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ON THE STRUCTURE OF GLOBAL CONTINUA OF SOLUTIONS BIFURCATING FROM INFINITY OF SOME NONLINEAR FOURTH ORDER EIGENVALUE PROBLEMS

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Abstract. In this paper we consider bifurcation from infinity in some class of nonlinear eigenvalue problems for fourth order differential operators. We show the existence of two families of unbounded continua of nontrivial solutions, corresponding to the nodal properties and bifurcating from the intervals at infinity.

1. Introduction

We consider the following nonlinear eigenvalue problem

$$\ell y \equiv (py'')'' - (qy')' + ry = \lambda \tau y + f(x, y, y', y'', y''', \lambda),\ x \in (0, l), \tag{1.1}$$

$$\begin{array}{l} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \\ y(l)\cos\delta - Ty(l)\sin\delta = 0, \end{array} \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $Ty \equiv (py'')' - qy'$, the function p is twice continuously differentiable and positive on $[0, l]$, q is continuously differentiable and nonnegative on $[0, l]$, r is continuous on $[0, l]$ and τ is continuous and positive on $[0, l]$, α, β, γ, $\delta \in [0, \frac{\pi}{2}]$. The function f is continuous on $[0, l] \times \mathbb{R}^5$ satisfying the condition: there exist $M > 0$ and $c_0 > 1$ such that

$$\left| \frac{f(x, u, s, v, w, \lambda)}{u} \right| \le M,\ x \in [0, l],\ |u| + |s| + |v| + |w| \ge c_0,\ \lambda \in \mathbb{R}. \tag{1.3}$$

Since condition (1.3) holds, we can consider bifurcation from "$y = \infty$", i.e., the existence of solutions of (1.1)-(1.2) having arbitrarily large y. If nonlinear term f satisfies $o(|y| + |y'| + |y''| + |y'''|)$ condition, then the problem is said to be asymptotically linear and the existence of solutions (λ, y) of (1.1)-(1.2) with large y bifurcating from infinity may be discussed as in the papers [11, 13, 14]. The approach used in these papers is to transform the bifurcation from infinity problem to a problem involving bifurcation from zero at eigenvalues of the linearization of (1.1)-(1.2), and then apply the standard global bifurcation theory of Rabinowitz [10]. As equation (1.1) contains the nonlinear term f satisfying (1.3) problem

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(1.1)-(1.2) need not be asymptotically linear and the transformed problem may not have a linearization at $y = 0$. Thus the standard global bifurcation results are not immediately applicable and the proofs in [11] are not valid in this case. However, by extending the approximation technique from [5] and combining it with the global results in [1, 2, 3] we prove the existence, in this case, of global continua of solutions bifurcating from infinity which are similar to those obtained in [8, 9, 12].

2. Preliminary

Although problem (1.1)-(1.2) is not asymptotically linearizable (when $f \not\equiv 0$), it is nevertheless related to a fourth-order linear problem

$$\begin{cases} \ell(y)(x) = \lambda \tau(x) y(x), \ x \in (0, l), \\ y \in B.C., \end{cases} \tag{2.1}$$

where by $B.C.$ we denote the set of boundary conditions (1.2).

Let E be the Banach space of all continuously three times differentiable functions on $[0, l]$ which satisfy the conditions $B.C.$ and is equipped with its usual norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max\limits_{x\in[0,l]} |u(x)|$.

Let

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{u \in E : u^{(i)}(x) \neq 0, Tu(x) \neq 0, \ x \in [0, l], \ i = 0, \ 1, \ 2\}$$

and

$S_2 = \{u \in E :$ there exists $i_0 \in \{0, 1, 2\}$ and $x_0 \in (0, 1)$ such that $u^{(i_0)}(x_0) = 0$, or $Tu(x_0) = 0$ and if $u(x_0)u''(x_0) = 0$, then $u'(x)Tu(x) < 0$ in a neighborhood of x_0, and if $u'(x_0)Tu(x_0) = 0$, then $u(x)u''(x) < 0$ in a neighborhood of $x_0\}$.

