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ON MAGNETIC CURVES IN THE 3-DIMENSIONAL HEISENBERG GROUP

CIHAN ÖZGÜR

Abstract. We consider normal magnetic curves in 3-dimensional Heisenberg group H_3. We prove that γ is a normal magnetic curve in H_3 if and only if it is a geodesic obtained as an integral curve of e_3 or a non-Legendre slant circle or a Legendre helix or a slant helix. We obtain the parametric equations of normal slant magnetic curves in 3-dimensional Heisenberg group H_3.

1. Introduction

Let (M, g) be a Riemannian manifold and F a closed 2-form. Then F is called a *magnetic field* (see [1], [2] and [8]) if it is associated by the relation

$$g(\Phi X, Y) = F(X, Y), \quad \forall X, Y \in \chi(M) \tag{1.1}$$

to the Lorentz force Φ which is defined as a skew symmetric endomorphism field on M. Let ∇ be the Levi-Civita connection associated to the metric g and $\gamma : I \to M$ a smooth curve. Then γ is called a *magnetic curve* or a *trajectory* for the magnetic field F if it is solution of the Lorentz equation

$$\nabla_{\gamma'(t)}\gamma'(t) = \Phi(\gamma'(t)). \tag{1.2}$$

The Lorentz equation generalizes the equation of geodesics. A curve which satisfies the Lorentz equation is called *magnetic trajectory*. It is well-known that the magnetic curves have constant speed. When the magnetic curve γ is arc length parametrized, it is called a *normal magnetic curve* [9].

In [4], magnetic curves in Sasakian 3-manifolds were considered. In [15], the classification of Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ was given. In [16], the authors prove that a normal magnetic curve on the Sasakian sphere $\mathbb{S}^{2n+1}$ lies on a totally geodesic sphere S^3. In [9], magnetic curves in a $(2n + 1)$-dimensional Sasakian manifold was studied. In [6], Killing magnetic curves in three-dimensional almost paracontact manifolds were considered. In [14], magnetic curves on flat para-Kähler manifolds were studied. In [18], magnetic curves in $3D$ semi-Riemannian manifolds was considered. In [13], magnetic trajectories in an almost contact metric manifold $\mathbb{R}^{2N+1}$ were studied. Magnetic curves in cosymplectic manifolds were studied in [10]. Periodic magnetic curves in Berger spheres were considered in [12]. Some closed magnetic curves on a 3-torus were investigated in [17].

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Moreover, in [19], Legendre curves in 3-dimensional Heisenberg group were investigated.

Motivated by the above studies, in the present paper, we consider normal magnetic curves in 3-dimensional Heisenberg group H_3. We prove that γ is a normal magnetic curve in H_3 if and only if it is a geodesic obtained as an integral curve of e_3 or a non-Legendre slant circle with curvature $\kappa = |q|\sin\alpha$ and of constant contact angle $\alpha = \arccos(-\frac{\lambda}{2q})$, where $-\frac{\lambda}{2q} \in [-1,1]$ or a Legendre helix with $\kappa = |q|$ and $\tau = \frac{\lambda}{2}$ or a slant helix with $\kappa = |q|\sin\alpha$ and $\tau = \frac{\lambda}{2} + \cos\alpha$. Moreover, we obtain the parametric equations of normal slant magnetic curves in 3-dimensional Heisenberg group H_3.

2. Preliminaries

Let $M^{2n+1} = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and Ω the fundamental 2-form of M^{2n+1} defined by

$$\Omega(X,Y) = g(\varphi X, Y). \tag{2.1}$$

If $\Omega = d\eta$, then M^{2n+1} is called a contact metric manifold [3].

The magnetic field Ω on M^{2n+1} can be defined by

$$F_q(X,Y) = q\Omega(X,Y),$$

where X and Y are vector fields on M^{2n+1} and q is a real constant. F_q is called the contact magnetic field with strength q [13]. If $q = 0$ then the magnetic curves are geodesics of M^{2n+1}. Because of this reason we shall consider $q \neq 0$ (see [4] and [9]).

From (2.1) and (1.1), the Lorentz force Φ associated to the contact magnetic field F_q can be written as

$$\Phi_q = q\varphi.$$

So the Lorentz equation (1.2) can be written as

$$\nabla_{\gamma'(t)}\gamma'(t) = q\varphi(\gamma'(t)), \tag{2.2}$$

where $\gamma : I \subseteq R \to M^{2n+1}$ is a smooth curve parametrized by arc length (see [9] and [13]).

