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POINTWISE BERNSTEIN-WALSH-TYPE INEQUALITIES IN REGIONS WITH CUSPS IN THE LEBESGUE SPACE

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Abstract. In this work, we investigate the order of growth of the modulus of an arbitrary algebraic polynomials in the weighted Lebesgue space, where the contour and the weight functions have some singularities. In particular, we obtain pointwise Bernstein-Walsh type estimation for algebraic polynomials in the unbounded regions with piecewise smooth boundary having exterior and interior zero angles.

1. Introduction and Main Results

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Omega := \text{ext}L := \overline{\mathbb{C}} \backslash \overline{G}$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\Delta := \{w : |w| > 1\}$ and let $\wp_n$ denote the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$. Let $h(z)$ be a weight function. For any $p > 0$ we introduce:

$$\begin{aligned} \|P_n\|_{A_p} &:= \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \left| P_n(z) \right|^p d\sigma_z \right)^{1/p}, \ 0 < p < \infty; \\ \|P_n\|_{A_\infty} &:= \|P_n\|_{A_\infty(1,G)} := \max_{z \in \overline{G}} \left| P_n(z) \right|, \ p = \infty, \end{aligned}$$

where σ_z is the two-dimensional Lebesgue measure,

$$\begin{aligned} \|P_n\|_{\mathcal{L}_p} &:= \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left| P_n(z) \right|^p |dz| \right)^{1/p}, \ 0 < p < \infty; \quad (1.1) \\ \|P_n\|_{\mathcal{L}_\infty} &:= \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} \left| P_n(z) \right|, \ p = \infty, \end{aligned}$$

when L is rectifiable and

$$\|P_n\|_{C(\overline{G})} := \max_{z \in \overline{G}} \left| P_n(z) \right|.$$

Clearly, $\|\cdot\|_{A_p}$ and $\|\cdot\|_{\mathcal{L}_p}$ are the quasinorms (i.e., norms for $1 \le p \le \infty$ and $p-$norms for $0 < p < 1$).

Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ normalized by $\Phi(\infty) = \infty$, $\lim_{z \to \infty} \frac{\Phi(z)}{z} > 0$, and $\Psi := \Phi^{-1}$. For $t \ge 1$, we set:

$$L_t := \{z : |\Phi(z)| = t\}, L_1 \equiv L, G_t := \text{int}L_t, \Omega_t := \text{ext}L_t.$$

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Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on curve L which is located in positive direction. For some fixed number R_0, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider a generalized Jacobi weight function $h(z)$ defined as follows:

$$h(z) := h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \qquad z \in G_{R_0}, \tag{1.2}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., m$, and h_0 is uniformly separated from zero in G_{R_0}, that is, there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that for all $z \in G_{R_0}$

$$h_0(z) \geq c_0 > 0.$$

Well known Bernstein -Walsh Lemma [20] says that for any $R > 1$

$$|P_n(z)| \leq R^n \|P_n\|_{C(\overline{G})}, \qquad z \in G_R. \tag{1.3}$$

An analogous estimation in terms of the quasinorm (1.1) for $p > 0$ was obtained in [13] for $h(z) \equiv 1$ and in [5, Lemma 2.4] for $h(z)$ defined as in (1.2), as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\{0; \gamma_j : j \leq \ m\}. \tag{1.4}$$

In [4, Theorem1.1] a similar problem has been studied for $A_p(1, G)$-norm, $p > 0$; in that theorem for an arbitrary Jordan region the following result has been obtained: for any $p > 0$, $P_n \in \wp_n$, $R_1 = 1 + \frac{1}{n}$ and arbitrary R, $R > R_1$, the following is true:

$$\|P_n\|_{A_p(1,G_R)} \leq c \cdot R^{n+\frac{2}{p}} \|P_n\|_{A_p(1,G_{R_1})},$$

where $c = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$. Note that the constant c is sharp.

To give a similar estimation to (1.3) and (1.4) inequalities with respect to pair (G, G_R), first we will give some definitions and notations.

Following [14, p.97], [18], the Jordan curve (or arc) L is called $K-$quasiconformal $(K \geq 1)$, if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment). On the other hand, in [14, p.97], [18] some geometric criteria of quasiconformality of the curves were also given (see, [9, p.81], [15], [17, p.286]). We give one of them. Let z_1, z_2 be arbitrary points on L and $L(z_1, z_2)$ denotes the subarc of L of shorter diameter with endpoints z_1 and z_2. The curve L is a quasicircle if and only if the quantity

$$\frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|}$$

is bounded for all z_1, $z_2 \in L$ and $z \in L(z_1, z_2)$.

