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SIMULTANEOUS APPROXIMATION IN LEBESGUE SPACE WITH VARIABLE EXPONENT

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Abstract. In this paper we deal with the simultaneous approximations by trigonometric polynomials of 2π periodic functions in the variable exponent Lebesgue spaces. In terms of the higher moduli of smoothness the direct and inverse theorems of simultaneous approximation are proved. Moreover, the generalized variable exponent Lipschitz classes are defined and the constructive characterizations of these classes are obtained.

1. Introduction

Let $p(x) : \mathbb{T} := [0, 2\pi] \to [0, \infty)$ be a Lebesgue measurable 2π periodic function such that

$$1 \le p_- := ess \inf_{x\in\mathbb{T}} p(x) \le ess \sup_{x\in\mathbb{T}} p(x) := p^+ < \infty.$$

Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ with a variable exponent $p(\cdot)$ consists of all 2π periodic Lebesgue measurable functions f satisfying the condition $\int_0^{2\pi} |f(x)|^{p(x)} dx < \infty$. It is a linear space and equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_0^{2\pi} |f(x)/\lambda|^{p(x)} dx \le 1 \right\}$$

becomes a Banach space.

The variable exponent spaces first of all were thought in [24] by Orlicz. Later in [22] Nakano introduced the variable exponent Lebesgue spaces as specific examples of modular spaces, and their properties were further developed in [23]. These spaces on the real line were developed independently by Tsenov, Portnov and Sharapudinov. The comprehensive presentation of these results can be found in the monograph [27].

Interest in the variable exponent Lebesgue spaces has increased since the 1990s, because of their use in a variety of applications, especially, for mathematical modeling of electrorheological fluids, to model some physical problems and also

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to study of image processing. Nowadays there are sufficiently wide investigations relating to the fundamental problems of these spaces, in view of potential theory, maximal and singular integral operator theory and others. The detailed information about these investigations can be found in the monographs [25] , [6], [8] and also in the works [9], [18], [19], [26]. But the approximation problems in the variable exponent Lebesgue spaces, aren't investigated sufficiently wide. Meanwhile, some of the fundamental problems of approximation theory in the variable exponent Lebesgue spaces of periodic and non periodic functions defined on the intervals of real line were studied and solved by Sharapudinov. The detailed presentation of these results can be found in his book [27] (see also: [28], [29]).

The variable exponent Lebesgue spaces are a generalization of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(\cdot)$. It plays an important role for constitution of the structural properties of these spaces. For this purpose we define some classes of variable exponent functions $p(\cdot)$, which are needed for the effective researches in the variable exponent Lebesgue spaces.

Definition 1.1. *If the function $p(\cdot)$ satisfies the condition*

$$|p(x)-p(y)|\ln(2\pi/|x-y|)\leq c,\quad \forall x,y\in[0,2\pi]$$

with a positive constant c, then we say that $p(\cdot)\in\mathcal{P}(\mathbb{T})$.

The set of all variable exponents $p(\cdot)\in\mathcal{P}(\mathbb{T})$ with $p_->1$, we denote by $\mathcal{P}_0(\mathbb{T})$.

Let $W_k^{p(\cdot)}(\mathbb{T})$, $k=1,2,...$, be a space containing the Lebesgue measurable 2π periodic and $k-1$ times continuously differentiable functions such that $f^{(k)}\in L^{p(\cdot)}(\mathbb{T})$. So $W_k^{p(\cdot)}(\mathbb{T})$ becomes a Banach space with the norm $\|f\|_{W_k^{p(\cdot)}(\mathbb{T})}=\|f\|_{L^{p(\cdot)}}+\left\|f^{(k)}\right\|_{L^{p(\cdot)}}$.

Let Π_n be the class of trigonometric polynomials of degree not exceeding n and let $E_n(f)_{p(\cdot)}:=\inf\{\|f-T_n\|_{L^{p(\cdot)}}:T_n\in\Pi_n\}$, $n=0,1,2,...$, be the *best approximation number* for $f\in L^{p(\cdot)}(\mathbb{T})$.

We benefit from the notations $T_n^0:=T_n^0(f)$ and $T_n^*:=T_n^*(f)$ for trigonometric polynomials pertain to Π_n the *best* and *near-best approximating* to f, respectively, i.e., $\left\|f-T_n^0\right\|_{L^{p(\cdot)}}=E_n(f)_{p(\cdot)}$, $\|f-T_n^*\|_{L^{p(\cdot)}}\leq cE_n(f)_{p(\cdot)}$, $n=0,1,2,...$, for some constant $c>0$, independent of n.

One of the essential difference (in view of the approximation theory, especially) between the classical Lebesgue space $L^p(\mathbb{T})$ and variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is the invariance property of $L^p(\mathbb{T})$ with respect to the shift operator $f(\cdot+h)$; $L^{p(\cdot)}(\mathbb{T})$ space is non invariant [30] with respect to this shift. This property has a crucial importance in harmonic analysis and in approximation theory, especially for the constructions of the moduli of smoothness in $L^{p(\cdot)}(\mathbb{T})$. Using different moduli of smoothness some results on approximation theory in the variable exponent Lebesgue spaces were obtained in [11], [10], [1], [2],[3], [4], [5], [16], [17]. Here, these results were proved under the condition of $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$. The last condition is essentially in the above cited investigations, because it is necessary for boundedness of the maximal operators, considered in these works.