The Heisenberg group H_3 can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{H_3} = dx^2 + dy^2 + \eta \otimes \eta,$$

where (x,y,z) are standard coordinates in $\mathbb{R}^3$ and

$$\eta = dz + \frac{\lambda}{2}(ydx - xdy),$$

where λ is a non-zero real number. If $\lambda = 1$, then the Heisenberg group H_3 is frequently referred as the model space Nil_3 of the Nil geometry in the sense of Thurston [20]. The Heisenberg group is a multiplicative group, and this is essential for the construction of a left-invariant orthonormal basis. The readers would acknowledge to know the expression of the product. Since $\lambda \neq 0$, the 1-form η satisfies $d\eta \wedge \eta = -\lambda dx \wedge dy \wedge dz$. Hence η is a contact form. In [11], J.

Inoguchi obtained the Levi-Civita connection ∇ of the metric g with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}. \tag{2.3}$$

He obtained

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \frac{\lambda}{2} e_3, & \nabla_{e_1} e_3 = -\frac{\lambda}{2} e_2, \\ \nabla_{e_2} e_1 = -\frac{\lambda}{2} e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \frac{\lambda}{2} e_1, \\ \nabla_{e_3} e_1 = -\frac{\lambda}{2} e_2, & \nabla_{e_3} e_2 = \frac{\lambda}{2} e_1, & \nabla_{e_3} e_3 = 0. \end{array} \tag{2.4}$$

We also have the Heisenberg brackets

$$[e_1, e_2] = \lambda e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

Let φ be the $(1,1)$-tensor field defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$ and $\varphi(e_3) = 0$. Then using the linearity of φ and g we have

$$\eta(e_3) = 1, \quad \varphi^2(X) = -X + \eta(X)e_3, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

We also have

$$d\eta(X,Y) = \frac{\lambda}{2} g(X, \varphi Y)$$

for any $X, Y \in \chi(M)$. Then for $\xi = e_3$, (φ, ξ, η, g) defines an almost contact metric structure on H_3. If $\lambda = 2$, then (φ, ξ, η, g) is a contact metric structure and the Heisenberg group H_3 is a Sasakian space form of constant holomorphic sectional curvature -3 (see [11]). For arbitrary $\lambda \neq 0$, we do not work in contact Riemannian geometry. However, the fundamental 2-form is closed and hence it defines a magnetic field.

Let $\gamma : I \to H_3$ be a Frenet curve parametrized by arc length s. The contact angle $\alpha(s)$ is a function defined by $\cos\alpha(s) = g(T(s), \xi)$. The curve γ is said to be *slant* if its contact angle $\alpha(s)$ is a constant [7]. Slant curves of contact angle $\frac{\pi}{2}$ are traditionally called *Legendre curves* [3].

For $(H_3, \varphi, \xi, \eta, g)$, the Lorentz equation (1.2) can be written as

$$\nabla_{\gamma'(t)} \gamma'(t) = q\varphi(\gamma'(t)), \tag{2.5}$$

(see [9]).

3. Magnetic Curves in 3-dimensional Heisenberg Group H_3

Let $\gamma : I \to H_3$ be a curve parametrized by arc length. We say that γ is a Frenet curve if one of the following three cases holds:

i) γ is of osculating order 1. In this case, $\nabla_{\gamma'}\gamma' = 0$, which means that γ is a geodesic.

ii) γ is of osculating order 2. In this case, there exist two orthonormal vector fields $T = \gamma'$, N and a positive function κ (curvature) along γ such that $\nabla_T T = \kappa N$, $\nabla_T N = -\kappa T$.

iii) γ is of osculating order 3. In this case, there exist three orthonormal vector fields $T = \gamma'$, N, B and a positive function κ (curvature) and τ (torsion) along γ such that

$$\nabla_T T = \kappa N,$$
$$\nabla_T N = -\kappa T + \tau B,$$

$$\nabla_T B = -\tau N,$$

where $\kappa = \|\nabla_T T\|$. A *circle* is a Frenet curve of osculating order 2 such that κ is a non-zero positive constant; a *helix* is a Frenet curve of osculating order 3 such that κ and τ are non-zero constants (see [19]).