An estimate of the Bernstein-Walsh type for the regions G with quasiconformal boundary and weight function $h(z)$, as defined in (1.2) with $\gamma_j > -2$ for any $p > 0$ was found in [2] (see, also [3]) as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 \ R^{*^{n+\frac{1}{p}}} \|P_n\|_{A_p(h,G)},$$

where $R^* := 1 + c_2(R-1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ are constants, independent of n and R.

In [19] N. Stylianopoulos replaced the norm $\|P_n\|_{C(\overline{G})}$ with the norm $\|P_n\|_{A_2(G)}$ on the right hand side of (1.3) and found a new version of the Bernstein-Walsh

Lemma: *For any quasiconformal and rectifiable curve L there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \leq c_3(L)\frac{\sqrt{n}}{d(z,L)}\,\|P_n\|_{A_2(G)}\,|\Phi(z)|^{n+1}\,, \quad z \in \Omega, \tag{1.5}$$

holds for every $P_n \in \wp_n$, where $c_3(L) > 0$ is a constant independent of n and z, $d(z,L) := \operatorname{dist}(z,L) := \inf\{|\zeta - z| : \ \zeta \in L\}$.

Further, analogous results (1.5) for the quasinorm $\|.\|_{\mathcal{L}_p(h,L)}$ at the right hand side for some regions and the weight function $h(z)$ defined as in (1.2) with $\gamma_j > -1$ were obtained in [5] for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary with interior and exterior zero angles, in [6] for $p > 0$ and for regions bounded by piecewise quasiconformal boundary with interior and exterior zero angles, in [7] for $p > 1$ and for regions bounded by piecewise smooth boundary with exterior zero angles (without interior zero angles), in [8] for $p > 0$ and for regions bounded by piecewise quasismooth boundary with interior and exterior zero angles and in others.

In this work, we investigate similar problems for $z \in \Omega$ in regions bounded by piecewise smooth curve having interior and exterior zero angles and for generalized Jacobi weight function $h\,(z)$ defined in (1.2), through $\|.\|_{\mathcal{L}_p(h,L)}$ −quasinorm and $p > 0$.

Let us give some definitions and notations that will be used later in the text.

Let S be a rectifiable Jordan curve or arc and let $z = z(s),\ s \in [0,\ |S|]\,,\ |S| := mes\ S$, denote the natural representation of S.

Definition 1.1. We say that a Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We will write a region $G \in C_\theta$ if $\partial G \in C_\theta$.

According to [18], we have the following facts:

Corollary 1.1. *If $S \in C_\theta$ then S is $(1+\varepsilon)-$quasiconformal for arbitrary small $\varepsilon > 0$.*

Corollary 1.2. *If S is an analytic curve or arc, then S is 1-quasiconformal.*

Now, we shall define a new class of regions with piecewise smooth boundary, which have at the boundary points, corners and exterior cusps simultaneously.

For any $j = 1, 2, ...$ and sufficiently small $\varepsilon_1 > 0$ we denote by $f_j : [0, \varepsilon_1] \to \mathbb{R}$ the twice differentiable functions such that $f_j(0) = 0,\ f_j^{(k)}(x) > 0,\ x > 0$ and $k = 0, 1, 2$.

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general.

Definition 1.2. We say that a Jordan region $G \in C_\theta(\lambda_i; f_j),\ 0 < \lambda_i \leq 2,\ i = \overline{0, m_1},\ f_j = f_j(x),\ j = \overline{m_1 + 1, m}$, if $L = \partial G$ consists of a union of finite number of $C_\theta-$ arcs $\{L_j\}_{j=0}^m$, connecting at the points $\{z_j\}_{j=0}^m \in L$, such that L is locally smooth at $z_0 \in L \backslash\ \{z_j\}_{j=1}^m$ and:

a) for every $z_i \in L,\ i = \overline{0, m_1},\ m_1 \leq m$, the region G has exterior (with respect to $\overline{G}$) angle $\lambda_i \pi,\ 0 < \lambda_i \leq 2$, at the corner z_i;

b) for every $z_j \in L$, $j=\overline{m_1+1,m}$, in the local co-ordinate system (x,y) with origin at z_j the following conditions are satisfied:

$b_1)$ $\{z=x+iy: |z|<\varepsilon_1, \ c_1 f_j(x)\leq y\leq c_2 f_j(x), \ 0\leq x\leq \varepsilon_1\}\subset \overline{\Omega}$,

$b_2)$ $\{z=x+iy: |z|<\varepsilon_1, \ |y|\geq \varepsilon_2 x, \ 0\leq x\leq \varepsilon_1\}\subset \overline{G}$

for some constants $-\infty<c_1<c_2<+\infty$, $0<\varepsilon_i<1$, $i=1,2$.

Here and in further, for any $k\geq 0$ and $m>k$ the notation $j=\overline{k,m}$ denotes $j=k,k+1,...,m$.

It is clear from Definition 1.2 that each region $G\in C_\theta(\lambda_i;f_j)$ may have exterior nonzero (may be interior zero) $\lambda_i\pi$, $0<\lambda_i\leq 2$, angles at the points $z_i\in L$, $i=\overline{1,m_1}$, (when $\lambda_i=2$ interior zero angles) and exterior zero angles at which the boundary arcs are touching under the $f_j(x)-$speed at the points $z_j\in L$, $j=\overline{m_1+1,m}$. If $m_1=m=0$, then the region G does not have such angles, and in this case we will write: $G\in C_\theta$; if $m_1=m\geq 1$, then G has only $\lambda_i\pi$, $0<\lambda_i\leq 2$, $i=\overline{1,m_1}$, exterior nonzero (may be interior zero) angles, and in this case we will write: $G\in C_\theta(\lambda_i;0)$; if $m_1=0$ and $m\geq 1$, then G has only exterior zero angles, and in this case we will write: $G\in C_\theta(1;f_j)$).

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^m\in L$ defined in (1.2) and Definition 1.2 are identical. Without loss of generality, we assume that these points on the curve $L=\partial G$ are located in the positive direction such that, G has $\lambda_j\pi$, $0<\lambda_j\leq 2$, $j=\overline{0,m_1}$, exterior angles (when $\lambda_j=2-$ interior zero angles (interior cusps)) at the points $\{z_j\}_{j=1}^{m_1}$, $m_1\leq m$, and has exterior zero angles (exterior cusps) on the points $\{z_j\}_{j=m_1+1}^{m}$ and $w_j:=\Phi(z_j)$.

For simplicity of exposition and in order to avoid cumbersome calculations, without loss of generality, we will take $m_1=1$, $m=2$. Then, after this assumption, in the future we will have region $G\in C_\theta(\lambda_1;f_2)$, such that at the point $z_1\in L$ G has exterior nonzero $\lambda_1\pi$, $0<\lambda_1\leq 2$, (when $\lambda_1=2-$ interior zero angles (interior cusps)), and at the point $z_2\in L$ - exterior zero angle which the boundary arcs are touching under the $f_2(x)-$speed, and we set $\lambda:=\lambda_1$, $f:=f_2$.

According to the "three-point" criterion [9, p.100], every piecewise smooth curve (without any cusps) is quasiconformal.

Now we can state our new results.

Theorem 1.1. *Let $p>0$; $G\in C_\theta(\lambda;f)$, for some $0<\lambda\leq 2$ and $f(x)=cx^{1+\alpha}$, $\alpha>0$; $h(z)$ be defined as in (1.2). Then, for any $P_n\in\wp_n$, $n\in\mathbb{N}$ and arbitrary small $\varepsilon>0$, we have:*

$$|P_n(z)|\leq c_1\frac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L)}A_{n;1}\left\|P_n\right\|_{\mathcal{L}_p(h,L)}, \ z\in\Omega, \tag{1.6}$$

where $c_1=c_1(G,\gamma_1,\gamma_2,\alpha, \ p,\varepsilon)>0$ is a constant, independent of z and n,

$$A_{n;1}:=\begin{cases} n^{\frac{(\gamma_1-1)\widehat{\lambda}}{p}}, & \text{if } \gamma_1>1+\frac{\gamma_2-1}{\widehat{\lambda}(1+\alpha)}, \gamma_2>1,\\ n^{\frac{\gamma_2-1}{p(1+\alpha)}+\varepsilon}, & \text{if } 1<\gamma_1\leq 1+\frac{\gamma_2-1}{\widehat{\lambda}(1+\alpha)}, \gamma_2>1,\\ (\ln n)^{\frac{1}{p}}, & \text{if } \gamma_1\leq 1, \gamma_2=1,\\ 1, & \text{if } -1<\gamma_1,\gamma_2<1,\end{cases} \tag{1.7}$$

and $\widehat{\lambda}:=\begin{cases}\max\{1;\lambda\}+\varepsilon, & \text{if } 0<\lambda<2,\\ 2, & \text{if } \lambda=2.\end{cases}$