After that defining a new modulus of smoothness

$$\Omega\left(f,\delta\right)_{p(\cdot)} := \sup_{0<h\leq\delta}\left\|\frac{1}{h}\int_0^h\left[f\left(\cdot\right)-f\left(\cdot+t\right)\right]dt\right\|_{L^{p(\cdot)}},$$

which is more sensitive than the moduli considered in the above cited works, some direct and inverse theorems of approximation theory in $L^{p(\cdot)}\left(\mathbb{T}\right)$ spaces under the condition of $p\left(\cdot\right)\in\mathcal{P}\left(\mathbb{T}\right)$, which is more general than the condition $p\left(\cdot\right)\in\mathcal{P}_0\left(\mathbb{T}\right)$, were proved in [30, 31, 32], and [13, 14].

In the variable exponent Lebesgue spaces, as in the classical Lebesgue spaces, for the construction of the approximation aggregates an important role play the Fourier series representation

$$f\left(x\right)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx+b_k\sin kx\right)=\sum_{k=-\infty}^{\infty}c_k e^{ikx}$$

of $f\in L^1\left(\mathbb{T}\right)$ and its partial sums $S_n\left(f\right)$, $n=1,2,3,...$, defined as

$$S_n\left(f\right)\left(x\right):=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right)=\sum_{k=-n}^{n}c_k e^{ikx}.$$

We use also the De la Vallée Poussin means of $f\in L^1\left(\mathbb{T}\right)$, defined by

$$V_n\left(f\right)\left(x\right):=V_n\left(f\right)=\frac{1}{n}\sum_{\upsilon=n}^{2n-1}S_\upsilon\left(f\right)\left(x\right)$$

and having the integral representation

$$V_n\left(f\right):=V_n\left(f\right)\left(x\right)=\int_{-\pi}^{\pi}f\left(x-t\right)\mathcal{K}_n\left(t\right)dt$$

with kernel

$$\mathcal{K}_n\left(t\right):=\frac{1}{\pi}\frac{\sin\left(3nt/2\right)\sin\left(nt/2\right)}{2n\sin^2(t/2)}.$$

Definition 1.2. *Let $f\in L^{p(\cdot)}\left(\mathbb{T}\right)$ with $p\left(\cdot\right)\in\mathcal{P}\left(\mathbb{T}\right)$ and let*

$$\Delta_t^r f\left(x\right):=\sum_{s=0}^{r}\left(-1\right)^{r+s}\binom{r}{s}f\left(x+st\right)\ ,\qquad r=1,2,...\ .$$

For every positive integer r we define the rth modulus of smoothness by

$$\Omega_r\left(f,\delta\right)_{p(\cdot)}:=\sup_{|h|\leq\delta}\left\|\frac{1}{h}\int_0^h\Delta_t^r f\left(x\right)dt\right\|_{L^{p(\cdot)}},\quad h>0.$$

If $f,\ f_1,\ f_2\in L^{p(\cdot)}\left(\mathbb{T}\right)$ with $p\left(\cdot\right)\in\mathcal{P}\left(\mathbb{T}\right)$, then the rth modulus of smoothness $\Omega_r\left(f,\delta\right)_{p(\cdot)}$ has the following properties :

i)$\Omega_r\left(f,\delta\right)_{p(\cdot)}$ *is non-negative, continuous and non-decreasing function of* $\delta>0$,

ii)$\Omega_r\left(f,\delta\right)_{p(\cdot)}$ *is uniformly bounded function in* $L^{p(\cdot)}\left(\mathbb{T}\right)$,

iii) $\lim_{\delta\to0}\Omega_r\left(f,\delta\right)_{p(\cdot)}=0$,

iv)$\Omega_r\left(f_1+f_2,\delta\right)_{p(\cdot)}\leq\Omega_r\left(f_1,\delta\right)_{p(\cdot)}+\Omega_r\left(f_2,\delta\right)_{p(\cdot)}$.

We note that , the properties $i)$ and $ii)$ were proved in [14, Lemmas 2 and 3].

In this work we investigate the simultaneous approximation and constructive characterization problems in $W_k^{p(\cdot)}(\mathbb{T})$. The results obtained in this work, were reported in *International Conference on Complex Analysis and Related Topics*, June 20-24 2016, Bucharest, Romania (see, [15, p.16]). We need to emphasize that, similar results by using the different type modulus of smoothness, were also given in [34]. For the completeness of the proofs of the reported results we will give also the proofs of *Theorems 1.1 and 1.5* formulated in below, and obtained independently by us in [34].

Throughout this work by $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot)$,..., $c(\cdot,\cdot)$, $c_1(\cdot,\cdot)$, $c_2(\cdot,\cdot)$, $c(\cdot,\cdot,\cdot)$, $c_1(\cdot,\cdot,\cdot)$, $c_2(\cdot,\cdot,\cdot)$ we denote the constants, depending in general on the parameters given in the corresponding brackets and independent of n.

Our new results obtained in this paper are following:

Theorem 1.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{T})$. Then there exists a positive constant $c(p,k)$ such that for every $f \in W_k^{p(\cdot)}(\mathbb{T})$ and $n \in \mathbb{N}$ the inequality*

$$\left\| f^{(k)} - (T_n^*)^{(k)} \right\|_{L^{p(\cdot)}} \leq c(p,k) E_n \left(f^{(k)} \right)_{p(\cdot)}$$

holds.

This theorem in the classical Lebesgue space $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, was proved in [7]. In the case of $k \in \mathbb{R}^+$ it was proved in the weighted Lebesgue and weighted Orlicz spaces in [35] and [4], respectively. *Theorem 1.1*, when $k \in \mathbb{R}^+$ and $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, was proved in [1]. *Theorem 1.1,* independently from us was also obtained in [34].