Theorem 3.1. *Let $(H_3, \varphi, \xi, \eta, g)$ be the Heisenberg group and consider the contact magnetic field F_q for $q \neq 0$ on H_3. Then γ is a normal magnetic curve associated to F_q in H_3 if and only if*

i) γ is a geodesic obtained as an integral curve of e_3 or

ii) γ is a non-Legendre circle with curvature $\kappa = |q| \sin\alpha$ and of constant contact angle $\alpha = \arccos(-\frac{\lambda}{2q})$, where $-\frac{\lambda}{2q} \in [-1, 1]$ or

iii) γ is a Legendre helix with $\kappa = |q|$ and $\tau = \frac{\lambda}{2}$ or

iv) γ is a slant helix with $\kappa = |q| \sin\alpha$ and $\tau = \frac{\lambda}{2} + q\cos\alpha$, where α is a constant such that $\alpha \in (0, \pi)$.

Proof. If the magnetic curve γ is a geodesic, then $\varphi T = 0$, which means that T is collinear to e_3. Then being unitary we must have $T = \mp e_3$. So γ is a geodesic obtained as an integral curve of ξ.

Since γ is parametrized by arc-length, we can write

$$T = \sin\alpha\cos\beta e_1 + \sin\alpha\sin\beta e_2 + \cos\alpha e_3, \tag{3.1}$$

where $\alpha = \alpha(s)$ and $\beta = \beta(s)$. Using (2.4) we have

$$\begin{aligned}\nabla_T T = &\left(\alpha' \cos\alpha\cos\beta - \sin\alpha\sin\beta\left(\beta' - \lambda\cos\alpha\right)\right) e_1 \\ &+ \left(\alpha' \cos\alpha\sin\beta + \sin\alpha\cos\beta\left(\beta' - \lambda\cos\alpha\right)\right) e_2 \\ &-\alpha' \sin\alpha e_3. \end{aligned} \tag{3.2}$$

On the other hand, by the use of (3.1), it follows that

$$\varphi T = -\sin\alpha\sin\beta e_1 + \sin\alpha\cos\beta e_2. \tag{3.3}$$

Since γ is a magnetic curve

$$\nabla_T T = q\varphi(T),$$

which gives us

$$\alpha' \cos\alpha\cos\beta - \sin\alpha\sin\beta\left(\beta' - \lambda\cos\alpha\right) = -q\sin\alpha\sin\beta, \tag{3.4}$$

$$\alpha' \cos\alpha\sin\beta + \sin\alpha\cos\beta\left(\beta' - \lambda\cos\alpha\right) = q\sin\alpha\cos\beta, \tag{3.5}$$

$$\alpha' \sin\alpha = 0. \tag{3.6}$$

From (3.6), we find $\alpha' = 0$ or $\sin\alpha = 0$. If $\sin\alpha$ =0, then $\varphi T = 0$. So by the discussion of the beginning of the proof, it follows that γ is a geodesic obtained as an integral curve of e_3. If $\alpha' = 0$, then α is a constant, this means that γ is a slant curve. So we can assume that $\sin\alpha > 0$, which means that $\alpha \in (0, \pi)$.

Since α is a constant, from (3.4) or (3.5), we obtain $\beta' - \lambda\cos\alpha = q$. Hence

$$\beta(s) = (\lambda\cos\alpha + q)\, s + c, \tag{3.7}$$

where c is an arbitrary real number.

Substituting $\alpha' = 0$ and $\beta' - \lambda\cos\alpha = q$ into (3.2), we find

$$\nabla_T T = -q\sin\alpha\sin\beta e_1 + q\sin\alpha\cos\beta e_2. \tag{3.8}$$

Now let $\{T,N,B\}$ denote the Frenet frame of γ. Since $\nabla_T T=\kappa N$, from (3.8) we obtain

$$\kappa=|q|\sin\alpha=\text{constant}. \tag{3.9}$$

By (3.8) and (3.9), it follows that

$$N=sgn(q)\left(-\sin\beta e_1+\cos\beta e_2\right). \tag{3.10}$$

Then by the use of (3.10), (2.4) and $\beta'-\lambda\cos\alpha=q$, we find

$$\nabla_T N=sgn(q)\bigg(-\cos\beta\left(\frac{\lambda}{2}\cos\alpha+q\right)e_1$$
$$-\sin\beta\left(\frac{\lambda}{2}\cos\alpha+q\right)e_2+\frac{\lambda}{2}\sin\alpha e_3\bigg).$$

