in\Omega_r\colon \mathcal{M}(|Du|^2)>1\}\right) & \\ +\sum_{i=1}^{k+1} \varepsilon_1^i\, w\left(\left\{x\in\Omega_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2\lambda_1^{2(k+1-i)}\right\}\right). & \end{aligned}$$

□

Noting that because of the arbitrary choice of the point $y_0 \in \Omega$, the above estimates hold locally for any $\Omega_r = C_r(y) \cap \Omega$ with $y \in \Omega$.

Lemma 4.3. *Let $f \in L_1(\Omega)$ be a nonnegative function, w be an A_q-weight, $q \in (1,\infty)$, φ be a weight satisfying (2.7), and $\theta > 0$ and $\lambda > 1$ be constants. Then $f \in M_w^{q,\varphi}(\Omega)$ if and only if*

$$S := \sup_{y\in\Omega,\ r>0} \sum_{k\geq 1} \frac{\lambda^{kq} w(\{x \in \Omega_r : f(x) > \theta\lambda^k\})}{\varphi(C_r(y))^q w(C_r(y))} < \infty.$$

Moreover,

$$\frac{1}{c} S \leq \|f\|^q_{M_w^{q,\varphi}(\Omega)} \leq c(1+S),$$

for some universal constant $c = c(\theta, \lambda, q, k, \varphi, w, \Omega)$.

Proof. Consider $\Omega_r = \Omega \cap C_r(y)$ with $y \in \Omega$, then

$$\begin{aligned}
&\frac{1}{\varphi(C_r(y))^q} \frac{1}{w\big(C_r(y)\big)} \int_{\Omega_r} f(x)^q w(x)\,dx \\
&= \frac{1}{\varphi(C_r(y))^q} \frac{1}{w\big(C_r(y)\big)} \int_{\{x\in\Omega_r : f\leq\theta\lambda\}} f(x)^q w(x)dx \\
&+ \sum_{k\geq 1} \frac{1}{\varphi(C_r(y))^q} \frac{1}{w\big(C_r(y)\big)} \int_{\{x\in\Omega_r : \theta\lambda^k < f \leq \theta\lambda^{k+1}\}} f(x)^q w(x)dx \\
&\leq (\theta\lambda)^q \frac{w(\Omega_r)}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \\
&+ \sum_{k\geq 1} \frac{(\theta\lambda^{k+1})^q}{\varphi(C_r(y))^q w\big(C_r(y)\big)} w\big(\{x \in \Omega_r : f(x) > \theta\lambda^k\}\big) \\
&\leq (\theta\lambda)^q \left[\frac{w(\Omega_r)}{\varphi(C_r(y))^q w\big(C_r(y)\big)} + \sum_{k\geq 1} \frac{\lambda^{kq} w\big(\{x \in \Omega_r : f(x) > \theta\lambda^k\}\big)}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \right].
\end{aligned}$$

Taking the supremum over $C_r(y)$ and making use of (2.10), we get

$$\|f\|^q_{M_w^{q,\varphi}(\Omega)} \leq c(1+\mathcal{S}).$$

On the other hand

$$\begin{aligned}
&\frac{1}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \int_{\Omega_r} f(x)^q w(x)\,dx \\
&= \frac{q}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \int_{\Omega_r} \left(\int_0^{f(x)} \xi^{q-1} d\xi \right) w(x)\,dx \\
&= \frac{q}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \int_0^{\infty} w\big(\{x \in \Omega_r : f(x) > \xi\}\big) \xi^{q-1}\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{q}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \sum_{k\geq 1} w\big(\{x\in\Omega_r : f(x)>\theta\lambda^k\}\big) \int_{\theta\lambda^{k-1}}^{\theta\lambda^k} \xi^{q-1}\,d\xi \\
&= \theta^q(1-\lambda^{-q}) \frac{1}{\varphi(C_r(y))^q w\big(C_r(y)\big)} \sum_{k\geq 1} \lambda^{kq} w\big(\{x\in\Omega_r : f(x)>\theta\lambda^k\}\big)\,.
\end{aligned}
$$

Taking again the supremum over $C_r(y)$ we get

$$
\|f\|^q_{M^{q,\varphi}_w(\Omega)} \geq \frac{1}{c}\mathcal{S}
$$

with a positive constant $c=c(\theta,\lambda,w,q)$. □

We are in a position now to prove Theorem 3.1.

