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BEREZIN SYMBOLS, HÖLDER-MCCARTHY AND YOUNG INEQUALITIES AND THEIR APPLICATIONS

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Abstract. We give in terms of Berezin symbols some refinements of Hölder-McCarthy inequality and Young inequality for positive operators on the reproducing kernel Hilbert space. By applying these inequalities we prove some new estimates for the Berezin number of operators. We also discuss the power inequalities $ber\left(A^{n}\right) \leq cber\left(A\right)^{n}$ and $ber\left(A\right)^{n} \leq Cber\left(A^{n}\right)$ for integers $n \geq 1$, which is not well studied yet.

1. Introduction and Background

In this article, we give in terms of Berezin symbols some refinements of Hölder-McCarthy inequality and Young inequality for positive operators on the reproducing kernel Hilbert space (shortly, RKHS). We also apply these inequalities to prove some power inequalities for Berezin number of some operators.

Recall that a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}\left(\Omega\right)$ is the Hilbert space of complex-valued functions on some set Ω such that the evaluation functionals $f \to f\left(\lambda\right), \ \lambda \in \Omega$, are continuous on $\mathcal{H}$. Then by the classical Riesz representation theorem for any $\lambda \in \Omega$ there exists a unique function $k_{\lambda} \in \mathcal{H}$ such that

$$f\left(\lambda\right) = \left\langle f, k_{\lambda}\right\rangle \text{ for all } f \in \mathcal{H}.$$

The collection $\{k_{\lambda} : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. The reproducing kernel k_{λ} has the following representation in terms of any orthonormal basis $\left(e_{n}\right)_{n\geq 0}$ of the space $\mathcal{H}\left(\Omega\right)$ as follows (see Aronzajn [1]):

$$k_{\lambda}\left(z\right) = \sum_{n=0}^{\infty} \overline{e_{n}\left(\lambda\right)} e_{n}\left(z\right).$$

It follows from this representation in particular that if $\mathcal{H}\left(\Omega\right) = H^{2}\left(\mathbb{D}\right)$ (the Hardy-Hilbert space) then $k_{\lambda}\left(z\right) = \frac{1}{1-\overline{\lambda}z} \quad \left(\lambda, z \in \mathbb{D}\right)$, because $\left(z^{n}\right)_{n\geq 0}$ is the orthonormal basis for $H^{2}\left(\mathbb{D}\right)$; and if $\mathcal{H}\left(\Omega\right) = L_{a}^{2}\left(\mathbb{D}\right)$ (the Bergman-Hilbert space) then $k_{\lambda}\left(z\right) = \frac{1}{\left(1-\overline{\lambda}z\right)^{2}} \quad \left(\lambda, z \in \mathbb{D}\right)$, since $\left(\sqrt{n+1}z^{n}\right)_{n\geq 0}$ is the orthonormal basis of $L_{a}^{2}\left(\mathbb{D}\right)$ (see Aronzajn [1], Hedenmalm, Korenblum and Zhu [14]). Let $\widehat{k}_{\lambda}$ denote the normalized reproducing kernel defined by $\widehat{k}_{\lambda} = \frac{k_{\lambda}}{\left\|k_{\lambda}\right\|}$. For any operator

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$A\in\mathcal{B}\left(\mathcal{H}\left(\Omega\right)\right)$ (the Banach algebra of all bounded linear operators on $\mathcal{H}\left(\Omega\right)$) its Berezin symbol $\widetilde{A}$ is defined by (see [2, 3, 5, 22])

$$\widetilde{A}\left(\lambda\right):=\langle A\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle,\ \lambda\in\Omega.$$

The author introduced [15, 16] the following two new numerical values for operators on $\mathcal{H}\left(\Omega\right)$ as follows:

$$\begin{aligned} Ber\left(A\right) &: = Range\left(\widetilde{A}\right) \quad \text{(the Berezin set of } A \text{)} \\ ber\left(A\right) &: = \sup_{\lambda\in\Omega}\left|\widetilde{A}\left(\lambda\right)\right| \quad \text{(the Berezin number of } A \text{).} \end{aligned}$$