We can take individual cases when the curve L in the both points have the same type of angle: exterior nonzero or exterior zero angle. In this case, from Theorem 1.1, we obtain the following:

Corollary 1.3. *Let $p > 0$; G have exterior nonzero angles $\lambda_j\pi$ at the points z_j for some $0 < \lambda_j \leq 2$, $j = 1,2$; $h(z)$ be defined as in (1.2). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$ and arbitrary small $\varepsilon > 0$, we have:*

$$|P_n(z)| \leq c_2 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L)} A_{n;2} \|P_n\|_{\mathcal{L}_p(h,L)}, \ z \in \Omega,$$

where $c_2 = c_2(G, \gamma_1, \gamma_2, p, \varepsilon) > 0$ is a constant, independent of z and n,

$$A_{n;2} := \begin{cases} n^{\frac{(\gamma^*-1)\widehat{\lambda}^*}{p}}, & \text{if } \gamma_1, \gamma_2 > 1, \\ (\ln n)^{\frac{1}{p}}, & \text{if } \gamma_1 \leq 1, \gamma_2 = 1, \\ 1, & \text{if } -1 < \gamma_1, \gamma_2 < 1, \end{cases}$$

and $\gamma^* := \max\{\gamma_1; \gamma_1\}$, $\widehat{\lambda}^* := \max\left\{\widehat{\lambda}_1^*; \widehat{\lambda}_2^*\right\}$.

Corollary 1.4. *Let $p > 0$; G have at the points z_j exterior zero angles $f_j(x) = cx^{1+\alpha_j}$ for some $\alpha_j > 0, j = 1,2$; $h(z)$ be defined as in (1.2). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and arbitrary small $\varepsilon > 0$, we have:*

$$|P_n(z)| \leq c_3 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L)} A_{n;3} \|P_n\|_{\mathcal{L}_p(h,L)}, \qquad z \in \Omega,$$

where $c_3 = c_3(G, \gamma_1, \gamma_2, p, \varepsilon) > 0$ is a constant, independent of z and n and

$$A_{n;3} := \begin{cases} n^{\frac{\gamma^*-1}{p(1+\alpha^*)}+\varepsilon} & \text{if } \gamma_1, \gamma_2 > 1, \\ (\ln n)^{\frac{1}{p}}, & \text{if } \gamma_1 \leq 1, \gamma_2 = 1, \\ 1, & \text{if } -1 < \gamma_1, \gamma_2 < 1, \end{cases}$$

$\alpha^* := \min\{\alpha_1; \alpha_2\}$.

Now, we give correspondingly estimate for the $|P_n(z)|$ for the points $z \in G$. In this case the following holds.

Theorem 1.2. *Let $p > 0$; $G \in C_\theta(\lambda; f)$, for some $0 < \lambda \leq 2$ and $f(x) = cx^{1+\alpha}$, $\alpha > 0$; $h(z)$ be defined as in (1.2). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$ and arbitrary small $\varepsilon > 0$ we have:*

$$|P_n(z)| \leq c_4 \frac{A_{n;1}}{d^{2/p}(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)}, \qquad z \in G, \tag{1.8}$$

where $c_4 = c_4(G, \gamma_1, \gamma_2, p, \varepsilon) > 0$ is a constant, independent from z and n and $A_{n;1}$ defined as in (1.7).

