Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

UPPER AND LOWER SOLUTIONS METHOD FOR FRACTIONAL OSCILLATION EQUATIONS

RABAH KHALDI AND ASSIA GUEZANE-LAKOUD

Abstract. This paper concerns the existence of solution to an oscillation equation involving Riemann-Liouville fractional derivative with initial conditions. The main tools for this study are the upper and lower solutions method and Schauder's fixed point theorem. For this, we reduce the posed problem to a first order ordinary initial value problem, then we give an explicit expression for the upper and lower solutions. The obtained results are illustrated by an example.

1. Introduction

The main advantage of fractional calculus in comparison with the classical integer-order is their ability to describe memory properties of materials and consequently have a better representation of physical models. Due to the intensive development of the theory of fractional calculus, the study of nonlinear differential equations of fractional order attracted more attention and many papers and monographs are devoted to this subject, see [2-10]. In general, exact solutions of these fractional equations are not available, therefore, different techniques for proving the existence of solutions for such problems were developed. Recent commonly used techniques are the method of upper and lower solutions, Mawhin's coincidence degree theory, fixed point theorems, the monotone iterative technique, etc.

The purpose of this paper is to discuss the fractional oscillation equation involving fractional time derivatives of the Riemann-Liouville type with initial conditions that we denote by (P1):

$$(P1)\begin{cases} D_{0^+}^{q}u(t)+\omega u(t)=f\left(t,u(t),D_{0^+}^{q-1}u(t)\right),0\leq t\leq 1,\omega>0\\ u(0)=0,D_{0^+}^{q-1}u(0)=0,\end{cases}$$

where $1<q<2$, u is the unknown function and $f\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$. The study is based on Schauder fixed point theorem and the method of lower and upper solutions. Moreover, we give explicit expressions for the lower and upper solutions and some results on the localization of the solution of the problem in

2010 *Mathematics Subject Classification.* 34A08, 26A33, 34A12.
Key words and phrases. Initial value problem, Fractional derivative, Upper and lower solutions method, Existence of solution.

question. Since physical phenomena, describing differential equations are mainly nonlinear in nature, some papers proved the existence of solution for nonlinear differential equations, see [1-10], imposing strong assumptions on the nonlinear term.

The organization of this paper is as follows. Section 2 is devoted to some definitions on fractional calculus and lemmas that will be used later. Upper and lower solutions for the problem in question is also cited. In section 3, we convert the problem (P1) to an equivalent first order initial value problem that we modify to conclude the existence of solutions for problem (P1). In addition, we construct the upper and lower solutions for (P1). An example is given to illustrate the theoretical results.

2. Preliminaries

Let us recall some basic definitions and some known results that can be found in [5,7,8].

Definition 2.1. *Let* $\alpha > 0$, *then the Riemann-Liouville fractional integral of a function* g *is defined by*

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.2. *The Riemann fractional derivative of order* q *of* g *is defined by*

$$D_{0+}^{q} g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds$$

where $n = [q] + 1.([q]$ *is the integer part of* q).

Lemma 2.1. *Let* $p, q \geq 0,\ f \in L_1[0,1]$. *Then:*

1- $I_{0+}^{q} D_{0+}^{q} f(t) = f(t) + \sum_{j=1}^{n} c_j t^{q-j}$, *for almost everywhere on* $[0,1]$.

2- *If* $p > 0$, $m \in \mathbb{N}$ *and* $D = \frac{d}{dt}$, *then* $D^m D_{0+}^{p} f(t) = D_{0+}^{p+m} f(t)$.

Now we define the lower and upper solutions for problem (P1):

Definition 2.3. The functions $\alpha,\ \beta \in AC^2[0,1]$ are called lower and upper solutions of problem (P1) respectively, if

a) $D_{0+}^{q}\alpha(t) + \omega\alpha(t) \leq f\left(t, \alpha(t), D_{0+}^{q-1}\alpha(t)\right)$, for $0 \leq t \leq 1$,

$\alpha(0) \leq 0$, $D_{0+}^{q-1}\alpha(0) \leq 0$.

b) $D_{0+}^{q}\beta(t) + \omega\beta(t) \geq f\left(t, \beta(t), D_{0+}^{q-1}\beta(t)\right)$, for $0 \leq t \leq 1$,

$\beta(0) = 0$, $D_{0+}^{q-1}\beta(0) \geq 0$.

