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# CONVERGENCE OF BIORTHOGONAL EXPANSION OF A FUNCTION FROM THE CLASS $W_2^1(G)$ IN EIGEN AND ASSOCIATED FUNCTIONS OF EVEN ORDER ORDINARY DIFFERENTIAL OPERATOR

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**Abstract**. In the paper an ordinary differential operator of $2m$-th order is considered. Absolute and uniform convergence of biorthogonal expansion of a function $f(x)$ from the class $W_2^1(G)$, $G=(0,1)$, satisfying the condition $f(0)=f(1)=0$, in eigen and associated functions of this operator is studied, and the rate of uniform convergence of this expansion in $\overline{G}=[0,1]$ is estimated.

## 1. Introduction and formulation of results

Consider on the interval $G=(0,1)$ a formal differential operator

$$Lu=u^{(2m)}+p_2(x)\,u^{(2m-2)}+\cdots+p_{2m}(x)\,u$$

with complex-valued coefficients $p_l(x)\in L_1(G)$, $l=\overline{2,2m}$.

Denote by $D_{2m}(G)$ a class of functions absolutely continuous together with their derivatives of order $\leq 2m-1$ on the closed interval $\overline{G}=[0,1]$ ($D_{2m}(G)\equiv W_1^{2m}(G)$).

Following [2] under an eigenfunction of the operator $L$ corresponding to a complex eigenvalue $\lambda$, we will understand any complex-valued function $\overset{\circ}{u}(x)\in D_{2m}(G)$ not identically equal to zero and satisfying almost everywhere in $G$ the equation $L\overset{\circ}{u}+\lambda\overset{\circ}{u}=0$. In the similar way, under an associated function of order $r\,(r\geq 1)$ of the operator $L$, corresponding to the same eigenvalue $\lambda$ and eigenfunction $\overset{\circ}{u}(x)$ we will understand any complex-valued function $\overset{r}{u}(x)\in D_{2m}(G)$ satisfying almost everywhere on $G$ the equation $L\overset{r}{u}+\lambda\overset{r}{u}=\overset{r-1}{u}$.

We will consider each eigen function an associated function of zero order. The highest order of the root (associated) functions corresponding to the given eigen function will be said to be the rank of this eigen function.

Consider an arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$ consisting of eigen and associated functions of the operator $L$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding eigenvalues. We require that, together with each associated function of order $r\geq 1$ the system $\{u_k(x)\}_{k=1}^{\infty}$ contain the corresponding eigenfunction and all associated functions

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of order less than $r$, and the ranks of eigenfunctions be uniformly bounded. This means that $u_k(x) \in D_{2m}(G)$ and satisfies almost everywhere in $G$ the equation $Lu_k + \lambda_k u_k = \theta_k u_{k-1}$, where $\theta_k$ is equal either to 0 (in this case $u_k(x)$ is an eigenfunction) or 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k(x)$ an associated function).

Denote $\mu_k = \left[(-1)^{m+1}\lambda_k\right]^{1/2m}$, where $\left(\rho e^{i\varphi}\right)^{1/2m} = \rho^{1/2m} e^{i\varphi/(2m)}$, $-\pi < \varphi \leq \pi$. Obviously, $Re\mu_k \geq 0$.

Denote $L_p(G)$, $1 \leq p \leq \infty$, a space of functions $f(x)$ with the norm $\|f\|_p = \left(\int\limits_G |f(x)|^p dx\right)^{1/p}$, moreover for $p = \infty$

$$\|f\|_\infty = vrai\sup_{x\in\overline{G}} |f(x)|.$$

Let $p_l(x) \in W_1^{2m-l}(G)$, $l = \overline{2, 2m}$. Therewith $W_1^0(G) \equiv L_1(G)$.

By $L^*$ we denote the formally adjoint operator of the operator $L$, i.e.

$$L^* y = (-1)^n y^{(n)} + (-1)^{n-2}\left(\overline{p_2(x)} y\right)^{(n-2)} + \cdots + \overline{p_n(x)} y,$$

where $n = 2m$.