Now we define the cross product $\times$ by $e_1\times e_2=e_3$ and we compute $B=T\times N$. Then we obtain

$$B=sgn(q)\left(-\cos\alpha\cos\beta e_1-\cos\alpha\sin\beta e_2+\sin\alpha e_3\right). \tag{3.11}$$

Since $\nabla_T N=-\kappa T+\tau B$, we find

$$\frac{\lambda}{2}sgn(q)=-|q|\cos\alpha+sgn(q)\tau. \tag{3.12}$$

If γ is Legendre then from (3.12), it is a Legendre helix with $\kappa=|q|$ and $\tau=\frac{\lambda}{2}$. If γ is non-Legendre then from (3.12), it is a slant helix with $\kappa=|q|\sin\alpha$ and $\tau=\frac{\lambda}{2}+q\cos\alpha$.

If the osculating order is 2, then from (3.12), $\cos\alpha=-\frac{\lambda}{2q}$. So γ is a circle with $\kappa=|q|\sin\alpha$ and of constant contact angle $\alpha=\arccos(-\frac{\lambda}{2q})$, where $-\frac{\lambda}{2q}\in[-1,1]$.

Conversely, assume that γ is a slant helix with $\kappa=|q|\sin\alpha$ and $\tau=\frac{\lambda}{2}+q\cos\alpha$, where α is the contact angle between γ and e_3. Then $\cos\alpha=g(T,e_3)$. Hence T is of the form (3.1). Taking the covariant derivative of (3.1) with respect to T, since α is a constant, we have

$$\nabla_T T=\left(\beta'-\lambda\cos\alpha\right)\left[-\sin\alpha\sin\beta e_1+\sin\alpha\cos\beta e_2\right]=\kappa N$$

So we find $g(e_3,N)=0$. Hence e_3 can be written as

$$e_3=\cos\alpha T+\mu B, \tag{3.13}$$

where $\mu=\mp\sin\alpha$ is a real constant since $\|e_3\|=1$. By (3.13), by a covariant differentiation, we have

$$\frac{\lambda}{2}\varphi T=(\tau\mu-\kappa\cos\alpha)N, \tag{3.14}$$

which gives us

$$\frac{\lambda^2}{4}g(\varphi T,\varphi T)=\frac{\lambda^2}{4}\sin^2\alpha=(\tau\mu-\kappa\cos\alpha)^2. \tag{3.15}$$

Since $\kappa=|q|\sin\alpha$ and $\tau=\frac{\lambda}{2}+q\cos\alpha$, from the equation (3.15), we find $\mu=sgn(q)\sin\alpha$. Then the equality (3.14) turns into

$$\varphi T=sgn(q)\sin\alpha N.$$

Using Frenet formulas

$$\nabla_T T=\kappa N=|q|\sin\alpha N=q\varphi T.$$

Then the Lorentz equation (2.5) is satisfied. Hence γ is a magnetic curve.

If γ is a Legendre helix with $\kappa = |q|$ and $\tau = \frac{\lambda}{2}$, then taking $\alpha = \frac{\pi}{2}$ in the above case, we have

$$\varphi T = sgn(q)N$$

and

$$\nabla_T T = \kappa N = |q|\, N = q\varphi T,$$

which means that γ is a magnetic curve.

If γ is a non-Legendre circle with curvature $\kappa = |q|\sin\alpha$ and of constant contact angle $\alpha = \arccos(-\frac{\lambda}{2q})$, then taking $\tau = 0$ and $\cos\alpha = -\frac{\lambda}{2q}$ we have again $\nabla_T T = q\varphi T$. This implies that γ is a magnetic curve.

Then we get the result as required. □

4. Explicit Formulas for Magnetic Curves in 3-dimensional Heisenberg Group H_3

In [5], R. Caddeo, C. Oniciuc and P. Piu obtained the parametric equations of all non-geodesic biharmonic curves in Heisenberg group Nil_3. Using the similar method of [5], we can state a result analogous to [Theorem 3.5, [9]]:

Theorem 4.1. *The normal slant magnetic curves on H_3, described by* (2.2) *have the parametric equations*

a)

$$x(s) = \frac{1}{\upsilon}\sin\alpha\sin(\upsilon s + c) + d_1,$$

$$y(s) = -\frac{1}{\upsilon}\sin\alpha\cos(\upsilon s + c) + d_2,$$

$$\begin{aligned} z(s) &= \left(\cos\alpha + \frac{\lambda}{2\upsilon}\sin^2\alpha\right)s - \frac{\lambda}{2\upsilon}d_1\sin\alpha\cos(\upsilon s + c) \\ &\quad -\frac{\lambda}{2\upsilon}d_2\sin\alpha\sin(\upsilon s + c) + d_3, \end{aligned}$$

where $\upsilon = \lambda\cos\alpha + q \neq 0$ and c, d_1, d_2, d_3 are real numbers and α denotes the contact angle which is a constant such that $\alpha \in (0, \pi)$ or

b)

$$x(s) = (\sin\alpha\cos c)\, s + d_4,$$

$$y(s) = (\sin\alpha\sin c)\, s + d_5$$

and

$$z(s) = \left(-\frac{q}{\lambda} + \frac{\lambda}{2}\sin\alpha\,(d_4\sin c - d_5\cos c)\right)s + d_6,$$

where c, d_4, d_5 and d_6 are real numbers and α denotes the contact angle which is a constant such that $\alpha = \arccos(-\frac{q}{\lambda})$, where $-\frac{q}{\lambda} \in [-1, 1]$.

Proof. Let $\gamma(s) = (x(s), y(s), z(s))$. Then using the equations (2.3), the equation (3.1) can be written as

$$T = \sin\alpha\cos\beta(s)\left(\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z}\right) + \sin\alpha\sin\beta(s)\left(\frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z}\right) + \cos\alpha\frac{\partial}{\partial z}$$

$$= (\sin\alpha\cos\beta(s))\,\frac{\partial}{\partial x} + (\sin\alpha\sin\beta(s))\,\frac{\partial}{\partial y}$$

$$+\left(\frac{\lambda}{2}x(s)\sin\alpha\sin\beta(s) - \frac{\lambda}{2}y(s)\sin\alpha\cos\beta(s) + \cos\alpha\right)\frac{\partial}{\partial z}, \tag{4.1}$$

where $\beta(s) = (\lambda\cos\alpha + q)\,s{+}c$. To find the explicit equations, we should integrate the system $\frac{d\gamma}{ds} = T$. Then using (4.1), we have

$$\frac{dx}{ds} = \sin\alpha\cos(\upsilon s + c), \tag{4.2}$$

$$\frac{dy}{ds} = \sin\alpha\sin(\upsilon s + c) \tag{4.3}$$

and

$$\frac{dz}{ds} = \left(\cos\alpha + \frac{\lambda}{2}x(s)\sin\alpha\sin(\upsilon s + c) - \frac{\lambda}{2}y(s)\sin\alpha\cos(\upsilon s + c)\right), \tag{4.4}$$

where $\upsilon = \lambda\cos\alpha + q$.

Assume that $\upsilon \neq 0$. So the integration of the equations (4.2) and (4.3) gives us

$$x(s) = \frac{1}{\upsilon}\sin\alpha\sin(\upsilon s + c) + d_1 \tag{4.5}$$

and

$$y(s) = -\frac{1}{\upsilon}\sin\alpha\cos(\upsilon s + c) + d_2, \tag{4.6}$$

where d_1 and d_2 are real constants. Then substituting the equations (4.5) and (4.6) in (4.4) we get

$$\frac{dz}{ds} = \cos\alpha + \frac{\lambda}{2\upsilon}\sin^2\alpha + \frac{\lambda}{2}d_1\sin\alpha\sin(\upsilon s + c) - \frac{\lambda}{2}d_2\sin\alpha\sin(\upsilon s + c).$$

Hence the solution of the last differential equation gives us

$$z(s) = \left(\cos\alpha + \frac{\lambda}{2\upsilon}\sin^2\alpha\right)s - \frac{\lambda}{2\upsilon}d_1\sin\alpha\cos(\upsilon s + c)$$
$$-\frac{\lambda}{2\upsilon}d_2\sin\alpha\sin(\upsilon s + c) + d_3,$$

where d_3 is a real constant.