Here

$$AC^2[0,1] = \left\{u \in C^1[0,1], u' \text{ absolutely continuous function on } [0,1]\right\}.$$

3. Main results

First we solve a fractional problem of order $(q-1)$. Let (P2) denote the following problem

$$(P2)\begin{cases} D_{0^+}^{q-1}u(t) = v(t), 0 \le t \le 1 \\ u(0) = 0. \end{cases}$$

Lemma 3.1. *For* $1 < q < 2$*, the solution of problem (P2) is given by*

$$u(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} v(s)\, ds.$$

Proof. Applying $I_{0^+}^{q-1}$ to both sides of the differential equation in (P2), then using Lemma 2.1, we get

$$u(t) = I_{0^+}^{q-1} v(t) + ct^{q-2},$$

multiplying both sides of the resultant equation by t^{2-q} then using the initial condition $u(0) = 0$, it yields $c = 0$. Consequently $u(t) = I_{0^+}^{q-1} v(t)$. This completes the proof. □

Let $E = C([0,1], \mathbb{R})$ equipped with the uniform norm $||u|| = \max_{t\in[0,1]} |u(t)|$. Define the operator T on E by

$$Tv(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} v(s)\, ds = I_{0^+}^{q-1} v(t), t \in [0,1].$$

In view of Lemma 3.1, we get $u(t) = Tv(t)$. Taking the condition $D_{0^+}^{q-1} u(0) = 0$ into account, we see that problem (P1) is equivalent to the following first order initial value problem that we denote by (P3)

$$(P3)\begin{cases} v'(t) + \omega Tv(t) = f(t, Tv(t), v(t)), 0 \le t \le 1 \\ v(0) = 0 \end{cases}$$

The following Lemmas will be used in the sequel

Lemma 3.2. *Assume that there exists a constant* $A \ge 0$ *such that* $f(t,x,y) \le A$*, for* $0 \le t \le 1$*,* $0 \le x \le \frac{A}{\Gamma(q+1)}$ *and* $0 \le y \le A$*. Then problem (P1) has an upper solution.*

Proof. Setting $\varphi(t) = At$, it yields

$$\begin{aligned} 0 \le\ & T\varphi(t) = \frac{A}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} s ds \\ =\ & \frac{A}{\Gamma(q+1)} t^q \le \frac{A}{\Gamma(q+1)}, \end{aligned}$$

then $\varphi'(t) + \omega T\varphi(t) - f(t, T\varphi(t), \varphi(t)) \ge 0$ and $\varphi(0) \ge 0$. As a consequence we have $\beta(t) = T\varphi(t)$ is an upper solution of problem (P1). □

Lemma 3.3. *Assume that there exists a constant* $B \le 0$*, such that* $f(t,x,y) \ge B$*, for* $0 \le t \le 1$*,* $\frac{B}{\Gamma(q+1)} \le x \le 0$ *and* $B \le y \le 0$*, then problem (P1) has a lower solution.*

Proof. Let $\psi(t) = Bt$, so,

$$\begin{aligned} 0 \ge\ & T\psi(t) = \frac{B}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} s ds \\ =\ & \frac{B}{\Gamma(q+1)} t^q \ge \frac{B}{\Gamma(q+1)}, \end{aligned}$$

so, $\psi'(t)+\omega T\psi(t)-f(t,T\psi(t),\psi(t))\leq 0$ and $\psi(0)\leq 0$, consequently, $\alpha(t)=T\psi(t)$ is a lower solution of problem (P1). □

Lemma 3.4. *Under the assumptions of Lemmas 3.2 and 3.3, the upper and lower solutions of problem (P1) satisfy*

$$\alpha(t)\leq\beta(t),\ D_{0^+}^{q-1}\beta(t)\geq D_{0^+}^{q-1}\alpha(t),0\leq t\leq 1.$$

Proof. Since $\beta(t)=T\varphi(t)$ and $\alpha(t)=T\psi(t)$ are upper and lower solutions of problem (P1) respectively, then from

$$\begin{aligned}\beta(t) &= \frac{A}{\Gamma(q+1)}t^q\geq 0,\\ \alpha(t) &= \frac{B}{\Gamma(q+1)}t^q\leq 0,\end{aligned}$$

we get

$$D_{0^+}^{q-1}\beta(t)=\varphi(t)=At\geq Bt=\psi(t)=D_{0^+}^{q-1}\alpha(t),$$

that completes the proof. □

Define the operator $F:E\to E$, by

$$\begin{aligned}(Fv)(t) &= -\omega T(\min[\varphi,(\max(v,\psi))])+\\ &\quad f(t,T(\min[\varphi,(\max(v,\psi))]),\min[\varphi,(\max(v,\psi))]),\\ 0 &\leq t\leq 1.\end{aligned}$$