We require the system $\{u_k(x)\}_{k=1}^\infty$ satisfy the following conditions, which we call the conditions $A$:

1) the system $\{u_k(x)\}_{k=1}^\infty$ is complete and minimal in $L_2(G)$;

2) the Carleman and "sum of units" conditions are valid:

$$|Im\mu_k| \leq const, \qquad k = 1, 2, ..., \tag{1.1}$$

$$\sum_{\tau \leq \rho_k \leq \tau+1} 1 \leq const, \qquad \forall \tau \geq 0, \tag{1.2}$$

where $\rho_k = Re\mu_k$.

3) the system $\{v_k(x)\}_{k=1}^\infty$ biorthogonal to $\{u_k(x)\}_{k=1}^\infty$ is the system of eigen and associated functions of the formally adjoint operator $L^*$, i.e. $L^* v_k + \overline{\lambda}_k v_k = \theta_{k+1} v_{k+1}$;

4) the following antiapriori estimations are valid:

$$\theta_k \|u_{k-1}\|_2 \leq const\,(1 + |\mu_k|)^{2m-1} \|u_k\|_2, \tag{1.3}$$

$$\theta_{k+1} \|v_{k+1}\|_2 \leq const\,(1 + |\mu_k|)^{2m-1} \|v_k\|_2, \tag{1.4}$$

5) there exists a constant $C_0$ such that

$$\|u_k\|_2 \|v_k\|_2 \leq C_0, \quad k = 1, 2, ..., \tag{1.5}$$

6) for any $\tau \geq 0$ the following estimations are valid:

$$\sum_{0 \leq \rho_k \leq \tau} \|u_k\|_\infty^2 \|u_k\|_2^{-2} \leq const\,(1 + \tau), \tag{1.6}$$

$$\sum_{0 \leq \rho_k \leq \tau} \|v_k\|_\infty^2 \|v_k\|_2^{-2} \leq const\,(1 + \tau). \tag{1.7}$$

For an arbitrary function $f(x) \in W_2^1(G)$ we introduce the partial sum

$$\sigma_\nu(x, f) = \sum_{\rho_k \leq \nu} f_k u_k(x), \quad \nu > 0,$$

where

$$f_k = (f, v_k) = \int_0^1 f(x)\,\overline{v_k(x)}dx.$$

In the paper we prove the following theorem on the an absolute and uniform convergence of biorthogonal expansion.

**Theorem 1.1.** *Let the conditions A be fulfilled, a function $f(x) \in W_2^1(G)$ satisfy the condition $f(0) = f(1) = 0$. Then biorthogonal expansion of the function $f(x)$ converges absolutely and uniformly on the segment $\overline{G} = [0, 1]$ and the following estimation is valid:*

$$\|\sigma_\nu(\cdot, f) - f\|_{C[0,1]} \le$$

$$\le const \left\{ \nu^{-\frac{1}{2}} \left\|f'\right\|_2 + \sum_{l=2}^{2m-1} \nu^{\frac{1}{2}-l} \|q_l f\|_2 + \nu^{1-2m} \|q_{2m} f\|_1 \right\}, \quad \nu \ge 2, \tag{1.8}$$

*where*

$$q_l(x) = -\sum_{s=0}^{l-2} (-1)^{2m-l+s}\, C_{2m-l+s}^{s} p_{l-s}^{(s)}(x), \quad l = \overline{2, m},$$

*and the const is independent of the function $f(x)$.*

**Corollary 1.1.** *Let the conditions A be fulfilled. Then biorthogonal expansion of the function $f(x) \in W_2^1(G)$, $f(0) = f(1) = 0$, converges absolutely and uniformly, and the following estimations are valid:*

$$\|\sigma_\nu(\cdot, f) - f\|_{C[0,1]} \le const\nu^{-\frac{1}{2}} \|f\|_{W_2^1(G)}, \quad \nu \ge 2, \tag{1.9}$$

$$\|\sigma_\nu(\cdot, f) - f\|_{C[0,1]} = o\left(\nu^{-\frac{1}{2}}\right), \quad \nu \to +\infty, \tag{1.10}$$

*where the const is independent of $f(x)$, the symbol "o" is dependent on the function $f(x)$.*

Note that similar results for the Sturm-Liouville operator were established in papers [5], [6], [9], while for an operator of the fourth order in the paper [7].