Note that if $u \in S$ then the Jacobian $J = \rho^3 \cos\psi \sin\psi$ (see [1, 2, 4]) of the Prüfer-type transformation

$$\begin{cases} y(x) = \rho(x) \sin\psi(x) \cos\theta(x), \\ y'(x) = \rho(x) \cos\psi(x) \sin\varphi(x), \\ (py'')(x) = \rho(x) \cos\psi(x) \cos\varphi(x), \\ Ty(x) = \rho(x) \sin\psi(x) \sin\theta(x), \end{cases} \tag{2.2}$$

does not vanish in $(0, l)$.

For each $u \in S$ we define $\rho(u, x)$, $\theta(u, x)$, $\varphi(y, x)$ and $w(u, x)$ to be the continuous functions on $[0, l]$ satisfying

$$\rho(u, x) = u^2(x) + u'^2(x) + (p(x)u''(x))^2 + (Tu(x))^2,$$

$$\theta(u, x) = \operatorname{arctg} \frac{Tu(x)}{u(x)}, \ \theta(u, 0) = \beta - \pi/2\,,$$

$$\varphi(u, x) = \operatorname{arctg} \frac{u'(x)}{(pu'')(x)}, \ \varphi(u, 0) = \alpha\,,$$

$$w(u, x) = \operatorname{ctg}\psi(u, x) = \frac{u'(x) \cos\theta(u, x)}{u(x) \sin\varphi(u, x)}, \ w(u, 0) = \frac{u'(0) \sin\beta}{u(0) \sin\alpha}\,,$$

and $\psi(u,x) \in (0,\pi/2)$, $x \in (0,l)$, in the cases $u(0)u'(0) > 0$; $u(0) = 0$; $u'(0) = 0$ and $u(0)u''(0) > 0$, $\psi(u,x) \in (\pi/2,\pi)$, $x \in (0,l)$, in the cases $u(0)u'(0) < 0$; $u'(0) = 0$ and $u(0)u''(0) < 0$; $u'(0) = u''(0) = 0$, $\beta = \pi/2$ in the case $\psi(u,0) = 0$ and $\alpha = 0$ in the case $\psi(u,0) = \pi/2$.

It is apparent that ρ, θ, $\varphi, w : S \times [0,1] \to \mathbb{R}$ are continuous.

Remark 2.1. By (2.2) for each $u \in S$ the function $w(u,x)$ can be determined from one of the following relations

$$\text{a) } w(y,x) = \operatorname{ctg}\psi(y,x) = \frac{(py'')(x)\cos\theta(y,x)}{y(x)\cos\varphi(y,x)},\ w(y,0) = \frac{(py'')(0)\sin\beta}{y(0)\cos\alpha},$$

$$\text{b) } w(y,x) = \operatorname{ctg}\psi(y,x) = \frac{(py'')(x)\sin\theta(y,x)}{Ty(x)\cos\varphi(y,x)},\ w(y,0) = -\frac{(py'')(0)\cos\beta}{Ty(0)\cos\alpha},$$

$$\text{c) } w(y,x) = \operatorname{ctg}\psi(y,x) = \frac{y'(x)\sin\theta(y,x)}{Ty(x)\sin\varphi(y,x)},\ w(y,0) = -\frac{y'(0)\cos\beta}{Ty(0)\sin\alpha}.$$

For each $k \in \mathbb{N}$ and $\nu \in \{+,-\}$ let by S_k^ν denote the subset of $y \in S$ such that

1) $\theta(y,l) = (2k-1)\pi/2 - \delta$, where $\delta = \pi/2$ in the case $\psi(y,l) = 0$;