Denote by (P4) the modified problem:

$$(P4)\begin{cases} v'(t)=(Fv)(t),0\leq t\leq 1\\ v(0)=0.\end{cases}$$

We have the following result

Lemma 3.5. *If v is a solution of problem (P4) then $u=Tv$ is a solution of problem (P1) satisfying*

$$\begin{aligned}\alpha(t) &\leq u(t)\leq\beta(t),\\ D_{0^+}^{q-1}\alpha(t) &\leq D_{0^+}^{q-1}u(t)\leq D_{0^+}^{q-1}\beta(t),0\leq t\leq 1.\end{aligned}$$

Proof. Let us prove that $\psi(t)\leq v(t)\leq\varphi(t)$, for $t\in[0,1]$ where v is a solution of problem (P4). If we suppose the contrary, i.e. there exists $t_1\in(0,1]$ such that $v(t_1)>\varphi(t_1)$. Putting $\epsilon(t)=v(t)-\varphi(t)$, and using the initial conditions $v(0)=\varphi(0)=0$, it yields $\epsilon(0)=0$. Since ϵ is continuous, we conclude that there exist $t_2\in[0,t_1)$ and $t_3\in[t_1,1]$ such that $\epsilon(t_2)=0$ and $\epsilon(t)\geq 0$, $t\in[t_2,t_3]$. Now, differentiating and applying Definition 2.3, we get for $t\in[t_2,t_3]$,

$$\begin{aligned}\epsilon'(t) &= v'(t)-\varphi'(t)=-\omega T(\min[\varphi,(\max(v,\psi))])\\ &\quad +f(t,T(\min[\varphi,(\max(v,\psi))]),\min[\varphi,(\max(v,\psi))])-D_{0^+}^{q}\beta(t)\\ &\leq 0,\end{aligned}$$

which implies that ϵ is decreasing on $[t_2,t_3]$. Since $\epsilon(t_2)=0$ we conclude that $v(t)\leq\varphi(t),t\in[t_2,t_3]$, this contradicts the fact $v(t_1)>\varphi(t_1)$. Similarly we prove that $\psi(t)\leq v(t)$, $t\in[0,1]$. From the above discussion, it yields

$$v'(t)=(Fv)(t)=-\omega Tv(t)+f(t,Tv(t),v(t))$$

that implies that v is solution of (P3) and therefore $u = Tv$ is a solution of (P1). Finally, in view of the monotony of the operator T, we obtain

$$T\psi(t) \le Tv(t) \le T\varphi(t), t \in [0,1],$$

this achieves the proof. □

Now we give the existence theorem for the problem (P1):

Theorem 3.1. *Assume that there exist two constants A and B such that $A \ge 0$, $B \le 0$ and $A \ge |B|$ and the following hypotheses hold*

$(H1)$- $f(t,x,y) \le A$, for $0 \le t \le 1$, $0 \le x \le \frac{A}{\Gamma(q+1)}$ and $0 \le y \le A$,

$(H2)$- $f(t,x,y) \ge B$, for $0 \le t \le 1$, $\frac{B}{\Gamma(q+1)} \le x \le 0$ and $B \le y \le 0$,

then the problem (P1) has at least one solution u such that

$$\begin{aligned} \alpha(t) &\le u(t) \le \beta(t), \\ D_{0^+}^{q-1}\alpha(t) &\le D_{0^+}^{q-1}u(t) \le D_{0^+}^{q-1}\beta(t), 0 \le t \le 1. \end{aligned}$$

Proof. Define the operator $R : E \to E$, by

$$Rv(t) = \int_0^t (Fv)(s)\, ds, 0 \le t \le 1.$$

Let us remark that a fixed point v of R is a solution of (P4) and then it satisfies $\psi(t) \le v(t) \le \varphi(t)$, for $t \in [0,1]$. Put $\Omega = \{v \in C[0,1], \|v\| \le \left(\frac{\omega}{\Gamma(q+1)} + 1\right) A\}$. Since φ and $\psi \in \Omega$ then for $v \in \Omega$ and taking condition (H1) into account, it yields

$$\begin{aligned} |Rv(t)| = \left| \int_0^t (Fv)(s)\, ds \right| &\le \omega \int_0^t |T(\min[\varphi, (\max(v,\psi))])(s)|\, ds \\ &+ \int_0^t |f(s, T(\min[\varphi, (\max(v,\psi))])(s), \min[\varphi, (\max(v,\psi))(s)])|\, ds \\ &\le \left(\frac{\omega}{\Gamma(q+1)} + 1\right) A, \end{aligned}$$