## 2. Auxilary lemmas

Before starting the proof of theorem 1.1 we note that conditions $A$ provide Riesz basicity of each of the systems $\left\{u_k(x)\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{v_k(x)\|v_k\|_2^{-1}\right\}_{k=1}^{\infty}$ in $L_2(G)$ (see [1], [3], [4]). Therefore, for these systems the Bessel inequality is valid in $L_2(G)$.

Let the conditions of theorem 1.1 be fulfilled. Prove uniform convergence of the series

$$\sum_{k=1}^{\infty} |f_k|\, |u_k(x)|, \quad x \in \overline{G}. \tag{2.1}$$

For that we give auxiliary lemmas.

**Lemma 2.1.** *For the Fourier coefficient $f_k$ of the function $f(x) \in W_2^1(G)$, $f(0) = f(1) = 0$, the following represention is valid:*

$$f_k = (f, v_k) = \lambda_k^{-1} \sum_{i=0}^{m_k} \lambda_k^{-i} \left( f', v_{k+i}^{(2m-1)} \right)$$

$$+\lambda_k^{-1} \sum_{l=2}^{2m} \sum_{i=0}^{m_k} \lambda_k^{-i} \left( q_l f, v_{k+i}^{(2m-1)} \right), \quad \lambda_k \neq 0, \tag{2.2}$$

*where $m_k$ is the order of the associated function $v_k(x)$.*

*Proof.* By definition of the function $v_k(x)$ the equality $v_k = -\left(\overline{\lambda}_k\right)^{-1} L^* v_k + \left(\overline{\lambda}_k\right)^{-1} \theta_{k+1} v_{k+1}$ holds for $\lambda_k \neq 0$.

Taking this into account, we get

$$(v_k, f) = -\left(\overline{\lambda}_k\right)^{-1} (L^* v_k, f) + \left(\overline{\lambda}_k\right)^{-1} \theta_{k+1} (v_{k+1}, f) = -\left(\overline{\lambda}_k\right)^{-1} \left( v_k^{(2m)}, f \right)$$

$$+ \left(\overline{\lambda}_k\right)^{-1} \sum_{l=2}^{2m} \left( \overline{q}_l v_k^{(2m-l)}, f \right) + \left(\overline{\lambda}_k\right)^{-1} \theta_{k+1} (v_{k+1}, f)$$

$$= -\left(\overline{\lambda}_k\right)^{-1} \left( v_k^{(2m)}, f \right) + \left(\overline{\lambda}_k\right)^{-1} \sum_{l=2}^{2m} \left( v_k^{(2m-l)}, q_l f \right) + \theta_{k+1} \left(\overline{\lambda}_k\right)^{-1} (v_{k+1}, f).$$

From this recurrent relation, with regard to $\theta_{k+1} = \theta_{k+2} = \cdots = \theta_{k+m_k} = 1$, $\theta_{k+m_k+1} = 0$ we find

$$(v_k, f) = -\left(\overline{\lambda}_k\right)^{-1} \sum_{i=0}^{m_k} \left(\overline{\lambda}_k\right)^{-i} \left( v_{k+i}^{(2m)}, f \right) + \left(\overline{\lambda}_k\right)^{-1} \sum_{l=2}^{2m} \sum_{i=0}^{m_k} \left(\overline{\lambda}_k\right)^{-i} \left( v_{k+i}^{(2m-l)}, q_l f \right).$$

At first we conduct integration by parts in the expression $\left( v_{k+i}^{(2m)}, f \right)$, and then having taken the complex conjugation we get formula (2.2). Lemma 2.1 is proved. □