Clearly $Ber\left(A\right)\subset W\left(A\right)=\left\{\langle Ax,x\rangle : x\in\mathcal{H}\left(\Omega\right) \text{ and } \left\|x\right\|=1\right\}$ (the numerical range of A) and $ber\left(A\right)\leq w\left(A\right)=\sup\left\{\left|\langle Ax,x\rangle\right| : x\in\mathcal{H}\left(\Omega\right) \text{ and } \left\|x\right\|=1\right\}$ (the numerical radius of A). Some first results concerning to the study of these new numerical characteristics of operators are obtained by the author in [17, 18]; for further results, the reader can be found in [8, 9, 10, 11, 12]. Here, in particular, we prove new results for the Berezin number of some positive operators.

First we prove in terms of Berezin symbols an analog of McCarthy inequality [21], Hölder-McCarthy inequality [6] and Young inequality [7] for positive operators. After these, as biproduct, we prove some inequalities for Berezin number of operators by applying these inequalities.

2. Analogues of some classical inequalities for the Berezin symbols

Recall that a bounded linear operator A acting on a Hilbert space H is said to be positive, denoted by $A\geq 0$, if $\langle Ax,x\rangle\geq 0$ for all $x\in H$. In his paper [21], McCarthy proved the following inequalities for a positive operator $A\in\mathcal{B}\left(\mathcal{H}\right)$ (the Banach algebra of all bounded linear operators on H):

1) $\langle A^{\mu}x,x\rangle \leq \langle Ax,x\rangle^{\mu}\left\|x\right\|^{2(1-\mu)}$ for $\mu\in[0,1]$ and $x\in H$.

2) $\langle A^{\mu}x,x\rangle \geq \langle Ax,x\rangle^{\mu}\left\|x\right\|^{2(1-\mu)}$ for $\mu>1$ and $x\in H$.

1) and 2) can be simplified to the following 3) and 4), respectively:

3) $\langle A^{\mu}x,x\rangle \leq \langle Ax,x\rangle^{\mu}$ for $\mu\in[0,1]$ and $x\in H$.

4) $\langle A^{\mu}x,x\rangle \geq \langle Ax,x\rangle^{\mu}$ for $\mu>1$ and $x\in H$.

Note that the proofs of inequalities 1) and 2) use the integral representation of A and Hölder inequality, and therefore they are called the Hölder-McCarthy inequality. The following inequality is named as the Young inequality (see Furuta [7]):

For $A,B\geq 0$,

$$\mu A+\left(1-\mu\right)B\geq B\#\mu A \text{ for } 0\leq\mu\leq 1,$$

where $B\#\mu A:=B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{\mu}B^{1/2}$ is the $\mu-$ operator geometric mean. Its simplified form is as follows: for $A\geq 0$,

$$\mu A+\left(1-\mu\right)\geq A^{\mu} \text{ for } 0\leq\mu\leq 1.$$

It is well known [6, 7] that the Hölder-McCarthy inequality 3) and Young inequality are equivalent. Some refinement and generalization of Young inequalities are proved by Kittaneh and Manasrah [19].

Theorem 2.1. *Let A be a positive operator on a reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ and $\lambda \in \Omega$ with $A\widehat{k}_\lambda \neq 0$.*

a) If $f(\mu) := \frac{\widetilde{A^\mu(\lambda)}}{\widetilde{A}(\lambda)^\mu}$, then $f(\mu)$ is a convex function on $\mathbb{R}_+ := [0,\infty)$.

b) If $\widetilde{A}(\lambda) \geq \delta > 0$ for all $\lambda \in \Omega$ and some $\delta > 0$, then

$$\frac{ber\left(A^{\frac{\mu+\nu}{2}}\right)}{ber(A)^{\frac{\mu+\nu}{2}}} \leq \frac{1}{2}\left[\frac{ber(A^\mu)}{\delta^\mu} + \frac{ber(A^\nu)}{\delta^\nu}\right].$$