1.1. Sharpness of estimates. The sharpness of the estimations (1.6) and (1.8) for some special cases can be discussed by comparing them with the following result:

Remark 1.1. For any $n \in \mathbb{N}$ there exist polynomials $P_n^* \in \wp_n$, region $G^* \subset \mathbb{C}$ and constant $c_5 = c_5(G) > 0$, such that

$$|P_n^*(z)| \geq c_5 |\Phi(z)|^{n+1} \|P_n^*\|_{\mathcal{L}_2(\partial G^*)}, \ \forall z \in F \subset C\overline{G^*}.$$

2. Some auxiliary results

Recall that, as noted above throughout this work, $c, c_0, c_1, c_2, ...$ are positive constants (generally, different in different relations), which depend on G in general. Further, for the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \prec b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1. [1] *Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) *The statements $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent.*

Then $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$;

b) *If $|z_1 - z_2| \prec |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}},$$

Corollary 2.1. *Under the assumptions of Lemma 2.1, for $z_3 \in L_{r_0}$ ($z_3 \in L_{R_0}$)*

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}.$$

Corollary 2.2. *If $L \in C_\theta$, then for all $\varepsilon > 0$*

$$|w_1 - w_2|^{1+\varepsilon} \prec |z_1 - z_2| \prec |w_1 - w_2|^{1-\varepsilon}.$$

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_1 - z_2|\}$, we put $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min\limits_{1\leq j\leq m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{2} \Omega(z_j, \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j, \delta))$, $\widehat{\Delta}(\delta) := \Delta \backslash \Delta(\delta)$.

The following lemma is a consequence of the results given in [12], [16], [21] and estimation for the $\left|\Psi'\right|$ (see, for example, [11, Th.2.8]):

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}.$$

Lemma 2.2. [21]. *Let $G \in C_\theta(\lambda)$, $0 < \lambda_j < 2$, $j = \overline{1, m_1}$. Then for all $\varepsilon > 0$:*

i. *for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j+\varepsilon} \prec |\Psi(w) - \Psi(w_j)| \prec |w - w_j|^{\lambda_j-\varepsilon}$, and $|w - w_j|^{\lambda_j-1+\varepsilon} \prec |\Psi'(w)| \prec |w - w_j|^{\lambda_j-1-\varepsilon}$,*

ii. *for any $w \in \overline{\Delta}\backslash\Delta_j$, $(|w| - 1)^{1+\varepsilon} \prec d(\Psi(w), L)| \prec (|w| - 1)^{1-\varepsilon}$, and $(|w| - 1)^{\varepsilon} \prec |\Psi'(w)| \prec (|w| - 1)^{-\varepsilon}$*

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (1.2):

Lemma 2.3. [5, Lemma 3.5]. *Let L be a rectifiable Jordan curve; $h(z)$ be defined as in (1.2). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$ we have*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ p > 0, \tag{2.1}$$

where $\gamma^* = \max\left\{0; \gamma_k : k = \overline{1, m}\right\}$.

Remark 2.1. In the case $h(z) \equiv 1$, the estimate (2.1) has been proved in [13].

3. Proofs

3.1. Proof of Theorem 1.1.

Proof. Suppose that $G \in C_\theta(\lambda; f)$, for some $0 < \lambda \leq 2$ and $f(x) = cx^{1+\alpha}$, $\alpha > 0$ and $h(z)$ is defined as in (1.2).Let $\{\xi_j\}$, $1 \leq j \leq m \leq n$, be the zeros (if any) of $P_n(z)$ lying on Ω. Lets define the function Blaschke with respect to the zeros $\{\xi_j\}$ of the polynomial $P_n(z)$:

$$\widetilde{B}_j(z) := \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)}, \qquad z \in \Omega, \tag{3.1}$$

and let

$$B_m(z) := \prod_{j=1}^{m} \widetilde{B}_j(z), \qquad z \in \Omega. \tag{3.2}$$

It is easy to see that $B_m(\xi_j) = 0$, $|B_m(z)| \equiv 1$ at $z \in L$ and $|B_m(z)| < 1$ at $z \in \Omega$.

For any $p > 0$ and $z \in \Omega$ let us set:

$$T_{n,p}(z) := \left[\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)} \right]^{p/2}.$$

The function $T_{n,p}(z)$ is analytic in Ω, continuous on $\overline{\Omega}$, $T_{n,p}(\infty) = 0$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $T_{n,p}(z)$ and for this branch, we maintain the same designation.