Theorem 1.2. *Let $f \in W_k^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, and let $r = 1, 2, \ldots$. If*

$$\|f - T_n\|_{L^{p(\cdot)}} \leq \frac{c}{n^k} \Omega_r \left(f^{(k)}, 1/n \right)_{p(\cdot)} \qquad n = 1, 2, \ldots$$

for a trigonometric polynomial $T_n \in \Pi_n$, then there exists a positive constant $c(p,k,r)$ such that for $m = 0, 1, 2, \ldots, k$ the inequality

$$\left\| f^{(m)} - T_n^{(m)} \right\|_{L^{p(\cdot)}} \leq \frac{c(p,k,r)}{n^{k-m}} \Omega_r \left(f^{(k)}, 1/n \right)_{p(\cdot)}$$

holds.

The following theorem characterizes the simultaneous approximation property of De la Vallée Poussin means.

Theorem 1.3. *Let $f \in W_k^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$. Then there exists a positive constant $c(p,k,r)$ such that for $m = 0, 1, 2, \ldots, k$ the inequality*

$$\left\| f^{(m)} - V_n^{(m)}(f) \right\|_{L^{p(\cdot)}} \leq \frac{c(p,k,r)}{n^{k-m}} \Omega_r \left(f^{(k)}, 1/n \right)_{p(\cdot)}$$

holds.

Theorem 1.3, in the case of $m = 0$ and $r = 1$ was proved by Sharapudinov in [32].

Theorem 1.4. *If $f\in W_{r-k}^{p(\cdot)}(\mathbb{T})$, $k\leq r$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a positive constant $c(p,r)$ such that for every $n=1,2,...$, the inequality*

$$\Omega_r(f,1/n)_{p(\cdot)}\leq\frac{c(p,r)}{n^{r-k}}\left\|f^{(r-k)}\right\|_{L^{p(\cdot)}}$$

holds.

The results similar to *Theorem 1.4* by using the different type moduli of smoothness in the classical nonweighted and weighted Lebesgue spaces were obtained in [21, 20] and [35], respectively. In the weighted Orlicz spaces it was proved in [12]. In the case of $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ and $k=0$ by using the different type modulus of smoothness this estimation was proved in [1].

Theorem 1.5. *Let $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$. If $\sum\limits_{\nu=1}^{\infty}\nu^{k-1}E_\nu(f)_{p(\cdot)}<\infty$ for some $k=0,1,...$, then $f\in W_k^{p(\cdot)}(\mathbb{T})$ and there exists a positive constant $c(p,k)$ such that the inequality*

$$E_n\left(f^{(k)}\right)_{p(\cdot)}\leq c(p,k)\left\{n^kE_n(f)_{p(\cdot)}+\sum_{\nu=n+1}^{\infty}\nu^{k-1}E_\nu(f)_{p(\cdot)}\right\},\quad n=0,1,2,...$$

holds.

Theorem 1.6. *Let $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$. If $\sum\limits_{\nu=1}^{\infty}\nu^{k-1}E_\nu(f)_{p(\cdot)}<\infty$, for some $k=0,1,...$, then $f\in W_k^{p(\cdot)}(\mathbb{T})$ and there exists a positive constant $c(p,k,r)$ such that the inequality*

$$\begin{aligned}&\Omega_r\left(f^{(k)},1/n\right)_{p(\cdot)}\\ \leq\;& c(p,k,r)\left\{\frac{1}{n^r}\sum_{\nu=1}^{n}\nu^{k+r-1}E_\nu(f)_{p(\cdot)}+\sum_{\nu=n+1}^{\infty}\nu^{k-1}E_\nu(f)_{p(\cdot)}\right\},\quad n=1,2,...\end{aligned}$$

holds.

Theorems, similar to *Theorems 1.5* and *1.6* in the case of $p(\cdot)\in\mathcal{P}_0(\mathbb{T})$ were obtained in [1] and [2], respectively. In case of $p(\cdot)\in\mathcal{P}(\mathbb{T})$ *Theorem 1.5,* independently from us was also proved in [34]. The result, in the term of the different type modulus of smoothness, which is similar to *Theorem 1.6,* can be found in [34]. In the classical cases *Theorems 1.5* and *1.6* were proved in [33]. Similar results in the Orlicz spaces were obtained in [4].

From Theorem 1.6 we have

Corollary 1.1. *Let $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$. If $E_n(f)_{p(\cdot)}=\mathcal{O}\left(n^{-k-\alpha}\right)$, $n=1,2,$ for some $k=0,1,...$ and $\alpha>0$, then $f\in W_k^{p(\cdot)}(\mathbb{T})$ and for every $\delta>0$*

$$\Omega_r\left(f^{(k)},1/n\right)_{p(\cdot)}=\begin{cases}\mathcal{O}\left(n^{-\alpha}\right) & ,r>\alpha\\ \mathcal{O}\left(n^{-\alpha}\log n\right) & ,r=\alpha\\ \mathcal{O}\left(n^{-r}\right) & ,r<\alpha.\end{cases}$$

If we define a generalized variable exponent Lipschitz classes for $r := [\alpha]+1 > 0$ by

$$Lip_{k,\alpha}^{p(\cdot)}(\mathbb{T}) := \left\{ f \in W_k^{p(\cdot)}(\mathbb{T}) : \Omega_r\left(f^{(k)}, \delta\right)_{p(\cdot)} = \mathcal{O}\left(\delta^{\alpha}\right),\ \delta > 0 \right\},$$

then *Corollary 1.1* implies

Corollary 1.2. *Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$. If*

$$E_n(f)_{p(\cdot)} = \mathcal{O}\left(n^{-k-\alpha}\right) \qquad n = 0, 1, 2, ...$$

for some $k = 0, 1, ...$ and $\alpha > 0$, then $f \in Lip_{k,\alpha}^{p(\cdot)}(\mathbb{T})$.