Proof. (a) The proof is similar to the proof in [6]. Indeed, first of all, we note that $\widetilde{A^\mu(\lambda)} = \langle A^\mu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle$ is log-convex, i.e.,

$$\langle A^{\frac{\mu+\nu}{2}} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \leq \langle A^\mu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2} \langle A^\nu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2}.$$

Since $A^{\frac{\mu}{2}}$ and $A^{\frac{\nu}{2}}$ are self-adjoint (because A is bounded positive operator on $\mathcal{H}(\Omega)$), we have by using Schwartz inequality that

$$\begin{aligned}
\widetilde{A^{\frac{\mu+\nu}{2}}(\lambda)} &= \langle A^{\frac{\mu+\nu}{2}} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle = \langle A^{\frac{\nu}{2}} \widehat{k}_\lambda, A^{\frac{\mu}{2}} \widehat{k}_\lambda \rangle \\
&\leq \left\| A^{\frac{\mu}{2}} \widehat{k}_\lambda \right\| \left\| A^{\frac{\nu}{2}} \widehat{k}_\lambda \right\| = \langle A^{\frac{\mu}{2}} \widehat{k}_\lambda, A^{\frac{\mu}{2}} \widehat{k}_\lambda \rangle^{1/2} \langle A^{\frac{\nu}{2}} \widehat{k}_\lambda, A^{\frac{\nu}{2}} \widehat{k}_\lambda \rangle^{1/2} \\
&= \langle A^\mu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2} \langle A^\nu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2}.
\end{aligned}$$

By considering this and arithmetic-geometric mean inequality, we obtain for any $\mu, \nu \in \mathbb{R}_+$ that

$$\begin{aligned}
\frac{1}{2}\left[\frac{\langle A^\mu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle^\mu} + \frac{\langle A^\nu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle^\nu}\right] &\geq \frac{\langle A^\mu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2} \langle A^\nu \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2}}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{\frac{\mu+\nu}{2}}} \\
&\geq \frac{\langle A^{\frac{\mu+\nu}{2}} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{\frac{\mu+\nu}{2}}} \\
&= \frac{\widetilde{A^{\frac{\mu+\nu}{2}}(\lambda)}}{\widetilde{A}(\lambda)^{\frac{\mu+\nu}{2}}},
\end{aligned}$$

and hence,

$$\frac{\widetilde{A^{\frac{\mu+\nu}{2}}(\lambda)}}{\widetilde{A}(\lambda)^{\frac{\mu+\nu}{2}}} \leq \frac{1}{2}\left[\frac{\widetilde{A^\mu(\lambda)}}{\widetilde{A}(\lambda)^\mu} + \frac{\widetilde{A^\nu(\lambda)}}{\widetilde{A}(\lambda)^\nu}\right],$$

that is, $f\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{2}\left[f(\mu) + f(\nu)\right]$, which proves (a).

(b) Since by condition $\widetilde{A}(\lambda) \geq \delta > 0$ for all $\lambda \in \Omega$, that is the Berezin symbol is away from zero and $\left|\widetilde{A}(\lambda)\right| \leq \left\|A\widehat{k}_\lambda\right\|$, we conclude that $A\widehat{k}_\lambda \neq 0$ for all $\lambda \in \Omega$. Then by item (a), we have

$$\frac{\widetilde{A^{\frac{\mu+\nu}{2}}(\lambda)}}{\widetilde{A}(\lambda)^{\frac{\mu+\nu}{2}}} \leq \frac{1}{2}\left[\frac{\widetilde{A^\mu(\lambda)}}{\widetilde{A}(\lambda)^\mu} + \frac{\widetilde{A^\nu(\lambda)}}{\widetilde{A}(\lambda)^\nu}\right]$$

for all $\lambda \in \Omega$. From this, by using the inequality,

$$\frac{\widetilde{A^{\frac{\mu+\nu}{2}}}(\lambda)}{\widetilde{A}(\lambda)^{\frac{\mu+\nu}{2}}} \leq \frac{1}{2}\left[\frac{ber(A^{\mu})}{\delta^{\mu}} + \frac{ber(A^{\nu})}{\delta^{\nu}}\right],$$