2) $\varphi(y,l) = (k+1)\pi - \gamma$ or $\varphi(u,l) = k\pi - \gamma$ in the case $\psi(y,0) \in [0,\pi/2)$; $\varphi(y,l) = \pi - \gamma$ for $k = 1$, $\varphi(y,l) = k\pi - \gamma$ or $\varphi(y,l) = (k-1)\pi - \gamma$ for $k \geq 2$ in the case $\psi(y,0) \in [\pi/2,\pi)$, where $\gamma = 0$ in the case $\psi(y,l) = \pi/2$;

3) for fixed function y and increasing x the function $\theta(y,x)$ ($\varphi(y,x)$) takes values of the form $m\pi/2$, $m \in \mathbb{Z}$ (of the form $s\pi$, $s \in \mathbb{Z}$, respectively) while strictly increasing; for decreasing x the function $\theta(y,x)$ ($\varphi(y,x)$) takes values of the form $m\pi/2$, $m \in \mathbb{Z}$ (of the form $s\pi$, $s \in \mathbb{Z}$, respectively) while strictly decreasing;

4) the function $\nu y(x)$ is positive in a punctured neighborhood of $x = 0$.

By [1, Theorem 1.2] (see also [4]) the eigenvalues of problem (2.1) (except the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$) which is a completely regular Sturmian system (see [1, 6]) are real and simple and form an infinitely increasing sequence

$$\lambda_1 < \lambda_2 < \ ...\ < \lambda_k < \ldots;$$

moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k lies in $S_k = S_k^- \cup S_k^+$ (consequently, $y_k(x)$ has has $k-1$ simple nodal zeros in the interval $(0,l)$). Hence, the sets S_k^ν, $k \in \mathbb{N}$, $\nu \in \{+,-\}$ are not empty. Moreover, it follows immediately from the definition of these sets that they are disjoint and open in E.

3. Global bifurcation of problem (1.1)-(1.2) from infinity

We denote by $\mathfrak{L}$ the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1.1)-(1.2) and by $\mathfrak{L}_k^\nu$, $k \in \mathbb{N}$, the closure in $\mathbb{R} \times E$ of the set of all solutions (λ, y) of (1.1)-(1.2) with $y \in \mathbb{R} \times S_k^\nu$.

We say that (λ, ∞) is a bifurcation point of problem (1.1)-(1.2) with respect to the set $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, if each neighborhood of this point has a nonempty intersection with $\mathfrak{L}_k^\nu$.

Lemma 3.1. *The set of bifurcation points from infinity of problem* (1.1)-(1.2) (*with respect to the set* S_k^ν, $k \in \mathbb{N}$) *is nonempty.*

Proof. For the proof we consider the following modified nonlinear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda\tau(x)y + \frac{f(x,|y|^\varepsilon y, y', y'', y''', \lambda)}{(|y|+|y'|+|y''|+|y'''|)^{2\varepsilon}}, & x \in (0,l), \\ y \in B.C., \end{cases} \tag{3.1}$$

where $\varepsilon \in (0,1)$. By (1.3) we obtain

$$\frac{f(x, |u|^\varepsilon u,\, s,\, v,\, w, \lambda)}{(|u|+|v|+|s|+|w|)^{2\varepsilon}} = o\left(|u|+|s|+|v|+|w|\right)$$

at $(u,s,v,w)=\infty$ uniformly in $x\in[0,l]$ and in $\lambda\in\Lambda$, for every bounded interval $\Lambda\subset\mathbb{R}$. Then it follows by [9, Theorem 2.4] that for each $k\in\mathbb{N}$ and each ν there exists an unbounded continuum $C_{k,\varepsilon}^\nu$ of solutions of (3.1) which contains (λ_k,∞) and satisfies conclusions of well known theorem of bifurcation from infinity of Rabinowitz [11]. Moreover, there exists a neighborhood $\mathcal{N}$ of (λ_k,∞) such that

$$(C_{k,\varepsilon}^\nu \cap \mathcal{N}) \subset (\mathbb{R}\times S_k^\nu) \cup \{(\lambda_k,\infty)\}.$$