Finally, combining *Corollaries 1.1 and 2.2* we can formulate the following constructive characterization of the generalized variable exponent Lipschitz classes $Lip_{k,\alpha}^{p(\cdot)}(\mathbb{T})$.

Theorem 1.7. *Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, and $\alpha > 0$. Then for every $k = 0, 1, 2, ...$, the following statements are equivalent*

$$i)\ f \in Lip_{k,\alpha}^{p(\cdot)}(\mathbb{T}), \qquad ii)\ E_n(f)_{p(\cdot)} = \mathcal{O}\left(n^{-k-\alpha}\right), \qquad n \in \mathbb{N}.$$

2. Auxiliary Results

For the construction of the moduli of smoothness some convolution operators are needed.

Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, and let $1 \le \lambda < \infty$. For the measurable 2π periodic kernel κ_λ we define a convolution operator

$$\mathcal{L}_\lambda(f) := \mathcal{L}_\lambda(f)(x) := \int_{-\pi}^{\pi} f(t)\,\kappa_\lambda(t-x)\,dt.$$

Theorem 2.1 ([31]). *Let a 2π periodic kernel κ_λ, $1 \le \lambda < \infty$, satisfies the conditions: a) $\int_{-\pi}^{\pi} |\kappa_\lambda(t)|\,dt \le c_1$, b) $\sup_{x\in[-\pi,\pi]} |\kappa_\lambda(x)| \le c_2\lambda^{\nu}$, c) $|\kappa_\lambda(x)| \le c_3$ for $\lambda^{-\gamma} \le |x| \le \pi$ with some constants $c_i\,(i = 1, 2, 3), \upsilon, \gamma > 0$, independent of λ. If $p(\cdot) \in \mathcal{P}(\mathbb{T})$, then the operators family $\{\mathcal{L}_\lambda(f)\}_{\lambda\ge1}$ is uniformly bounded in $L^{p(\cdot)}(\mathbb{T})$.*

Since the kernel $\mathcal{K}_n$ satisfies [7] the conditions: *a)* $\int_{-\pi}^{\pi} |\mathcal{K}_n(t)|\,dt \le c\log(3/2)$, *b)* $\sup_{x\in[-\pi,\pi]} |\mathcal{K}_n(x)| \le c_2 n$, *c)* $|\mathcal{K}_n(t)| \le c_3$ for $n^{-1/2} \le x \le 2\pi - n^{-1/2}$, with $\lambda = n$, $\gamma = 1/2$ and $\nu = 1$, by *Theorem 2.1* , the De la Vallée Poussin means $V_n(f)$ are uniformly bounded in $L^{p(\cdot)}(\mathbb{T})$. It means that $\|V_n(f)\|_{L^{p(\cdot)}} \le c(p)\,\|f\|_{L^{p(\cdot)}}$ for every $f \in L^{p(\cdot)}(\mathbb{T})$. Furthermore, for every function f and a trigonometric polynomial T_n we have $V_n^{(r)}(f) = V_n\left(f^{(r)}\right)$ and $V_n(T_n) = T_n$.

The following theorem states an approximation properties of De la Vallée Poussin means $V_n(f)$.

Theorem 2.2 ([31]). *If $f \in W_k^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, then there exists a constant $c(p) > 0$ such that for every $n = 1, 2, ...$*

$$\|f - V_n(f)\|_{L^{p(\cdot)}} \le \frac{c(p)}{n^k}\,E_n\left(f^{(k)}\right)_{p(\cdot)}.$$

By Consequence 2.1 in [31], properties *iii*) and *ii*) of $\Omega_r(f,\delta)_{p(\cdot)}$, where $r=1$ we have

Corollary 2.1. *If $f\in W_k^{p(\cdot)}(\mathbb{T})$ with $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a constant $c(p)>0$ such that for every $n=1,2,...$*

$$E_n(f)_{p(\cdot)}\leq\frac{c(p)}{n^k}\left\Vert f^{(k)}\right\Vert_{L^{p(\cdot)}}.$$

We use also the following Bernstein type inequality for trigonometric polynomials.

Theorem 2.3 ([30]). *Let $p(\cdot)\in\mathcal{P}(\mathbb{T})$ and let T_n be a trigonometric polynomial of degree n. Then there exists a constant $c(p)>0$ such that for every $n=1,2,...$*

$$\left\Vert T_n^{(r)}(x)\right\Vert_{L^{p(\cdot)}}\leq c(p)n^r\left\Vert T_n(x)\right\Vert_{L^{p(\cdot)}}.$$

The following direct and inverse theorems were proved in [14].

Theorem 2.4 ([14]). *If $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a constant $c(p)>0$ such that for every $n=1,2,...$*

$$E_n(f)_{p(\cdot)}\leq c(p)\ \Omega_r(f,1/n)_{p(\cdot)}.$$

Theorem 2.5 ([14]). *If $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a constant $c(p)>0$ such that for every $n=1,2,...$*

$$\Omega_r(f,1/n)_{p(\cdot)}\leq\frac{c(p)}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p(\cdot)}.$$

Combining *Theorem 2.2 and 2.4* we obtain

Corollary 2.2. *If $f\in W_k^{p(\cdot)}(\mathbb{T})$ with $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a constant $c(p,k)>0$ such that for every $n=1,2,...$*

$$\left\Vert f-V_n(f)\right\Vert_{L^{p(\cdot)}}\leq\frac{c(p,k)}{n^k}\ \Omega_r\left(f^{(k)},1/n\right)_{p(\cdot)}.$$