and hence, since $A^{\frac{\mu+\nu}{2}}$ is also positive, we have

$$\begin{aligned} \widetilde{A^{\frac{\mu+\nu}{2}}}(\lambda) &\leq \frac{1}{2}\widetilde{A}(\lambda)^{\frac{\mu+\nu}{2}}\left[\frac{ber(A^{\mu})}{\delta^{\mu}} + \frac{ber(A^{\nu})}{\delta^{\nu}}\right] \\ &\leq \frac{1}{2}ber(A)^{\frac{\mu+\nu}{2}}\left[\frac{ber(A^{\mu})}{\delta^{\mu}} + \frac{ber(A^{\nu})}{\delta^{\nu}}\right] \end{aligned}$$

for all $\lambda \in \Omega$. Consequently, by taking the supremum from the left hand, we obtain

$$ber\left(A^{\frac{\mu+\nu}{2}}\right) \leq \frac{1}{2}ber(A)^{\frac{\mu+\nu}{2}}\left[\frac{ber(A^{\mu})}{\delta^{\mu}} + \frac{ber(A^{\nu})}{\delta^{\nu}}\right],$$

or equivalently,

$$\frac{ber\left(A^{\frac{\mu+\nu}{2}}\right)}{ber(A)^{\frac{\mu+\nu}{2}}} \leq \frac{1}{2}\left[\frac{ber(A^{\mu})}{\delta^{\mu}} + \frac{ber(A^{\nu})}{\delta^{\nu}}\right],$$

as desired. □

Note that a fundamental power inequality for the numerical radius $w(A)$ of operators on a Hilbert space H is the following which originally was proved by Berger [4] (see also, Halmos [13] and Pearcy [23]):

$$w(A^n) \leq w(A)^n \quad (\forall n \geq 1).$$

However, the inequality

$$ber(A^n) \leq ber(A)^n \quad (\forall n \geq 1),$$

or its reverse

$$ber(A)^n \leq Cber(A^n)$$

for some constant $C > 0$, for the Berezin number of operators is not well investigated; for some particular results on these inequalities, see for instance in [10, 11]. The following corollary of Theorem 2.1 gives some particular results in these directions.

Corollary 2.1. *$ber(A^{\mu}) \leq ber(A)^{\mu}$ for $0 \leq \mu \leq 1$, and $ber(A^{\mu}) \geq ber(A)^{\mu}$ for $\mu \in (1, +\infty)$.*

Proof. Indeed, the convexity of $f(\mu)$ implies the desired inequalities. As a matter of fact, $f(\mu)$ defined as in above satisfies $f(0) = f(1) = 1$. Hence the convexity of it implies the required inequalities for the Berezin number of A. □

For our further results, we need to the following analog of a refinement of the Hölder-McCarthy inequality (see [6, Theorem 2.3]).

Theorem 2.2. *Let $A \geq 0$ and $\eta \geq 1$. Then*

$$\begin{aligned} m(\mu,\nu)\left(1-\left(\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}}\right)^{\eta}\right) &\leq 1-\left(\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}}\right)^{\eta} \\ &\leq M(\mu,\nu)\left(1-\left(\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}}\right)^{\eta}\right) \end{aligned}$$

for all $\mu,\nu \in (0,1)$; here $m(\mu,\nu) := \min\left\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\right\}$ and $M(\mu,\nu) := \max\left\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\right\}$. Moreover two inequalities in above are equivalent.