**Lemma 2.2.** *For any $\mu \geq 2$ the following estimations are valid:*

$$\sum_{Re\mu_k \geq \mu} |\mu_k|^{-(1+\delta)} \|u_k\|_\infty^2 \|u_k\|_2^{-2} \leq C_1(\delta) \mu^{-\delta}, \tag{2.3}$$

$$\sum_{Re\mu_k \geq \mu} |\mu_k|^{-(1+\delta)} \|v_k\|_\infty^2 \|v_k\|_2^{-2} \leq C_2(\delta) \mu^{-\delta}, \tag{2.4}$$

*where $\delta > 0$, $C_1(\delta)$, $C_2(\delta)$ are constants independent of $\mu$.*

*Proof.* Prove estimation (2.4). By conditions (1.1), (1.2), (1.7) and Abel's transformation for any fixed natural number $l$ we have

$$\sum_{\mu \leq \rho_k \leq [\mu]+l} |\mu_k|^{-(1+\delta)} \|v_k\|_\infty^2 \|v_k\|_2^{-2}$$

$$\leq \sum_{[\mu] \leq \rho_k \leq [\mu]+l} \rho_k^{-(1+\delta)} \|v_k\|_\infty^2 \|v_k\|_2^{-2} \leq \sum_{n=[\mu]}^{[\mu]+l} n^{-(1+\delta)} \sum_{n \leq \rho_k < n+1} \|v_k\|_\infty^2 \|v_k\|_2^{-2} \leq$$

$$\leq \sum_{n=[\mu]}^{[\mu]+l-1}\left(\sum_{1\leq\rho_k\leq n+1}\|v_k\|_\infty^2\|v_k\|_2^{-2}\right)\left(n^{-(1+\delta)}-(n+1)^{-(1+\delta)}\right)$$

$$+\left(\sum_{1\leq\rho_k\leq[\mu]+l}\|v_k\|_\infty^2\|v_k\|_2^{-2}\right)([\mu]+l)^{-(1+\delta)}$$

$$+\left(\sum_{1\leq\rho_k\leq[\mu]-l}\|v_k\|_\infty^2\|v_k\|_2^{-2}\right)[\mu]^{-(1+\delta)}\leq const\sum_{n=[\mu]}^{[\mu]+l-1}(n+1)\frac{(1+\delta)(n+1)^\delta}{(n(n+1))^{1+\delta}}$$

$$+const\,([\mu]+l)([\mu]+l)^{-1-d}+const\,([\mu]-1)[\mu]^{-1-\delta}$$

$$\leq const\left((1+\delta)\sum_{n=[\mu]}^{\infty}n^{-(1+\delta)}+[\mu]^{-\delta}\right)\leq C_2(\delta)\,\mu^{-\delta},$$

where $C_2(\delta)=const\left(1+\frac{1}{\delta}\right)$.

Hence, by arbitrariness of the natural number $l$ we get estimation (2.4). Estimation (2.3) is established in the same way. Lemma 2.2 is proved. □

**Lemma 2.3.** *(see [8]) Subject to conditions A, the systems* $\left\{v_k^{(l)}(x)\|v_k\|_2^{-1}\mu_k^{-l}\right\}$, $\mu_k\neq 0$, $l=\overline{0,2m-1}$ *are Bessel in* $L_2(G)$ *i.e. for an arbitrary* $f(x)\in L_2(G)$ *the following inequality is valid*

$$\left(\sum_{\mu_k\neq 0}\left|\left(f,v_k^{(l)}\|v\|_2^{-1}\mu_k^{-l}\right)\right|^2\right)^{1/2}\leq const\,\|f\|_2\,. \tag{2.5}$$