The following result can be proved easily by classical method and using *Corollary 2.1*

Corollary 2.3. *If $f\in W_k^{p(\cdot)}(\mathbb{T})$ with $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a constant $c(p,k)>0$ such that for every $n=1,2,...$*

$$E_n(f)_{p(\cdot)}\leq\frac{c(p)}{n^k}\ E_n\left(f^{(k)}\right)_{p(\cdot)}.$$

Combining *Corollary 2.3* and *Theorem 2.4* we obtain

Corollary 2.4. *If $f\in W_k^{p(\cdot)}(\mathbb{T})$ with $p(\cdot)\in\mathcal{P}(\mathbb{T})$, then there exists a constant $c(p,k)>0$ such that for every $n=1,2,...$*

$$E_n(f)_{p(\cdot)}\leq\frac{c(p,k)}{n^k}\ \Omega_r\left(f^{(k)},1/n\right)_{p(\cdot)}.$$

Lemma 2.1. *Let $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$. Let T_n be a trigonometric polynomial of degree n. If there are the sequences $\{M_n\}$ and $\{N_n\}$ of real numbers such that*

$$\|f-T_n\|_{L^{p(\cdot)}}\leq M_n \text{ and } \left\|T_n^{(r)}\right\|_{L^{p(\cdot)}}\leq N_n \qquad r=1,2,...$$

then there exists a constant $c(p)>0$ such that for every $\delta>0$ and $n=1,2,...$

$$\Omega_r(f,\delta)_{p(\cdot)}\leq c(p)\left\{M_n+\delta^r N_n\right\},\quad \delta>0.$$

Proof. Using the inequality $\Omega_r(T_n,\delta)_{p(\cdot)}\leq c(p)\delta^r\left\|T_n^{(r)}\right\|_{L^{p(\cdot)}}$, proved in [14] and the properties $iv)$ and $ii)$ of $\Omega_r(f,\delta)_{p(\cdot)}$ we have

$$\begin{aligned}\Omega_r(f,\delta)_{p(\cdot)} &\leq \Omega_r(f-T_n,\delta)_{p(\cdot)}+\Omega_r(T_n,\delta)_{p(\cdot)}\\ &\leq c_1(p)\|f-T_n\|_{L^{p(\cdot)}}+c(p)\delta^r\left\|T_n^{(r)}\right\|_{L^{p(\cdot)}}\leq c_2(p)\left\{M_n+\delta^r N_n\right\}.\end{aligned}$$

□

Lemma 2.2. *Let $f\in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}(\mathbb{T})$. If T_n^0 is the best approximation trigonometric polynomial to f, then there exists a positive constant $c(p,k)$ such that for every $n=1,2,..$*

$$\left\|\left(T_n^0\right)^{(k)}\right\|_{L^{p(\cdot)}}\leq c(p,k)\sum_{\nu=1}^{n}\nu^{k-1}E_\nu(f)_{p(\cdot)},\quad k=0,1,2,...\ .$$

Proof. Let T_n^0 be the best approximation polynomial to f. For $m\in\mathbb{N}$ satisfying the inequality $2^{m-1}\leq n<2^m$ we construct a sequence $\{n_j\}$ $j=1,2,...,m-1$ such that $n_0=0$, $n_j=2^j$ and $n_m=n$. For every $i\geq1$ the inequality

$$2^{(i+1)k}E_{2^i}(f)_{p(\cdot)}\leq 2^{2k}\sum_{\mu=2^{i-1}+1}^{2^i}\mu^{k-1}E_\mu(f)_{p(\cdot)}. \tag{2.1}$$

holds. Indeed, taking into account that $\left\{E_n(f)_{p(\cdot)}\right\}_{n=1}^{\infty}$ is a decreasing sequence, we have

$$\begin{aligned}2^{(i+1)k}E_{2^i}(f)_{p(\cdot)} &= 2^{2k}2^{(i-1)k}E_{2^i}(f)_{p(\cdot)}\\ &= 2^{2k}2^{i-1}\left(2^{i-1}\right)^{k-1}E_{2^i}(f)_{p(\cdot)}\\ &\leq 2^{2k}2^{i-1}\left(2^{i-1}+1\right)^{k-1}E_{2^i}(f)_{p(\cdot)}\\ &\leq 2^{2k}\sum_{\mu=2^{i-1}+1}^{2^i}\mu^{k-1}E_{2^i}(f)_{p(\cdot)}\\ &\leq 2^{2k}\sum_{\mu=2^{i-1}+1}^{2^i}\mu^{k-1}E_\mu(f)_{p(\cdot)}.\end{aligned}$$

Since $T_n^0(x) = T_0^0(x) + \sum_{j=1}^{m}\left(T_{n_j}^0(x) - T_{n_{j-1}}^0(x)\right)$, differentiating and applying *Theorem 2.3* and (2.1) we have

$$\begin{aligned}
\left\|\left(T_n^0\right)^{(k)}\right\|_{L^{p(\cdot)}} &\leq \sum_{j=1}^{m}\left\|\left(T_{n_j}^0\right)^{(k)} - \left(T_{n_{j-1}}^0\right)^{(k)}\right\|_{L^{p(\cdot)}} \\
&\leq \sum_{j=1}^{m} n_j^k \left\|T_{n_j}^0 - T_{n_{j-1}}^0\right\|_{L^{p(\cdot)}} \leq 2\sum_{j=1}^{m} n_j^k \; E_{n_{j-1}}(f)_{p(\cdot)} \\
&\leq c(p)\left\{2^k E_0(f)_{p(\cdot)} + \sum_{j=2}^{m-1} n_j^k E_{n_{j-1}}\;(f)_{p(\cdot)} + n_m^k E_{n_{m-1}}(f)_{p(\cdot)}\right\} \\
&\leq c(p)\left\{2^k E_0(f)_{p(\cdot)} + \sum_{j=2}^{m-1} 2^{jk} E_{n_{j-1}}(f)_{p(\cdot)} + 2^{mk} E_{n_{m-1}}(f)_{p(\cdot)}\right\}
\end{aligned}$$