Proof. Recall that $\mathbf{F}\in M^{p,\varphi}_w(\Omega)$, $w(x)\in A_{\frac{p}{2}}$, $p\in(2,\infty)$ with a weight φ satisfying (2.7). Because of the scaling invariance property of (3.1) under a normalization, we can assume that the norm of $\mathbf{F}$ is small enough. In fact, taking

$$
\bar u(x) = \frac{\delta u(x)}{\sqrt{\||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}_w(\Omega)}}} \qquad \text{and} \qquad \bar{\mathbf{F}}(x) = \frac{\delta \mathbf{F}(x)}{\sqrt{\||\mathbf{F}|^2\|_{M^{\frac{p}{2},\varphi}_w(\Omega)}}}
$$

instead of u and $\mathbf{F}$ in (3.1) we get $\||\bar{\mathbf{F}}|^2\|_{M^{\frac{p}{2},\varphi}_w(\Omega)} = \delta^2$. Then we need to prove boundedness of the norm of the gradient $|D\bar u|$. Hence, it is enough to get $\|\mathcal{M}(|D\bar u|^2)\|_{M^{\frac{p}{2},\varphi}_w(\Omega)} \leq c$. For this goal, we apply Lemma 4.3 with $f=\mathcal{M}(|D\bar u|^2)$, $\lambda=\lambda_1^2$, $\theta=1$ and $q=\frac{p}{2}$.

Take $\mathfrak{C}$ corresponding to the solution $\bar u$. We note that, for each $y\in\mathfrak{C}$,

$$
\begin{aligned}
\frac{w(\mathfrak{C}\cap C_1(y))}{w(C_1(y))} \leq & c w(\mathfrak{C}) = c w\big(\{x\in\Omega_r : \mathcal{M}(|D\bar u|^2)>\lambda_1^2\}\big) \\
\leq & c\int_{\Omega_r} \mathcal{M}(|D\bar u|^2)(x)w(x)dx \leq c\int_{\Omega_r} |D\bar u(x)|^2 w(x)dx \\
\leq & c\int_{\Omega} |D\bar u(x)|^2 w(x)dx \leq c\int_{\Omega} |\bar{\mathbf{F}}(x)|^2 w(x)dx \\
\leq & c\||\bar{\mathbf{F}}|^2\|_{L^{\frac{p}{2},\varphi}_w(\Omega)} \leq c\delta^2,
\end{aligned}
$$

according to (3.8) with a constant depending on n,p,φ,κ and Ω. Taking δ small enough, we get

$$
\Theta(1) = \frac{w(\mathfrak{C}\cap C_1(y))}{w(C_1(y))} \leq c\delta^2 < \varepsilon\,.
$$

Therefore Lemma 4.2 gives

$$\begin{aligned}
&\sum_{k\geq 1}\lambda_1^{2k\frac{p}{2}}\frac{w\big(\{x\in\Omega_r\colon \mathcal{M}(|D\bar u|^2)>\lambda_1^{2k}\}\big)}{\varphi(C_r(y))^{\frac{p}{2}}r^n}\\
&\quad\leq \sum_{k\geq 1}\lambda_1^{kp}\varepsilon_1^k\frac{w\big(\{x\in\Omega_r\colon \mathcal{M}(|D\bar u|^2)>1\}\big)}{\varphi(C_r(y))^{\frac{p}{2}}\,w\big(C_r(y)\big)}\\
&\quad+\sum_{k\geq 1}\sum_{i=1}^{k}\lambda_1^{kp}\varepsilon_1^i\frac{w\big(\{x\in\Omega_r\colon \mathcal{M}(|\bar{\mathbf F}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big)}{\varphi(C_r(y))^{\frac{p}{2}}\,w\big(C_r(y)\big)}\\
&\quad\leq \sum_{k\geq 1}\big(\lambda_1^p\varepsilon_1\big)^k\frac{w(\Omega_r)}{\varphi(C_r(y))^{\frac{p}{2}}\,w\big(C_r(y)\big)}\\
&\quad+\sum_{i\geq 1}\big(\lambda_1^p\varepsilon_1\big)^i\underbrace{\sum_{k\geq i}\lambda_1^{p(k-i)}\frac{w\big(\{x\in\Omega_r\colon \mathcal{M}(|\bar{\mathbf F}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big)}{\varphi^{\frac{p}{2}}(C_r(y))\,w\big(C_r(y)\big)}}_{\mathcal{S}'}\\
&\quad\leq \kappa_4\sum_{k\geq 1}\big(\lambda_1^p\varepsilon_1\big)^k+\sum_{i\geq 1}\big(\lambda_1^p\varepsilon_1\big)^i\mathcal{S}'
\end{aligned}$$