Cauchy integral representation for the unbounded region Ω is as follows:

$$T_n(z) = -\frac{1}{2\pi i} \int_L T_n(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in \Omega.$$

Since $|\Phi(\zeta)| = 1$, for $\zeta \in L$, from (3.1) and (3.2) we have

$$|P_n(z)|^{p/2} \leq \frac{|\Phi(z)|^{\frac{p(n+1)}{2}}}{2\pi d(z, L)} \int_L |P_n(\zeta)|^{p/2} |d\zeta|. \tag{3.3}$$

Set

$$A_n := \int_L |P_n(\zeta)|^{p/2} |d\zeta| = \sum_{i=1}^{2} \int_{L^i} |P_n(\zeta)|^{p/2} |d\zeta|. \tag{3.4}$$

Multiplying the numerator and the denominator by $h_0^{1/2}(\zeta) \prod_{j=1}^{2} |\zeta - z_j|^{\gamma_j/2}$ of the integrand, then applying the Holder inequality, from (1.2) we obtain:

$$\begin{aligned} A_n \ &\prec \ \sum_{i=1}^{2} \left(\int_{L^i} h(\zeta) |P_n(\zeta)|^p |d\zeta| \right)^{1/2} \cdot \left(\int_{L^i} \frac{|d\zeta|}{\prod_{i=1}^{2} |\zeta - z_i|^{\gamma_i}} \right)^{1/2} \\ &\prec \ \|P_n\|_p^{p/2} \cdot \left(\sum_{i=1}^{2} \int_{L^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}} \right)^{1/2} =: \|P_n\|_p^{p/2} \cdot \left(J_n^1 + J_n^2\right)^{1/2}, \end{aligned} \tag{3.5}$$

where

$$J_n^i := \int\limits_{L^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}}, \qquad i = 1, 2.$$

To simplify further calculations, we assume that $z_1 = -1$, $z_2 = 1$; $(-1, 1) \subset G$ and let local co-ordinate axis in Definition 1.2 be parallel to OX and OY in the co-ordinate system; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : \mathrm{Im} z \geq 0\}$, $L^- := \{z \in L : \mathrm{Im} z < 0\}$; Let $w_j = \Phi(z_j) = e^{i\varphi_j}$; $w^\pm := \left\{\tau = e^{i\theta_\pm} : \theta_\pm = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, $z^\pm = \Psi(w^\pm)$, $l_i^\pm := l_i^\pm(z_i, z^\pm)$ denote the arcs, connecting the points z_i with $z^\pm$, respectively; $\left| l_i^\pm \right| := mes\ l_i^\pm$, $i = 1, 2$; $L^1 := L^1(z^+, z_1, z^-)$ denote the arcs, connecting the points z^+ and z^-, passing through the point z_1, $L^2 := L^2(z^-, z_2, z^+)$ denote the arcs, connecting the points z^- and z^+, passing through the point z_2.

Let z_0 be an arbitrary point on L^+ (or on L^- subject to the chosen direction). To simplify further calculations, without loss of generality, we can take $z_0 = z^+$ (or $z_0 = z^-$).

Throughout this work, we will take $R = 1 + \frac{1}{n}$. Further, let $d_{i,R} := d(z_i, L_R)$, $E_1^{j,\pm} := \left\{\zeta \in L^j : \ |\zeta - z_j| < c_j d_{j,R}\right\}$, and $E_2^{j,\pm} := \{\zeta \in L^j : \ c_j d_{j,R} \leq |\zeta - z_j| < \left| l_j^\pm \right|\}$, $j = 1, 2$. Taking into account these notations, (3.5) can be written as

$$A_n \prec \|P_n\|_p^{p/2} \cdot \left(J_n^1 + J_n^2\right)^{1/2} =: \|P_n\|_p^{p/2} \sum_{i=1}^{2} \left[I_{n,1}^{i,\pm} + I_{n,2}^{i,\pm}\right]^{1/2},$$

where

$$I_{n,k}^{i,\pm} := I_{n,k}^{i,\pm}(E_k^{i,\pm}) := \int\limits_{E_k^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}}, \ i, k = 1, 2. \tag{3.6}$$

According to (3.3) and (3.4), it is sufficient to estimate the integrals $I_{n,k}^{i,\pm}$ for each $i = 1, 2$ and $k = 1, 2$.

Given the possible values of γ_i $(-1 < \gamma_i < 0, \ \gamma_i \geq 0, \ i = 1, 2)$, we will consider the estimates for the $I_{n,k}^{i,\pm}$ separately.