$$= c(p)\left\{2^k E_0(f)_{p(\cdot)} + \sum_{j=1}^{m-1} 2^{(j+1)k} E_{2^j}(f)_{p(\cdot)}\right\}$$

$$\leq c(p)\left\{2^k E_0(f)_{p(\cdot)} + \sum_{j=1}^{m-1} 2^{2k} \sum_{\nu=2^{j-1}+1}^{2^j} \nu^{k-1} E_\nu(f)_{p(\cdot)}\right\}$$

$$\leq c(p)\left\{2^{2k} E_0(f)_{p(\cdot)} + 2^{2k}\sum_{\nu=1}^{2^{m-1}} \nu^{k-1} E_\nu(f)_{p(\cdot)}\right\}$$

$$\leq c(p,k)\left\{E_0(f)_{p(\cdot)} + \sum_{\nu=1}^{2^{m-1}} \nu^{k-1} E_\nu(f)_{p(\cdot)}\right\} \leq c(p,k)\sum_{\nu=1}^{n} \nu^{k-1} E_\nu(f)_{p(\cdot)}.$$

□

3. Proofs of Main Results

Proof of Theorem 1.1. Let $f \in W_k^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$. Let also $T_n^0(f)$, $T_n^*(f) \in \Pi_n$ be the *best* and *near-best approximating* trigonometric polynomials for f, respectively. Then

$$\left\|f^{(k)} - T_n^{*(k)}(f)\right\|_{L^{p(\cdot)}} \leq \left\|f^{(k)} - V_n\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} + \left\|T_n^{*(k)}(V_n(f)) - T_n^{*(k)}(f)\right\|_{L^{p(\cdot)}}$$

$$+\left\|V_n\left(f^{(k)}\right) - T_n^{*(k)}(V_n(f))\right\|_{L^{p(\cdot)}} =: I_1 + I_2 + I_3.$$

By the boundedness of $V_n(f)$

$$\begin{aligned} I_1 &= \left\| f^{(k)} - V_n\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} \\ &\leq \left\| f^{(k)} - T_n^0\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} + \left\| T_n^0\left(f^{(k)}\right) - V_n\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} \\ &= E_n\left(f^{(k)}\right)_{p(\cdot)} + \left\| V_n\left(T_n^0\left(f^{(k)}\right)\right) - V_n\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} \\ &\leq E_n\left(f^{(k)}\right)_{p(\cdot)} + c_3(p)\left\| T_n^0\left(f^{(k)}\right) - f^{(k)}\right\|_{L^{p(\cdot)}} \leq c_4(p)E_n\left(f^{(k)}\right)_{p(\cdot)} \end{aligned}$$

and by *Theorems 2.3* , *2.2* and *Corollary 2.3*

$$\begin{aligned} I_2 &= \left\| T_n^{*(k)}\left(V_n(f)\right) - T_n^{*(k)}(f)\right\|_{L^{p(\cdot)}} \leq c_5(p)n^k \left\| T_n^*\left(V_n(f)\right) - T_n^*(f)\right\|_{L^{p(\cdot)}} \\ &\leq c_5(p)n^k \left\{ \left\| T_n^*\left(V_n(f)\right) - V_n(f)\right\|_{L^{p(\cdot)}} \right. \\ &\quad \left. + \left\| V_n(f) - f\right\|_{L^{p(\cdot)}} + \left\| f - T_n^*(f)\right\|_{L^{p(\cdot)}} \right\} \\ &\leq c_5(p)n^k \left\{ c(p)E_n\left(V_n(f)\right)_{p(\cdot)} + \left\| V_n(f) - f\right\|_{L^{p(\cdot)}} + c(p)E_n(f)_{p(\cdot)} \right\} \\ &\leq c_5(p)n^k \left\{ c(p)E_n\left(V_n(f)\right)_{p(\cdot)} + \frac{c(p)}{n^k}E_n\left(f^{(k)}\right)_{p(\cdot)} + \frac{c(p)}{n^k}E_n\left(f^{(k)}\right)_{p(\cdot)} \right\}. \end{aligned}$$

Since by *Theorem 2.2*and *Corollary 2.3*

$$\begin{aligned} E_n\left(V_n(f)\right)_{p(\cdot)} &\leq \left\| V_n(f) - T_n^0(f)\right\|_{L^{p(\cdot)}} \\ &\leq \left\| V_n(f) - f\right\|_{L^{p(\cdot)}} + \left\| f - T_n^0(f)\right\|_{L^{p(\cdot)}} \\ &\leq \frac{c_1(p,k)}{n^k}E_n\left(f^{(k)}\right)_{p(\cdot)}, \end{aligned}$$

we obtain that $I_2 \leq c(p,k)E_n\left(f^{(k)}\right)_{p(\cdot)}$.