Now assume that $\upsilon = \lambda\cos\alpha + q = 0$. Then $\alpha = \arccos(-\frac{q}{\lambda})$, where $-\frac{q}{\lambda} \in [-1, 1]$. So from (4.2), (4.3) and (4.4), we have

$$\frac{dx}{ds} = \sin\alpha\cos c, \tag{4.7}$$

$$\frac{dy}{ds} = \sin\alpha\sin c \tag{4.8}$$

and

$$\frac{dz}{ds} = \left(-\frac{q}{\lambda} + \frac{\lambda}{2}x(s)\sin\alpha\sin c - \frac{\lambda}{2}y(s)\sin\alpha\cos c\right). \tag{4.9}$$

Similar to the solution of the previous case, we find

$$x(s) = (\sin\alpha\cos c)\,s + d_4,$$

$$y(s) = (\sin\alpha\sin c)\,s + d_5$$

and

$$z(s) = \left(-\frac{q}{\lambda} + \frac{\lambda}{2}\sin\alpha\,(d_4 \sin c - d_5 \cos c)\right) s + d_6,$$

where d_4, d_5 and d_6 are real constants. This completes the proof of the theorem. □

Remark 4.1. For $\lambda = 1$, the Heisenberg group H_3 is frequently referred as the model space Nil_3. Hence Theorem 3.1 and Theorem 4.1 can be restated taking $\lambda = 1$ for the Nil space Nil_3.

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References

[1] T. Adachi, Curvature bound and trajectories for magnetic fields on a Hadamard surface, *Tsukuba J. Math.* **20** (1996), 225–230.

[2] M. Barros, A. Romero, J. L. Cabrerizo and M. Fernández, The Gauss-Landau-Hall problem on Riemannian surfaces, *J. Math. Phys.* **46** (2005), no. 11, 112905, 15 pp.

[3] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds,* Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010.

[4] J. L. Cabrerizo, M. Fernández and J. S. Gómez, On the existence of almost contact structure and the contact magnetic field, *Acta Math. Hungar.* **125** (2009), 191–199.

[5] R. Caddeo, C. Oniciuc and P. Piu, Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group, *Rend. Sem. Mat. Univ. Politec. Torino* **62** (2004), 265–277.

[6] G. Calvaruso, M. I. Munteanu and A. Perrone, Killing magnetic curves in three-dimensional almost paracontact manifolds, *J. Math. Anal. Appl.* **426** (2015), 423–439.

[7] J. T. Cho, J. Inoguchi and J-E. Lee, On slant curves in Sasakian 3-manifolds, *Bull. Austral. Math. Soc.* **74** (2006), 359–367.

[8] A. Comtet, On the Landau levels on the hyperbolic plane, *Ann. Physics* **173** (1987), 185–209.

[9] S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor, Magnetic curves in Sasakian manifolds, *J. Nonlinear Math. Phys.* **22** (2015), 428–447.

[10] S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor, Magnetic curves in cosymplectic manifolds, *Rep. Math. Phys.* **78** (2016), 33–48.

[11] J. Inoguchi, Minimal surfaces in the 3-dimensional Heisenberg group, *Differ. Geom. Dyn. Syst.* **10** (2008), 163–169.

[12] J. Inoguchi and M. I. Munteanu, Periodic magnetic curves in Berger spheres, *Tohoku Math. J.* **69** (2017), 113–128.

[13] M. Jleli, M. I. Munteanu and A. I. Nistor, Magnetic trajectories in an almost contact metric manifold $\mathbb{R}^{2N+1}$, *Results Math.* **67** (2015), 125–134.

[14] M. Jleli and M. I. Munteanu, Magnetic curves on flat para-Kähler manifolds, *Turkish J. Math.* **39** (2015), 963–969.

[15] M. I. Munteanu and A. I. Nistor, The classification of Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$, *J. Geom. Phys.* **62** (2012), 170–182.

[16] M. I. Munteanu and A. I. Nistor, A note on magnetic curves on $\mathbb{S}^{2n+1}$, *C. R. Math. Acad. Sci. Paris* **352** (2014), 447–449.

[17] M. I. Munteanu and A. I. Nistor, On some closed magnetic curves on a 3-torus, *Math. Phys. Anal. Geom.* **20** (2017), no. 2, Art. 8, 13 pp.

[18] Z. Özdemir, İ. Gök, Y. Yaylı and N. Ekmekci, Notes on magnetic curves in $3D$ semi-Riemannian manifolds, *Turkish J. Math.* **39** (2015), 412–426.

[19] A. Sarkar and D. Biswas, Legendre curves on three-dimensional Heisenberg groups, *Facta Univ. Ser. Math. Inform.* **28** (2013), 241–248.

[20] W. M. Thurston, *Three-dimensional Geometry and Topology I*, Second edition. Princeton Math. Series. 35 (Princeton University Press, Princeton N. J., 1997).

Cihan Özgür
Department of Mathematics, Balıkesir University, Balıkesir, Turkey.
E-mail address: `cozgur@balikesir.edu.tr`

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