where we have used (3.8) for the last inequality. Let us note that

$$\begin{aligned}
\mathcal{S}'&=\sum_{k\geq i}\lambda_1^{p(k-i)}\frac{w\big(\{x\in\Omega_r\colon \mathcal{M}(|\bar{\mathbf F}|^2)>\delta^2\lambda_1^{2(k-i)}\}\big)}{\varphi(C_r(y))^{\frac{p}{2}}w\big(C_r(y)\big)}\\
&=\sum_{k\geq i}\big(\lambda_1^{2(k-i)}\big)^{\frac{p}{2}}\frac{w\big(\{x\in\Omega_r\colon \mathcal{M}\big(\frac{|\bar{\mathbf F}|^2}{\delta^2}\big)>\lambda_1^{2(k-i)}\}\big)}{\varphi(C_r(y))^{\frac{p}{2}}w\big(C_r(y)\big)}\\
&\leq \frac{k_p}{\varphi(C_r(y))^{\frac{p}{2}}\,w\big(C_r(y)\big)}\left(w(\Omega_r)+\int_{\Omega_r}\mathcal{M}\Big(\frac{|\bar{\mathbf F}|^2}{\delta^2}\Big)^{\frac{p}{2}}(x)w(x)\,dx\right)\\
&\leq \frac{k_p}{\varphi(C_r(y))^{\frac{p}{2}}\,w\big(C_r(y)\big)}\left(w(\Omega_r)+\int_{\Omega_r}\Big(\frac{|\bar{\mathbf F}|^2}{\delta^2}\Big)^{\frac{p}{2}}(x)w(x)\,dx\right).
\end{aligned}$$

Taking again the supremum over $y\in\Omega$ and $r>0$ and making use of (3.8) we get

$$\mathcal{S}'\leq c\left(1+\left\|\left|\frac{\bar{\mathbf F}}{\delta}\right|^2\right\|^{\frac{p}{2}}_{M_w^{\frac{p}{2},\varphi}(\Omega)}\right)\leq c\left(1+\frac{1}{\delta^p}\,\big\||\bar{\mathbf F}|^2\big\|^{\frac{p}{2}}_{M_w^{\frac{p}{2},\varphi}(\Omega)}\right)\leq c\,.$$

Taking ε, and the corresponding δ, small enough such that $0<\lambda_1^p\varepsilon_1<1$ we get

$$\sum_{k\geq 1}\lambda_1^{2k\frac{p}{2}}\frac{|\{x\in\Omega_r:\mathcal{M}(|D\bar u|^2)>\lambda_1^{2k}\}|}{\varphi(C_r(y))^{\frac{p}{2}}\,w\big(C_r(y)\big)}\leq c\sum_{k\geq 1}\big(\lambda_1^p\varepsilon_1\big)^k\leq c.$$

Taking again the supremum over $y\in\Omega$, $r>0$ in the estimates above and making use of Lemma 4.2 we find that

$$\|\mathcal{M}(|D\bar u|^2)\|_{M_w^{\frac{p}{2},\varphi}(\Omega)}\leq c.$$

This way, (2.7) and the definition of $\bar{u}$ imply

$$\||Du|^2\|_{M_w^{\frac{p}{2},\varphi}(\Omega)} \le c\||\mathbf{F}|^2\|_{M_w^{\frac{p}{2},\varphi}(\Omega)}$$

with constant depending on known quantities. □

Acknowledgements

The research of M.N. Omarova and Sh.A. Muradova is partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan.

The authors would like to express their gratitude to the referee for his (her) very valuable comments and suggestions.

References

[1] A. Akbulut, V.S. Guliyev, R. Mustafayev, On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces, Math. Bohem. **137** (1) (2012) 27-43.

[2] V. Burenkov, A. Gogatishvili, V.S. Guliyev, R. Mustafayev, Boundedness of the fractional maximal operator in local Morrey-type spaces, Complex Var. Elliptic Equ. **55** (8-10) (2010) 739-758.

[3] S. Amarantos, K. Tsimillis, C. Savvakis, J. Petropoulos, Kinetic analysis of transient perme-ation curves, J. Membr. Sci. **13** (1983) 259-272.

[4] S.-S. Byun, Elliptic equations with BMO coefficients in Lipschitz domains, Trans. Amer. Math. Soc. **357** (3) (2004) 1025-1046.

[5] S.-S. Byun, D. Palagachev, Weighted L_p estimates for elliptic equations with measurable coefficients in nonsmooth domains, Potential Anal. **41** (2014) 51-79.

[6] S.-S. Byun, L.G. Softova, Gradient estimates in generalized Morrey spaces for parabolic operators, Math. Nachr. **288** (14-15) (2015) 1602-1614.