Proof. It follows from Theorem 2.1 that $f^{\eta}(\mu)$ is a convex function by $\eta \geq 1$.
If $\nu \geq \mu$, then we have

$$\frac{f^{\eta}(\mu)-f^{\eta}(0)}{\mu-0} \leq \frac{f^{\eta}(\nu)-f^{\eta}(0)}{\nu-0},$$

that is,

$$\frac{f^{\eta}(\mu)-1}{\mu} \leq \frac{f^{\eta}(\nu)-1}{\nu},$$

whence

$$1-f^{\eta}(\mu) \geq \frac{\mu}{\nu}\left(1-f^{\eta}(\nu)\right).$$

Next, if $\mu \geq \nu$, then we have

$$\frac{f^{\eta}(1)-f^{\eta}(\mu)}{1-\mu} \geq \frac{f^{\eta}(1)-f^{\eta}(\nu)}{1-\nu},$$

that is,

$$1-f^{\eta}(\mu) \geq \frac{1-\mu}{1-\nu}\left(1-f^{\eta}(\nu)\right).$$

For the completing the proof of the theorem, it remains only to use the same arguments as in the proof of Theorem 2.3 of the paper [6] (we omit it). □

In the next result we consider Theorem 2.2 under tha case $\eta = 1$.

Proposition 2.1. *Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$ be an operator such that $A \geq 0$ and $\widetilde{A}(\lambda) \geq \delta > 0$ for all $\lambda \in \Omega$ and some $\delta > 0$. If $1 \geq \nu \geq \mu > 0$, then*

$$\frac{ber(A^{\mu})}{ber(A)^{\mu}} + \frac{\mu}{\nu}\left(1-\frac{ber(A^{\nu})}{\delta^{\nu}}\right) \leq 1. \tag{2.1}$$

Proof. Indeed, it follows again from the arithmetic-geometric mean inequality that

$$\begin{aligned} 1-\frac{\mu}{\nu}+\frac{\mu}{\nu}\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}} &\geq \left(\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}}\right)^{\frac{\mu}{\nu}} \\ &= \frac{\widetilde{A^{\nu}(\lambda)}^{\frac{\mu}{\nu}}}{\widetilde{A}(\lambda)^{\nu\frac{\mu}{\nu}}} \geq \frac{\widetilde{A^{\mu}(\lambda)}}{\widetilde{A}(\lambda)^{\mu}} \end{aligned}$$

by $\frac{\mu}{\nu} \in (0,1)$. Hence

$$1-\frac{\widetilde{A^{\mu}(\lambda)}}{\widetilde{A}(\lambda)^{\mu}} \geq \frac{\mu}{\nu}\left(1-\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}}\right).$$

From this

$$1-\frac{\mu}{\nu}+\frac{\mu}{\nu}\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}} \geq \frac{\widetilde{A^{\mu}(\lambda)}}{\widetilde{A}(\lambda)^{\mu}} \geq \frac{\widetilde{A^{\mu}(\lambda)}}{ber(A)^{\mu}},$$

and hence

$$1-\frac{\mu}{\nu}+\frac{\mu}{\nu}\frac{ber(A^{\nu})}{\widetilde{A}(\lambda)^{\nu}} \geq \frac{\widetilde{A^{\mu}(\lambda)}}{ber(A)^{\mu}}.$$

Now by using that $\widetilde{A}(\lambda) \geq \delta$ *for all* $\lambda \in \Omega$,we have from this inequality that

$$1-\frac{\mu}{\nu}+\frac{\mu}{\nu}\frac{ber(A^{\nu})}{\delta^{\nu}} \geq \frac{ber(A^{\mu})}{ber(A)^{\mu}},$$

which means inequality (2.1). The proof is finished. □

Next result proves the equivalence between refined Hölder-McCarthy type inequality and refined Young type inequality.

Theorem 2.3. *Refined Hölder-McCarthy type inequality and refined Young type inequality are equivalent, i.e.,*

$$1-\frac{\widetilde{A^{\mu}(\lambda)}}{\widetilde{A}(\lambda)^{\mu}} \geq m(\mu,\nu)\left(1-\frac{\widetilde{A^{\nu}(\lambda)}}{\widetilde{A}(\lambda)^{\nu}}\right) \tag{2.2}$$

and

$$\mu\widetilde{A}+1-\mu\widetilde{A^{\mu}} \geq m(\mu,\nu)\left(\nu\widetilde{A}+1-\nu-\widetilde{A^{\nu}}\right) \tag{2.3}$$

are equivalent for given $\mu,\nu \in (0,1)$, *where* $m(\mu,\nu)$ *is as in Theorem 2.2.*