**Lemma 2.4.** *Subject to conditions A, the following estimations are valid:*

$$I_r(\mu)=\sum_{\rho_k\geq\mu}\left|\lambda_k^{-1}\sum_{i=0}^{m_k}\lambda_k^{-i}\left(q_rf,v_{k+i}^{(2m-r)}\right)\right||u_k(x)|$$

$$\leq const\mu^{\frac{1}{2}-r}\|q_rf\|_2\,,\qquad r=\overline{2,2m-1};\quad \mu\geq 2, \tag{2.6}$$

$$I_{2m}(\mu)=\sum_{\rho_k\geq\mu}\left|\lambda_k^{-1}\sum_{i=0}^{m_k}\lambda_k^{-i}(q_{2m}f,v_{k+i})\right||u_k(x)|$$

$$\leq const\mu^{1-2m}\|q_{2m}f\|_1\,,\quad \mu\geq 2. \tag{2.7}$$

*Proof.* Applying anti-apriori estimation (1.4) and estimation (1.5), we get

$$I_r(\mu)=\sum_{\rho_k\geq\mu}\left|\lambda_k^{-1}\sum_{i=0}^{m_k}\lambda_k^{-i}\left(q_rf,v_{k+i}^{(2m-r)}\right)\right||u_k(x)|$$

$$\leq\sum_{\rho_k\geq\mu}|\mu_k|^{-r}\|u_k\|_\infty\sum_{i=0}^{m_k}|\mu_k|^{-2mi}\left|\left(q_rf,v_{k+i}^{(2m-r)}\|v_{k+i}\|_2^{-1}\mu_k^{r-2m}\right)\right|\|v_{k+i}\|_2$$

$$\leq const\sum_{\rho_k\geq\mu}|\mu_k|^{-r}\|v_k\|_2\|u_k\|_\infty\sum_{i=0}^{m_k}\left|\left(q_rf,v_{k+i}^{(2m-r)}\|v_{k+i}\|_2^{-1}\mu_k^{r-2m}\right)\right|$$

$$\leq const \sum_{\rho_k \geq \mu} \mu_k^{-r} \left\| u_k \right\|_\infty \left\| u_k \right\|_2^{-1} \sum_{i=0}^{m_k} \left| \left( q_r f, v_{k+i}^{(2m-r)} \left\| v_{k+i} \right\|_2^{-1} \mu_k^{r-2m} \right) \right| .$$

At first we apply the Cauchy-Bunyakovski inequality for the sum, and then take into account $\sup\limits_k m_k < \infty$ (this follows from condition (1.2)) and having used lemma 2.2 and 2.3, we get

$$I_r(\mu) \leq const \left( \sum_{\mu_k \geq \mu} \mu_k^{-2r} \left\| u_k \right\|_\infty^2 \left\| u_k \right\|_2^{-2} \right)^{1/2}$$

$$\times \left( \sum_{\rho_k \geq \mu} \left( \sum_{i=0}^{m_k} \left| \left( q_r f, v_{k+i}^{(2m-r)} \left\| v_{k+i} \right\|_2^{-1} \mu_k^{r-2m} \right) \right| \right)^2 \right)^{1/2}$$

$$\leq const\ \mu^{\frac{1}{2}-r} \left( \sup_k m_k \right) \left\| q_r f \right\|_2 \leq const\ \mu^{\frac{1}{2}-r} \left\| q_r f \right\|_2, \quad r = \overline{2, 2m-1}.$$

Estimation (2.6) is established.

For proving estimation (2.7) at first we use the Holder inequality, take into account $\sup\limits_k m_k < \infty$ and apply anti-apriori estimation (1.4). As a result we have

$$I_{2m}(\mu) = \sum_{\rho_k \geq \mu} \left| \lambda_k^{-1} \sum_{i=0}^{m_k} \lambda_k^{-i} \left( q_{2m} f, v_{k+i} \right) \right| \left| u_k(x) \right|$$

$$\leq \sum_{\rho_k \geq \mu} \left| \mu_k \right|^{-2m} \left\| u_k \right\|_\infty \left( \sum_{i=0}^{m_k} \left| \mu_k \right|_r^{-2mi} \left\| v_{k+i} \right\|_\infty \right) \left\| q_{2m} f \right\|_1$$

$$\leq const \left( \sum_{\rho_k \geq \mu} \left| \mu_k \right|^{-2m} \left\| u_k \right\|_\infty \left\| v_k \right\|_\infty \right) \left\| q_{2m} f \right\|_1 .$$