Ultimately applying *Bernstein inequality* to I_3, using *Theorem 2.2* and *Corollary 2.3* we get

$$\begin{aligned} I_3 &= \left\| V_n\left(f^{(k)}\right) - T_n^{*(k)}\left(V_n(f)\right)\right\|_{L^{p(\cdot)}} \\ &\leq c(p)\left(2n-1\right)^k \left\| V_n(f) - T_n^*\left(V_n(f)\right)\right\|_{L^{p(\cdot)}} \\ &\leq c_7(p)\left(2n-1\right)^k E_n\left(V_n(f)\right)_{p(\cdot)} \\ &\leq c_7(p)\frac{\left(2n-1\right)^k}{n^k}E_n\left(f^{(k)}\right)_{p(\cdot)} \leq c_8(p,k)E_n\left(f^{(k)}\right)_{p(\cdot)} \end{aligned}$$

and therefore

$$\left\| f^{(k)} - T_n^{*(k)}(f)\right\|_{L^{p(\cdot)}} \leq I_1 + I_2 + I_3 \leq c(p,k)E_n\left(f^{(k)}\right)_{p(\cdot)}.$$

□

Proof of Theorem 1.2. Let $f \in W_k^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, and let $T_n^0(f)$ be the best approximation polynomial to f. For every $m \in \mathbb{N}$ we have

$$\left\| f^{(m)} - T_n^{(m)}\right\|_{L^{p(\cdot)}} \leq \left\| f^{(m)} - \left(T_n^0\right)^{(m)}\right\|_{L^{p(\cdot)}} + \left\| \left(T_n^0\right)^{(m)} - T_n^{(m)}\right\|_{L^{p(\cdot)}}. \quad (3.1)$$

If the condition of *Theorem 1.2* is fulfilled for a polynomial $T_n \in \Pi_n$, then it is fulfilled also for $T_n^0(f)$. Therefore, applying successively *Theorem 1.1*, *Corollary 2.3* and *Theorem 2.4* we have

$$\begin{aligned} \left\| f^{(m)} - \left(T_n^0(f)\right)^{(m)} \right\|_{L^{p(\cdot)}} &\leq c(p,k) E_n\left(f^{(m)}\right)_{p(\cdot)} \\ &\leq \frac{c(p,k)}{n^{k-m}} E_n\left(f^{(k)}\right)_{p(\cdot)} \\ &\leq \frac{c(p,k)}{n^{k-m}} \Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)}. \end{aligned} \tag{3.2}$$

On other hand, applying *Theorem 2.3* and *Corollary 2.4* we obtain the inequality

$$\begin{aligned} \left\| \left(T_n^0\right)^{(m)} - T_n^{(m)} \right\|_{L^{p(\cdot)}} &\leq c_9(p) n^m \left\{ \|T_n - f\|_{L^{p(\cdot)}} + \left\| f - T_n^0 \right\|_{L^{p(\cdot)}} \right\} \\ &\leq c_9(p) n^m \left\{ \frac{c(p,r)}{n^k} \Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)} + E_n(f)_{p(\cdot)} \right\} \\ &\leq \frac{c_{10}(p,r)}{n^{k-m}} \Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)}. \end{aligned} \tag{3.3}$$

Now, using (3.2) and (3.3) in (3.1) we conclude that

$$\left\| f^{(m)} - T_n^{(m)} \right\|_{L^{p(\cdot)}} \leq \frac{c(p,k,r)}{n^{k-m}} \Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)}.$$

□

Proof of Theorem 1.3. The proof is obtained in analogy to proof of *Theorem 1.2* using property *iii)* of $\Omega_r(f,\delta)_{p(\cdot)}$. □

Proof of Theorem 1.4. Let $f \in W_{r-k}^{p(\cdot)}(\mathbb{T})$, $k \leq r$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$. For the best approximation polynomial $T_n^0(f)$ we have [14] :
$\Omega_r\left(T_n^0(f), 1/n\right)_{p(\cdot)} \leq c(p,r) n^{-r} \left\| \left(T_n^0(f)\right)^{(r)} \right\|_{L^{p(\cdot)}}$, which by Bernstein inequality implies that

$$\begin{aligned} \Omega_r\left(T_n^0(f), 1/n\right)_{p(\cdot)} &\leq c(p,r) n^{-r} \left\| \left(T_n^0(f)\right)^{(r)} \right\|_{L^{p(\cdot)}} \\ &\leq c_{11}(p,r) n^{k-r} \left\| \left(T_n^0(f)\right)^{(r-k)} \right\|_{L^{p(\cdot)}}. \end{aligned}$$

Hence, using the properties *ii)* and *iv)* of $\Omega_r(f,\delta)_{p(\cdot)}$ and *Corollary 2.1* we have that

$$\begin{aligned} \Omega_r(f, 1/n)_{p(\cdot)} &\leq \Omega_r\left(f - T_n^0(f), 1/n\right)_{p(\cdot)} + \Omega_r\left(T_n^0(f), 1/n\right)_{p(\cdot)} \\ &\leq c(p,r) E_n(f)_{p(\cdot)} + c_{11}(p,r) n^{k-r} \left\| \left(T_n^0(f)\right)^{(r-k)} \right\|_{L^{p(\cdot)}} \\ &\leq \frac{c_{12}(p,r)}{n^{r-k}} \left(\left\| f^{(r-k)} \right\|_{L^{p(\cdot)}} + \left\| \left(T_n^0(f)\right)^{(r-k)} \right\|_{L^{p(\cdot)}} \right). \end{aligned} \tag{3.4}$$

On the other hand, by *Theorem 1.1* and *Corollary 2.1*

$$\begin{aligned}\left\|\left(T_n^0(f)\right)^{(r-k)}\right\|_{L^{p(\cdot)}} &\leq \left\|\left(T_n^0(f)\right)^{(r-k)} - f^{(r-k)}\right\|_{L^{p(\cdot)}} + \left\|f^{(r-k)}\right\|_{L^{p(\cdot)}} \\ &\leq c_{13}(p,r)E_n\left(f^{(r-k)}\right)_{p(\cdot)} + \left\|f^{(r-k)}\right\|_{L^{p(\cdot)}} \\ &\leq c_{14}(p,r)\left\|f^{(r-k)}\right\|_{L^{p(\cdot)}}. \qquad (3.5)\end{aligned}$$

Combining (3.4) and (3.5) we obtain the desired inequality.