Now we show that for any sufficiently small $\delta>0$ there exists $c_\delta > c_0$ such that the problem (3.1) with $\varepsilon\in(0,1)$ has no nontrivial solution (λ, y) which satisfies the following conditions:

$$\operatorname{dist}\{\lambda, I_k\} > \delta,\ y\in S_k^\nu,\ ||y||_3 > c_\delta.$$

Indeed, otherwise there exists $\delta_0>0$ and a sequence $\{(\lambda_n, y_n, \varepsilon_n)\}_{n=1}^\infty$ of solutions of (3.1), with

$$\operatorname{dist}\{\lambda_n, I_k\} > \delta_0,\ y_n\in S_k^\nu,\ ||y_n||_3 > n.$$

Clearly, (λ_n, y_n) solves the linear problem

$$\begin{cases} \ell(y) + h_n(x)\, y = \lambda\tau(x)y, & x\in(0,l), \\ y\in B.C., \end{cases} \tag{3.2}$$

where

$$h_n(x) = \begin{cases} -\frac{f(x,(\tau_n(x))^{\varepsilon_n} y_n(x), y_n'(x), y_n''(x), y_n'''(x), \lambda_n)}{(\tau_n(x))^{2\varepsilon_n} y_n(x)} & \text{if } y_n(x)\neq 0, \\ 0, & \text{if } y_n(x) = 0, \end{cases} \tag{3.3}$$

and $\tau_n(x) = |y_n(x)| + |y_n'(x)| + |y_n''(x)| + |y_n'''(x)|$. Taking (1.3) into account from (3.3) for sufficiently large $n\in\mathbb{N}$ we obtain

$$|h_n(x)| \le \frac{M}{(\tau_n(x))^{\varepsilon_n}} < M,\ x\in[0,l]. \tag{3.4}$$

Since $h_n(x)$ has a finite number of zeros on $(0,l)$ and is bounded on the closed interval [0, l], Remark 4.1 from [1] shows that the result of [1, Theorem 1.2] holds for problem (3.2). Then, taking (3.4) into account it follows from [1, formula (4.2)] that $\lambda_n\in I_k$, which yields the equality $\operatorname{dist}\{\lambda_n, I_k\} = 0$, contradicting $\operatorname{dist}\{\lambda_n, I_k\} > \delta_0$.

Now let small $\delta_1 > 0$ is fixed. Then there exists $c_1 > c_0$ such that for any $c \in (c_1, \infty)$ problem (3.1) with $\varepsilon \in (0,1)$ has a solution $(\lambda_{c,\varepsilon}, y_{c,\varepsilon})$ satisfying conditions:

$$\text{dist}\{\lambda_{c,\varepsilon} : I_k\} \leq \delta_1,\ y_{c,\varepsilon} \in S_k^\nu,\ ||y_{c,\varepsilon}||_3 = c.$$

Problem (3.1) shows that the set of points $(\lambda_{c,\varepsilon}, y_{c,\varepsilon})$ is bounded in $\mathbb{R} \times C^4[0,l]$ independently of ε. Hence there exists a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ such that $\varepsilon_n \to 0$ and $(\lambda_{c,\varepsilon_n}, y_{c,\varepsilon_n})$ converges in $\mathbb{R} \times E$ to a nontrivial solution (λ_c, y_c) of problem (1.1)-(1.2). It is obvious that $\lambda_c \in I_k(\delta_0) = [\lambda_k - \frac{M}{\tau_0} - \delta_0, \lambda_k + \frac{M}{\tau_0} + \delta_1]$, $y \in \overline{S_k^\nu} = S_k^\nu \cup \partial S_k^\nu$ and $||y_c||_3 = c$. Since $||y_c||_3 = c$, [1, Lemma 1.1] shows that $y_c \in S_k^\nu$. (In fact c_1 is chosen so that $(I_k \times (E \setminus \overline{B}_{c_1})) \subset \mathcal{N}$, where B_{c_1} is the open ball in E of radius c_1 centered at 0 and $\overline{B_{c_1}}$ is the closure of B_{c_1} in E.)