Hence by the Cauchy-Bunyakovskii inequality, condition (1.5) and lemma 2.2 it follows that

$$I_{2m}(\mu) \leq \left( \sum_{\rho_k \geq \mu} \left| \mu_k \right|^{-2r} \left\| u_k \right\|_{\infty,m}^2 \left\| u_k \right\|_2^{-2} \right)^{1/2}$$

$$\times \left( \sum_{\rho_k \geq \mu} \left| \mu_k \right|^{-2m} \left\| v_k \right\|_{\infty,m}^2 \left\| v_k \right\|_2^{-2} \right)^{1/2} \left\| q_{2m} f \right\|_1 \leq const\ \mu^{1-2m} \left\| q_{2m} f \right\|_1 .$$

Lemma 2.4 is proved. □

## 3. Proof of the results

We prove uniform convergence of series (2.1) on $\overline{G}=[0,1]$. For that we represent it in the form

$$\sum_{k=1}^{\infty}|f_k|\,|u_k(x)|=\sum_{0\le\rho_k<2}|f_k|\,|u_k(x)|+\sum_{\rho_k\ge 2}|f_k|\,|u_k(x)|=J_1+J_2.$$

By condition (1.2) for the sum $J_1$ it is fulfilled the estimation $J_1\le const\,\|f\|_1$, and by representation (2.2)

$$J_2\le\sum_{\rho_k\ge 2}|\mu_k|^{-2m}\left(\sum_{i=0}^{m_k}\left|\mu_k^{-2mi}\left(f',v_{k+i}^{(2m-1)}\right)\right|\right)|u_k(x)|$$

$$+\sum_{r=2}^{2m}I_r(2)=I_1(2)+\sum_{r=2}^{m}I_r(2)\,.$$

Uniform convergence of the series $I_r(2)$, $r=\overline{2,2m}$, follows from lemma 2.4 for $\mu=2$. Prove uniform convergence of the series $I_1(2)$. Transform the series $I_1(2)$ in the following form

$$I_1(2)=\sum_{\rho_k\ge 2}|\mu_k|^{-1}\left(\sum_{i=0}^{m_k}\left|\left(f',v_{k+i}^{(2m-1)}\,\|v_{k+i}\|_2^{-1}\,\mu_k^{1-2m}\right)\right|\,\|v_{k+i}\|_2\,|\mu_k|^{-2mi}\right)|u_k(x)|\,.$$

Having applied estimation (1.4) and condition (1.5), we have

$$I_1(2)\le const\sum_{\rho_k\ge 2}|\mu_k|^{-1}\,\|u_k\|_\infty\left(\sum_{i=0}^{m_k}\left|\left(f',v_{k+i}^{(2m-1)}\,\|v_{k+i}\|_2^{-1}\,\mu_k^{1-2m}\right)\right|\right)\|v_k\|_2$$

$$\le const\sum_{\rho_k\ge 2}\|u_k\|_\infty\,\|u_k\|_2^{-1}\,|\mu_k|^{-1}\left(\sum_{i=0}^{m_k}\left|\left(f',v_{k+i}^{(2m-1)}\,\|v_{k+i}\|_2^{-1}\,\mu_k^{1-2m}\right)\right|\right).$$

Hence by the Cauchy-Bunyakovskii inequality, lemma 2.2 and lemma 2.3 we get

$$I_1(2)\le const\left(\sum_{\rho_k\ge 2}\|u_k\|_\infty^2\,\|u_k\|_2^{-2}\,|\mu_k|^{-2}\right)^{1/2}$$

$$\times\left(\sum_{\rho_k\ge 2}\left(\sum_{i=0}^{m_k}\left|\left(f',v_{k+i}^{(2m-1)}\,\|v_{k+i}\|_2^{-1}\,\mu_k^{1-2m}\right)\right|\right)^2\right)^{1/2}$$

$$\le const2^{-1/2}\left(\sum_{\rho_k\ge 2}m_k\sum_{i=0}^{m_k}\left|\left(f',v_{k+i}^{(2m-1)}\,\|v_{k+i}\|_2^{-1}\,\mu_k^{1-2m}\right)\right|^2\right)^{1/2}$$

$$\le const\left(\sup_k m_k\right)^{1/2}2^{-\frac{1}{2}}\left\|f'\right\|_2<\infty.$$