thus $R(\Omega)$ is uniformly bounded and $R(\Omega) \subset \Omega$. For $0 \le t_1 < t_2 \le 1$, we have

$$\begin{aligned} & |Rv(t_1) - Rv(t_2)| \\ \le\ & \omega \int_{t_1}^{t_2} |T(\min[\varphi, (\max(v,\psi))])(s)|\, ds + \\ & \int_{t_1}^{t_2} |f(s, T(\min[\varphi, (\max(v,\psi))])(s), \min[\varphi, (\max(v,\psi))(s)])|\, ds \\ \le\ & \left(\frac{\omega}{\Gamma(q+1)} + 1\right) A(t_2 - t_1), \end{aligned}$$

so, $R(\Omega)$ is equicontinuous. From Arzela-Ascoli Theorem, we conclude that R is completely continuous. Applying Schauder fixed point theorem we conclude that R has a fixed point $v \in \Omega$, and so $u = Tv$ is a solution of (P1) satisfying from Lemma 3.5 $\alpha(t) \le u(t) \le \beta(t)$ and $D_{0^+}^{q-1}\alpha(t) \le D_{0^+}^{q-1}u(t) \le D_{0^+}^{q-1}\beta(t)$, $0 \le t \le 1$. The proof is completed. □

Example 3.1. Consider the problem (P1) with

$$\begin{aligned} f\left(t, u(t), D_{0^+}^{q-1}u(t)\right) &= \frac{e^t}{4} + \frac{1}{4}u^3(t) + \frac{\left(D_{0^+}^{q-1}u(t)\right)^2}{16}, \\ q &= \frac{3}{2}, 0 \leq t \leq 1, \end{aligned}$$

then if we choose $A = 1$, we get $f(t, x, y) \leq \frac{e^1}{4} + \frac{1}{4}\left(\frac{1}{\Gamma(\frac{5}{2})}\right)^3 + \frac{1}{16} = 0.84849 \leq A$, for $0 \leq t \leq 1$, $0 \leq x \leq \frac{A}{\Gamma(q+1)}$ and $0 \leq y \leq A$. For $B = -1$, it yields $f(t, x, y) \geq \frac{1}{4} + \frac{1}{4}\left(\frac{-1}{\Gamma(\frac{5}{2})}\right)^3 - \frac{1}{16} = 0.088 \geq B$, then all assumptions of Theorem 3.1 hold. Consequently problem (P1) has a solution u such that $(-t) = \alpha(t) \leq u(t) \leq \beta(t) = t$.

References

[1] C. De Coster, P. Habets, *Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results,* Chapter, Non Linear Analysis and Boundary Value Problems for Ordinary Differential Equations, Volume 371 of the series International Centre for Mechanical Sciences pp 1-78.

[2] A. Guezane-Lakoud, R. Khaldi, and D. F. M. Torres, On a fractional oscillator equation with natural boundary conditions, *Prog. Frac. Diff. Appl.*(Forthcoming), https://arxiv.org/pdf/1701.08962.

[3] M. Jia, X. Liu, Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions, *Applied Mathematics and Computation* **232** (2014), 313–323.

[4] R. Khaldi and A. Guezane-Lakoud, Upper and lower solutions method for higher order boundary value problems, *Progr. Fract. Differ. Appl.* **1** (2017), 53–57.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science, Amsterdam, The Netherlands, 2006.

[6] X. Liu, M. Jia, Multiple solutions for fractional differential equations with nonlinear boundary conditions, *Computers & Mathematics with Applications* **8** (2010), 2880–2886.

[7] I. Podlubny, *Fractional Differential Equation*, Academic Press, Sain Diego (1999).

[8] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[9] A. Shi, S. Zhang, Upper and lower solutions method and a fractional differential equation boundary value problem, *Electronic Journal of Qualitative Theory of Differential Equations* **30** (2009), 1–13.

[10] S. Zhang, X. Su, The existence of a solution for a frational differential equation with nonlinear boundary conditions considered using upper and lower solutions in reversed order, *Compu. Math. Appl.* **62** (2011) 1269–1274.

Rabah Khaldi

Laboratory of Advanced Materials, Department of Mathematics, Badji Mokhtar-Annaba University P.O.Box 12, 23000 Annaba, Algeria.

E-mail address: `rkhadi@yahoo.fr`

Assia Guezane-Lakoud
Laboratory of Advanced Materials, Department of Mathematics, Badji Mokhtar-Annaba University P.O.Box 12, 23000 Annaba, Algeria.
E-mail address: a_guezane@yahoo.fr

Received: October 21, 2016; Accepted: April 27, 2017