Let $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. In this case for the integral J_n^1, we get:

$$I_{n,1}^{1,\pm} = \int\limits_{E_1^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} \prec \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1}} \prec \begin{cases} d_{1,R}^{1-\gamma_1}, & \gamma_1 > 1, \\ 1, & -1 < \gamma_1 \leq 1; \end{cases} \tag{3.7}$$

$$I_{n,2}^{1,\pm} := \int\limits_{E_2^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} \prec \int\limits_{c_1 d_{1,R}}^{|l_1^\pm|} \frac{ds}{s^{\gamma_1}} \prec \begin{cases} d_{1,R}^{1-\gamma_1}, & \gamma_1 > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1 = 1, \\ 1, & -1 < \gamma_1 < 1. \end{cases}$$

Similar estimate for the integral J_n^2:

$$I_{n,1}^{2,\pm} = \int\limits_{E_1^{2,\pm}} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}} \prec \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2}} \prec \begin{cases} d_{2,R}^{1-\gamma_2}, & \gamma_2 > 1, \\ 1, & -1 < \gamma_2 \leq 1; \end{cases} \tag{3.8}$$

$$I_{n,2}^{2,\pm}=\int\limits_{E_2^{2,\pm}}\frac{|d\zeta|}{|\zeta-z_2|^{\gamma_2}}\prec\int\limits_{c_2d_{2,R}}^{|l_2^{\pm}|}\frac{ds}{s^{\gamma_2}}\prec\begin{cases}d_{2,R}^{1-\gamma_2}, & \gamma_2>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2=1,\\ 1, & -1<\gamma_2<1.\end{cases}$$

Let $\gamma_1<0$ and $\gamma_2<0$. Then, analogously to the (3.7) and (3.8), we get:

$$I_{n,1}^{1,\pm}=\int\limits_{E_1^{1,\pm}}|\zeta-z_1|^{(-\gamma_1)}|d\zeta|\prec d_{1,n}^{(-\gamma_1)}mesE_1^1\prec 1,\tag{3.9}$$

$$I_{n,2}^{1,\pm}=\int\limits_{E_2^{1,\pm}}|\zeta-z_1|^{(-\gamma_1)}|d\zeta|\prec\left|l_1^{\pm}\right|^{1-\gamma_1}\prec 1;$$

$$I_{n,1}^{2,\pm}\prec\int\limits_{E_1^{2,\pm}}|\zeta-z_2|^{(-\gamma_2)}|d\zeta|\prec d_{2,R}^{(-\gamma_2)}mesE_1^{2,\pm}\prec 1,\tag{3.10}$$

$$I_{n,2}^{2,\pm}\prec\int\limits_{E_2^{2,\pm}}|\zeta-z_2|^{(-\gamma_2)}|d\zeta|\prec\left|l_2^{\pm}\right|^{1-\gamma_2}\prec 1.$$

Therefore, in this case, from (3.6) - (3.10), we obtain:

$$A_n\prec\|P_n\|_p^{p/2}\cdot\begin{cases}d_{1,R}^{\frac{1-\gamma_1}{2}}+d_{2,R}^{\frac{1-\gamma_2}{2}}, & \gamma_1,\gamma_2>1,\\ \left[\ln\frac{1}{d_{1,R}}+\ln\frac{1}{d_{2,R}}\right]^{\frac{1}{2}}, & \gamma_1=\gamma_2=1,\\ 1, & -1<\gamma_1,\gamma_2<1.\end{cases}\tag{3.11}$$

Comparing (3.3), (3.4) and (3.11), we have:

$$|P_n(z)|\le c\frac{\left(A_{n,1}^0\right)^{1/p}}{d^{p/2}(z,L)}\|P_n\|_p|\Phi(z)|^{n+1},\tag{3.12}$$

where $c=c(G,p,\gamma_i)>0$, is the constant independent from n and z, and

$$A_{n,1}^0:=\begin{cases}d_{1,R}^{1-\gamma_1}+d_{2,R}^{1-\gamma_2}, & \gamma_1,\gamma_2>1,\\ \ln\frac{1}{d_{1,R}}+\ln\frac{1}{d_{2,R}}, & \gamma_1=\gamma_2=1,\\ 1, & -1<\gamma_1,\gamma_2<1.\end{cases}\tag{3.13}$$