Now let $\{c_n\}_{n=1}^\infty$ be a sequence converging to $+\infty$. Then for any $n \in \mathbb{N}$ there exists a solution (λ_n, y_n) of problem (1.1)-(1.2) such that $\lambda_n \in I_k(\delta_0)$, $y_n \in S_k^\nu$ and $||y_n||_3 = c_n$. From the sequence $\{\lambda_n\}_{n=1}^\infty$ we can select a subsequence $\{\lambda_{n_m}\}_{m=1}^\infty$ that converges to some $\lambda \in I_k(\delta_0)$. Therefore, there exists a sequence $\{(\lambda_{n_m}, y_{n_m})\}_{m=1}^\infty$ of solutions of problem (1.1)-(1.2) which converges to some (λ, ∞) in $\mathbb{R} \times E$, i.e. (λ, ∞) is a bifurcation point from infinity of problem (1.1)-(1.2) with respect to the set $\mathbb{R} \times S_k^\nu$. The lemma is proved.

Corollary 3.1. *If* (λ, ∞) *is a bifurcation point of problem* (1.1)-(1.2) *with respect to the set* $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, *then* $\lambda \in I_k$.

Proof. Assume the contrary, i.e. let $\lambda \notin I_k$. Let $\delta = \text{dist}\{\lambda, I_k\} > 0$. Since (λ, ∞) is a bifurcation point of problem (1.1)-(1.2), there exists a sequence $\{(\lambda_n, y_n)\}_{n=1}^\infty \subset \mathbb{R} \times S_k^\nu$ of solutions of (1.1)-(1.2) such that $(\lambda_n, y_n) \to (\lambda, \infty)$. Then there exists $n_\delta \in \mathbb{N}$ such that $|\lambda_n - \lambda| < \frac{\delta}{2}$ for $n > n_\delta$. Hence $\text{dist}\{\lambda_n, I_k\} > \frac{\delta}{2}$ for $n > n_\delta$.

It is obvious that (λ_n, y_n) solves the linear problem

$$\begin{cases} \ell(y) + \varphi_n(x)\, y = \lambda \tau(x) y, & x \in (0,l), \\ y \in B.C., \end{cases} \tag{3.5}$$

where

$$\varphi_n(x) = \begin{cases} -\frac{f(x, y_n(x), y_n'(x), y_n''(x), y_n'''(x), \lambda_n)}{y_n(x)} & \text{if } y_n(x) \neq 0, \\ 0, & \text{if } y_n(x) = 0. \end{cases} \tag{3.6}$$

Using (1.3) from (3.6) for any sufficiently large $n > n_\delta$ we obtain $|\varphi_n(x)| \leq M$. Hence, by [1, Remark 4.1] it follows from [1, formula (4.2)] that $\lambda_n \in I_k$, which contradicts the inequality $\text{dist}\{\lambda_n, I_k\} > \frac{\delta}{2}$. The proof of corollary is complete.

For $k \in \mathbb{N}$ and each ν let $\hat{D}_k^\nu$ denote the union of the connected components $\hat{D}_{k,\lambda}^\nu$ of the solution set of (1.1)-(1.2) emanating from bifurcation points $(\lambda, \infty) \in I_k \times \{\infty\}$ with respect to the set $\mathbb{R} \times S_k^\nu$. Clearly, $\hat{D}_k^\nu \neq \emptyset$. The set $\hat{D}_k^\nu$ may not be connected in $\mathbb{R} \times E$ although, by adding the "points at infinity" (λ, ∞), $\lambda \in I_k$, to $\mathbb{R} \times E$ and defining an appropriate topology on the resulting set, the set $D_k^\nu = \hat{D}_k^\nu \cup (I_k \times \{\infty\})$ is connected.