[7] S.-S. Byun, D.K. Palagachev, L.G. Softova, Global gradient estimate in weighted Lebesgue spaces for parabolic operators, Ann. Acad. Sci. Fenn. Math. **41** (1) (2016) 67-83.

[8] S. Byun, L. Wang, Elliptic equations with BMO coefficients in Reifenberg domains, Commun. Pur. Appl. Math. **57** (10) (2004) 1283-1310.

[9] S. Byun, L. Wang, Elliptic equations with measurable coefficients in Reifenberg domains, Adv. Math. 225 (5) (2010) 2648-2673.

[10] S. Byun, S. Ryu, L. Wang, Gradient estimates for elliptic systems with measurable coefficients in nonsmooth domains, Manuscripta Math. **133** (2010) 225-245.

[11] M. Chipot, D. Kinderlehrer, G. Vergara-Caffarelli, Smoothness of linear laminates, Arch. Ration. Mech. Anal. **96** (1986) 81-96.

[12] F. Chiarenza, M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. **7** (1987) 273-279.

[13] M. De Guzman, Differentiation of Integrals in R^n, Springer, Heidelberg (1975).

[14] J. Elschner, J. Rehberg, G. Schmidt, Optimal regularity for elliptic transmission problems including C^1 interfaces, Interfaces Free Bound. **9** (2) (2007) 233-252.

[15] H. Frisch, G. Forgacs, S. Chui, Time moments analysis of sorption and permeation in linear laminated media, J. Phys. Chem. **83** (21) (1979) 2787-2792.

[16] V.S. Guliyev, Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J. **3** (3) (2012) 33-61.

[17] V.S. Guliyev, Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl. Art. ID 503948 (2009).
[18] V.S. Guliyev, L. Softova, Generalized Morrey estimates for the gradient of divergence form parabolic operators with discontinuous coefficients, J. Differential Equations, **959** (2015), 2368-2387.
[19] V.S. Guliyev, L. Softova, Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients, Potential Anal. **38** (4) (2013), 843-862.
[20] V.S. Guliyev, L.G. Softova, Generalized Morrey regularity for parabolic equations with discontinuity data, Proc. Edinb. Math. Soc. **58** (1) (2015) 199-218.
[21] Y. Komori, S. Shirai, Weighted Morrey spaces and a singular integral operator, Math. Nachr. **282** (2) (2009) 219-231.
[22] N.V. Krylov, Parabolic and elliptic equations with VMO coefficients, Commun. Partial Differ. Equ. **32** (3) (2007) 453-475.
[23] Y. Li, L. Nirenberg, Estimates for elliptic systems from composite material, Commun. Pur. Appl. Math. **56** (7) (2003) 892-925.
[24] T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, (1991) 183-189.
[25] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938) 126-166.
[26] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. **165** (1972) 207-226.
[27] E. Nakai, Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, Math. Nachr. **166** (1994) 95-103.
[28] E. Reinfenberg, Solutions of the plateau problem for m-dimensional surfaces of varying topological type, Acta Math. **104** (1960) 1-92.
[29] L. Softova, The Dirichlet problem for elliptic equations with VMO coefficients in generalized Morrey spaces, in: Oper. Theory Adv. Appl., Springer Basel AG, **229**, (2013), 365-380.
[30] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.
[31] L.G. Softova, Morrey-type regularity of solutions to parabolic problems with discontinuous data, Manuscripta Math. **136** (3-4) (2011) 365-382.
[32] A. Torchinsky, Real-Variable Methods in Harmonic Analysis, Academic Press, Orlando, 1986.
[33] T. Toro, Doubling and flatness, geometry of measures, Not. Am. Math. Soc. **44** (9) (1997) 1087-1094.
[34] H. Triebel, Hybrid Function Spaces, Heat and Navier-Stokes Equations, EMS Tracts in Mathematics, **24** (2015).

Ahmet Eroglu
Department of Mathematics, Omer Halisdemir University, Nigde, Turkey
E-mail address: `aeroglu@ohu.edu.tr`

Mehriban N. Omarova
Institute of Mathematics and Mechanics, NAS of Azerbaijan, AZ 1141, Baku, Azerbaijan
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail address: `mehriban_omarova@yahoo.com`

Shemsiyye A. Muradova
Institute of Mathematics and Mechanics, NAS of Azerbaijan, AZ 1141, Baku, Azerbaijan
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail address: mshams01@yahoo.com

Received: December 2, 2016; Revised: April 13, 2017; Accepted: April 26, 2017