Proof. Suppose that (2.2) holds and $\lambda \in \Omega$ *is an arbitrary point. If* $\nu \geq \mu$, then we have

$$\mu\langle A\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle+1-\mu-\frac{\mu}{\nu}\left(\nu\langle A\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle+1-\nu-\langle A^{\nu}\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle\right)$$

$$= \frac{\nu-\mu}{\nu}+\frac{\mu}{\nu}\langle A^{\nu}\widehat{k}_{\lambda},\widehat{k}_{\lambda}\rangle.$$

On the other hand, by applying classical Young and Hölder-McCarthy inequalities, we obtain for all $\lambda \in \Omega$ *that*

$$\frac{\nu-\mu}{\nu}+\frac{\mu}{\nu}\widetilde{A^{\nu}(\lambda)} \geq \widetilde{A^{\nu}(\lambda)}^{\frac{\mu}{\nu}} \geq \widetilde{A^{\mu}(\lambda)},$$

which together with the last equality implies the desired inequality (2.3).

If now $\mu \geq \nu$, then for all $\lambda \in \Omega$ we have

$$\begin{aligned} & \mu\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + 1 - \mu - \frac{1-\mu}{1-\nu}\left(\nu\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + 1 - \nu - \langle A^\nu\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\right) \\ = & \left\langle\left(\frac{\mu-\nu}{1-\nu}A + \frac{1-\mu}{1-\nu}A^\nu\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \geq \langle A^{\frac{\mu-\nu}{1-\nu}}A^{\frac{\nu(1-\mu)}{1-\nu}}\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ = & \widetilde{A^\mu}(\lambda), \end{aligned}$$

which gives (2.3).

For the proof of reverse implication (2.3)⇒(2.2), we replace A by $\frac{A}{\widetilde{A(\lambda)}}$ in (2.3). Then we have

$$\begin{aligned} & \mu\frac{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle} + 1 - \mu - \frac{\langle A^\mu\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^\mu} \\ \geq & m(\mu,\nu)\left(\nu\frac{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle} + 1 - \nu - \frac{\langle A^\nu\widehat{k}_\lambda, \widehat{k}_\lambda\rangle}{\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^\nu}\right), \end{aligned}$$

which implies that

$$1 - \frac{\widetilde{A^\mu}(\lambda)}{\widetilde{A}(\lambda)^\mu} \geq m(\mu,\nu)\left(1 - \frac{\widetilde{A^\nu}(\lambda)}{\widetilde{A}(\lambda)^\nu}\right).$$

Since $\lambda \in \Omega$ is arbitrary, this means (2.2), which proves the theorem. □

In conclusion, we give one more inequalities related Berezin numbers of positive operators A, B and their fractional powers.

Proposition 2.2. *For any two positive operators A, B on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$, we have:*

(a) $\mu ber(A) + (1-\mu)ber(B) \geq ber(B\#\mu A)$ for all $\mu \in [0,1]$; here, as mentioned above, $B\#\mu A := B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^\mu B^{1/2}$ is the $\mu-$ operator geometric mean.

(b) $ber(A^\mu) - 1 \leq \mu(ber(A) - 1)$ for $0 \leq \mu \leq 1$.

Proof. The proof of this proposition is immediate from the well known Young inequalities (see Fruta [7]) for arbitrary two positive operators $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ (see Furuta [7]):

$$\mu A + (1-\mu)B \geq B\#\mu A \textit{ for } 0 \leq \mu \leq 1$$

and

$$\mu A + 1 - \mu \geq A^\mu \textit{ for } 0 \leq \mu \leq 1.$$

□

Next result follows from the Kantorovich inequality [7].

Proposition 2.3. *If $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is a positive operators such that $M \geq A \geq m > 0$, then*

$$ber\left(A^2\right) \leq \frac{(m+M)^2}{4mM}ber(A)^2.$$

This constant $\frac{(m+M)^2}{4mM}$ *is said to be the Kantorovich constant and it is easy to see that* $\frac{(m+M)^2}{4mM} \geq 1$.

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