For any set $B \in \mathbb{R} \times E$ we denote by $P_{\mathbb{R}}(B)$ the natural projection of B onto the set $\mathbb{R} \times \{0\}$.

Theorem 3.1. *For each $k \in \mathbb{N}$ and each ν for the set D_k^ν at least one of the following holds:*

(i) D_k^ν meets $I_{k'} \times \{\infty\}$ within the set $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$;

(ii) D_k^ν meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$;

(iii) $P_{\mathbb{R}}(D_k^\nu)$ is unbounded in $\mathbb{R}$.

In addition, if the union $D_k = D_k^+ \cup D_k^-$ does not satisfy (ii) or (iii) then it must satisfy (i) with $k' \neq k$.

Proof. For any nontrivial $(\lambda, v) \in \mathbb{R} \times E$ we define the function $\tilde{f}(\lambda, v) \in C[0, l]$ as follows:

$$\tilde{f}(\lambda, v)(x) = ||v||_3^2 f\left(x, \frac{v(x)}{||v||_3^2}, \frac{v'(x)}{||v||_3^2}, \frac{v''(x)}{||v||_3^2}, \frac{v'''(x)}{||v||_3^2}, \lambda\right),\ x \in [0, l].$$

Moreover, let $\tilde{f}(\lambda, 0) = 0$. By condition (1.3), the function $\tilde{f} : \mathbb{R} \times E \to C[0, l]$ is continuous and satisfies the condition

$$||\tilde{f}(\lambda, v)||_\infty \leq M||v||_3 \text{ for } ||v|| \leq c_0^{-1}. \tag{3.7}$$

Dividing (1.1)-(1.2) by $||y||_3^2$ and setting $v = \frac{y}{||y||_3^2}$ yields the problem

$$\begin{cases} \ell(v) = \lambda\tau(x)v + \tilde{f}(\lambda, v),\ x \in (0, l), \\ v \in B.C.\ , \end{cases} \tag{3.8}$$

since $||v||_3 = \frac{1}{||y||_3}$ and $y = \frac{v}{||v||_3^2}$. Note that the inversion $(\lambda, y) \to T(\lambda, y) = (\lambda, v)$ turns a "bifurcation at infinity" problem into a "bifurcation at zero" problem (see for example [3], [11]).

Let $\tilde{\mathfrak{L}}$ be the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of problem (3.8). Obviously, the inversion $(\lambda, y) \to T(\lambda, y)$ maps $\mathfrak{L}$ into $\tilde{\mathfrak{L}}$ and, heuristically, interchanges points at $y = 0$ (respectively, $y = \infty$) with points at $v = \infty$ (respectively, $v = 0$). Let $\tilde{D}_k^\nu$ be the union of all the components of $\tilde{\mathfrak{L}}$ which meet $I_k \times \{0\}$ within the set $\mathbb{R} \times S_k^\nu$. Then D_k^ν is the inverse image $T^{-1}(\tilde{D}_k^\nu)$ of $\tilde{D}_k^\nu$ under the inversion T. We now choose some fixed (arbitrary) $k_0 \in \mathbb{N}$ and ν_0, and we will prove the theorem for k_0 and ν_0. It should be noted that to prove the theorem it suffices to show that either $\tilde{D}_{k_0}^{\nu_0}$ meets some interval $I_k \times \{0\}$ within the set $\mathbb{R} \times S_k^\nu$, with $(k, \nu) \neq (k_0, \nu_0)$, or $\tilde{D}_{k_0}^{\nu_0}$ is unbounded in $\mathbb{R} \times E$ (the alternatives (ii) and (iii) stated in the theorem for $D_{k_0}^{\nu_0}$ correspond, via T, to the various ways in which $\tilde{D}_{k_0}^{\nu_0}$ can be unbounded).