Uniform convergence of the series $I_1(2)$ is established. Consequently, series (2.1) uniformly converges on $\overline{G}=[0,1]$. Hence, it follows uniform convergence of biorthogonal series itself. By the completeness of the system $\{v_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$ and absolute continuity of $f(x)$ on $\overline{G}$, we get that biorthogonal series of the function $f(x)$ converges uniformly just to $f(x)$, i.e. in the metric $C[0,1]$ it holds

$$f(x) = \sum_{k=1}^{\infty} f_k u_k(x), \qquad x \in \overline{G} \tag{3.1}$$

Now establish estimation (1.8). By equality (3.1)

$$|\sigma_\nu(x,f) - f(x)| = \left| \sum_{\rho_k > \nu} f_k u_k(x) \right| \le \sum_{\rho_k > \nu} |f_k| \, |u_k(x)| .$$

Taking into account the expression of the coefficient $f_k$ from lemma 2.1 we have

$$|\sigma_\nu(x,f) - f(x)| \le I_1(\nu) + \sum_{r=2}^{2m} I_r(\nu) . \tag{3.2}$$

Obviously, for $I_1(\nu)$ the following estimation (see the estimation of expression $I_1(2)$) is fulfilled:

$$I_1(\nu) \le const \nu^{-\frac{1}{2}} \left\| f' \right\|_2 . \tag{3.3}$$

For the sum $I_r(\nu)$, $r = \overline{2, 2m}$, by lemma 2.4 the following estimations are fulfilled

$$I_1(\nu) \le const \ \nu^{\frac{1}{2}-r} \left\| q_r f \right\|_2 , \quad r = \overline{2, 2m-1}. \tag{3.4}$$

$$I_{2m}(\nu) \le const \ \nu^{1-2m} \left\| q_{2m} f \right\|_1 . \tag{3.5}$$

Allowing for estimations (3.3)-(3.5) from (3.2), it follows that

$$\left\| \sigma_\nu(\cdot, f) - f \right\|_{C[0,1]}$$

$$\le const \left\{ \nu^{-\frac{1}{2}} \left\| f' \right\|_2 + \sum_{r=2}^{2m-1} \nu^{\frac{1}{2}-r} \left\| q_r f \right\|_2 + \nu^{1-2m} \left\| q_{2m} f \right\|_1 \right\} .$$

Theorem 1.1 is completely proved.

Estimation (1.9) directly follows from estimation (1.8) if we take into account $\|q_l f\|_2 \le \|q_l\|_2 \|f\|_\infty$, $l = \overline{2, 2m-1}$; $\|q_{2m} f\|_1 \le \|q_{2m}\|_1 \|f\|_\infty$, and for $f(x) \in W_2^1(G)$, $f(0) = f(1) = 0$ it holds $\|f\|_\infty \le \|f'\|_1 \le \|f'\|_2$.

For justification of estimation (1.10) we should pay attention to the estimation of the series $I_1(\nu)$. For it

$$I_1(\nu) \le const \ \nu^{-\frac{1}{2}} \left( \sum_{\rho_k \ge 0} m_k \sum_{i=0}^{m_k} \left| \left( f', \mu_k^{1-2m} \left\| v_{k+i} \right\|_2^{-1} v_{k+i}^{(2m-1)} \right) \right|^2 \right)^{1/2}$$

$$= const \ \nu^{-\frac{1}{2}} o(1) ,$$

is valid as $\nu \to +\infty$, for $\sup\limits_k m_k < \infty$ and the system $\left\{ \mu_k^{1-2m} \left\| v_{k+i} \right\|_2^{-1} v_{k+i}^{(2m-1)} \right\}_{\rho_k > 0}$ is Bessel in $L_2(G)$.

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