According to [10, Lemma 1.1] and Lemma 2.2, for the point z_1 we get:

$$d_{1,R}\succ n^{-\widehat{\lambda}_1}.\tag{3.14}$$

For the estimate $d_{2,R}$, let us set: $z_{2,R}\in L_R$ such that $d_{2,R}=|z_2-z_{2,R}|$; $\zeta^{\pm}\in L^{\pm}$ such that $d(z_{2,R},L^2\cap L^{\pm}):=d(z_{2,R},L^{+})$; $z_2^{\pm}:=\zeta\in L^2:\ |\zeta-z_2|=c_2d_{2,R}$. Under these notations, from Lemma 2.1, we obtain:

$$d_R^{\pm}:=d(z_{2,R},L^2\cap L^{\pm})\asymp\left|z_{2,R}-z_2^{\pm}\right|\asymp d_{2,R}^{1+\alpha}.\tag{3.15}$$

Hence, $d_{2,R} = \left(d_R^{\pm}\right)^{\frac{1}{1+\alpha}}$. On the other hand, according to Lemma 2.2 and [10, Lemma 1.1], we get: $d_R^{\pm} \succ n^{-1-\varepsilon}$. Therefore,

$$d_{2,R} \succ n^{\frac{\varepsilon-1}{1+\alpha}}, \tag{3.16}$$

for arbitrary small $\varepsilon > 0$. Comparing (3.12)-(3.16), we get:

$$|P_n(z)| \prec \frac{A_{n,1}}{d^{p/2}(z,L)} \|P_n\|_p |\Phi(z)|^{n+1}, \ z \in \Omega,$$

where

$$A_{n,1} := \begin{cases} n^{\frac{(\gamma_1-1)\widehat{\lambda}}{p}} + n^{\frac{\gamma_2-1}{p(1+\alpha)}+\varepsilon}, & \gamma_1, \gamma_2 > 1, \\ (\ln n)^{\frac{1}{p}}, & \gamma_1 = \gamma_2 = 1, \\ 1, & -1 < \gamma_1, \gamma_2 < 1, \end{cases} \tag{3.17}$$

and we complete the proof. □

3.2. Proof of Theorem 1.2.

Proof. Suppose that $G \in C_\theta(\lambda; f)$, for some $0 < \lambda \leq 2$ and $f(x) = cx^{1+\alpha}$, $\alpha > 0$ and $h(z)$ is defined as in (1.2). Let $w = \varphi(z)$ be denote the univalent conformal mapping from G onto the B, normalized by $\varphi(0) = 0$, $\varphi'(0) > 0$, and let $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, be zeros (if exists) of $P_n(z)$ lying on G. Let

$$B_m(z) := \prod_{j=1}^{m} \widetilde{B}_j(z) = \prod_{j=1}^{m} \frac{\varphi(z) - \varphi(\zeta_j)}{1 - \overline{\varphi(\zeta_j)}\varphi(z)},$$

denote a Blaschke function with respect to zeros $\{\zeta_j\}$, $1 \leq j \leq m \leq n$, of $P_n(z)$ ([20]). Clearly,

$$|B_m(z)| \equiv 1, \ z \in L, \ \text{and} \ \ |B_m(z)| < 1, \ z \in G.$$

For any $p > 0$ and $z \in G$, let us set:

$$Q_{n,p}(z) := \left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}, \qquad z \in G.$$

The function $Q_{n,p}(z)$ is analytic in G, continuous on $\overline{G}$ and does not have zeros in G. We take an arbitrary continuous branch of the $Q_{n,p}(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $Q_{n,p}(z)$ in G gives

$$Q_{n,p}(z) = \frac{1}{2\pi i} \int_L Q_{n,p}(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in G.$$

Hence,

$$|P_n(z)|^{p/2} \leq \frac{|B_m(z)|^{p/2}}{2\pi} \int_L \left|\frac{P_n(\zeta)}{B_m(\zeta)}\right|^{p/2} \frac{|d\zeta|}{|\zeta - z|} \leq \int_L |P_n(\zeta)|^{p/2} \frac{|d\zeta|}{|\zeta - z|},$$

since $|B_{m,R}(\zeta)| = 1$, for $\zeta \in L$. Then, since $|B_{m,R}(z)| < 1$, for $z \in G$ we have:

$$|P_n(z)|^{p/2} \prec \frac{1}{d(z,L)} \int_L |P_n(\zeta)|^{p/2} |d\zeta| =: \frac{1}{d(z,L)} A_n, \tag{3.18}$$

where A_n is defined as in (3.4). Combining relations (3.11), (3.13), (3.17) and (3.18), we complete the proof. □

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