Suppose that the above assertions for the set $\tilde{D}_{k_0}^{\nu_0}$ do not hold. Then $\tilde{D}_{k_0}^{\nu_0}$ is bounded in $\mathbb{R} \times E$ and we can choose a compact interval $\Lambda \subset \mathbb{R}$ such that $(P_{\mathbb{R}}(\tilde{D}_{k_0}^{\nu_0}) \cup I_{k_0}) \subset \Lambda$. Following [12, Theorem 1.3], we can find a neighborhood $\tilde{Q}$ of $\tilde{D}_{k_0}^{\nu_0}$ and sufficiently small $\delta_2 < \delta_1$ and c_2^{-2} $(c_2 > c_0)$ such that

$$\tilde{D}_{k_0}^{\nu_0} \subset \tilde{Q},\ \ ((I_{k_0}(\delta_2) \times B_{c_2^{-1}}) \cap (\mathbb{R} \times S_{k_0}^{\nu_0})) \subset \tilde{Q},\ \ \partial\tilde{Q} \cap \tilde{\mathfrak{L}} = \emptyset.$$

Along with problem (3.8), consider the following approximation problem

$$\begin{cases} \ell(v) = \lambda\tau(x)v + \tilde{f}(\lambda, ||v||_3^{\varepsilon}),\ x \in (0, l), \\ v \in B.C.\ , \end{cases} \tag{3.9}$$

where $\varepsilon \in (0, 1]$. For fixed $\varepsilon \in (0, 1)$ it follows from (3.7) that $||\tilde{f}(\lambda, ||v||_3^{\varepsilon}) = o(||v||_3)$ as $||v||_3 \to 0$ uniformly in $\lambda \in \Lambda$, so the global bifurcation results in

[1, 3] and [10] are applicable to this problem. Then for each fixed $\varepsilon \in (0,1)$ there exists a continuum $\tilde{D}^{\nu_0}_{k_0,\varepsilon}$ of solutions of problem (3.9) which meets $(\lambda_{k_0}, 0)$ within the set $\mathbb{R} \times S^{\nu_0}_{k_0}$ and either $\tilde{D}^{\nu_0}_{k_0,\varepsilon}$ is unbounded in $\mathbb{R} \times E$ or there is some $(k,\nu) \neq (k_0,\nu_0)$ such that $\tilde{D}^{\nu_0}_{k_0,\varepsilon}$ meets $(\lambda_k, 0)$ within the set $\mathbb{R} \times S^{\nu}_{k}$. Hence $\tilde{D}^{\nu_0}_{k_0,\varepsilon}$ intersects both $\tilde{Q}$ and the complement of $\tilde{Q}$, and consequently, $\tilde{D}^{\nu_0}_{k_0,\varepsilon} \cap \partial\tilde{Q} \neq \emptyset$. It follows that for each $\varepsilon \in (0,1)$ there exists a nontrivial solution $(\lambda_\varepsilon, v_\varepsilon) \in \partial\tilde{Q}$ of problem (3.9). Since $\tilde{Q}$ is bounded in $\mathbb{R} \times E$ it follows from (3.9) that the set $\{(\lambda_\varepsilon, v_\varepsilon) \in \mathbb{R} \times E : \varepsilon \in (0,1)\}$ is bounded in $\mathbb{R} \times C^4[0,l]$ independently of ε. Then there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$ such that $\varepsilon_n \to 0$ and $\{(\lambda_{\varepsilon_n}, v_{\varepsilon_n})\}_{n=1}^{\infty}$ converges to a nontrivial solution (λ_0, v_0) of (3.8) in $\mathbb{R} \times E$, which implies that $(\lambda_0, v_0) \in \partial\tilde{Q} \cap \tilde{\mathfrak{L}}$. This contradicts the relation $\partial\tilde{Q} \cap \tilde{\mathfrak{L}} = \emptyset$. The proof is complete.

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