□

Proof of Theorem 1.5. Let $T_n^0(f)$ be the best approximation polynomial to f. Choosing $2^m \leq n < 2^{m+1}$ for a given n and applying *Theorem 2.3* we have

$$\begin{aligned}\left\|\left(T_{2^{m+1}}^0\right)^{(k)} - \left(T_{2^m}^0\right)^{(k)}\right\|_{L^{p(\cdot)}} &\leq c(p)2^{(m+1)k}\left\|T_{2^{m+1}}^0 - T_{2^m}^0\right\|_{L^{p(\cdot)}} \\ &\leq c_{15}(p)2^{(m+1)k}E_{2^m}(f)_{p(\cdot)}.\end{aligned}$$

By (2.1) and by hypothesis of *Theorem 1.5*

$$\begin{aligned}\sum_{m=1}^{\infty}\left\|T_{2^{m+1}}^0 - T_{2^m}^0\right\|_{W_k^{p(\cdot)}(\mathbb{T})} &= \sum_{m=1}^{\infty}\left\|T_{2^{m+1}}^0 - T_{2^m}^0\right\|_{L^{p(\cdot)}} \\ &+ \sum_{m=1}^{\infty}\left\|\left(T_{2^{m+1}}^0\right)^{(k)} - \left(T_{2^m}^0\right)^{(k)}\right\|_{L^{p(\cdot)}} \\ &\leq c_{15}(p)\sum_{m=1}^{\infty}2^{(m+1)k}E_{2^m}(f)_{p(\cdot)} \\ &\leq c_{16}(p,k)\sum_{m=1}^{\infty}\sum_{\mu=2^{m-1}+1}^{2^m}\mu^{k-1}E_{\mu}(f)_{p(\cdot)} \\ &\leq c_{16}(p,k)\sum_{\mu=2}^{\infty}\mu^{k-1}E_{\mu}(f)_{p(\cdot)} < \infty.\end{aligned}$$

This inequality indicates that the sequence $\left\{T_{2^m}^0\right\}$ is a Cauchy sequence in the Banach space $W_k^{p(\cdot)}(\mathbb{T})$. Hence

$T_{2^m}^0 \to f \in W_k^{p(\cdot)}(\mathbb{T})$, $m \to \infty$, in $W_k^{p(\cdot)}(\mathbb{T})$. By *Theorem 1.3*, $V_n\left(f^{(k)}\right) \to f^{(k)}$ as $n \to \infty$ and then

$$\begin{aligned}E_n\left(f^{(k)}\right)_{p(\cdot)} &\leq \left\|f^{(k)} - V_n\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} \\ &\leq \left\|V_{2^{m+1}+1}\left(f^{(k)}\right) - V_n\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} \\ &+ \sum_{j=m+2}^{\infty}\left\|V_{2^j+1}f^{(k)} - V_{2^{j-1}+1}\left(f^{(k)}\right)\right\|_{L^{p(\cdot)}} \\ &= I_4 + I_5. \qquad (3.6)\end{aligned}$$

Taking into account that $V_n(f)$ is a trigonometric polynomial of degree $2n-1$ by *Theorem 2.3* we have

$$\begin{aligned} I_4 &= \left\| V_{2^{m+1}+1}\left(f^{(k)}\right) - V_n\left(f^{(k)}\right) \right\|_{L^{p(\cdot)}} \\ &\leq c_{17}(p)\left(2^{m+2}+1\right)^k \left\| V_{2^{m+1}+1}(f) - V_n(f) \right\|_{L^{p(\cdot)}} \\ &\leq c_{18}(p)\left(2^{m+3}\right)^k E_n(f)_{p(\cdot)} \\ &\leq c_{18}(p) 8^k n^k E_n(f)_{p(\cdot)} \\ &\leq c_{19}(p,k) n^k E_n(f)_{p(\cdot)} . \end{aligned} \tag{3.7}$$

On the other hand, by *Theorems 2.3*, *2.2* and (2.1)

$$\begin{aligned} I_5 &= \sum_{j=m+2}^{\infty} \left\| V_{2^j+1}^{(k)}(f) - V_{2^{j-1}+1}^{(k)}(f) \right\|_{L^{p(\cdot)}} \leq c_{20}(p,k) \sum_{j=m+2}^{\infty} \left(2^{j+1}+1\right)^k E_{2^j+1}(f)_{p(\cdot)} \\ &\leq c_{20}(p,k) \sum_{j=m+2}^{\infty} \left(2^{j+2}\right)^k E_{2^j+1}(f)_{p(\cdot)} \leq c_{20}(p,k) 2^k \sum_{j=m+2}^{\infty} \left(2^{j+1}\right)^k E_{2^j}(f)_{p(\cdot)} \\ &\leq c_{21}(p,k) \sum_{j=m+2}^{\infty} \sum_{\nu=2^{j-1}+1}^{2^j} \nu^{k-1} E_\nu(f)_{p(\cdot)} \leq c_{21}(p,k) \sum_{\nu=2^{m+1}+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot)} \\ &\leq c_{21}(p,k) \sum_{\nu=n+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot)} < \infty. \end{aligned} \tag{3.8}$$

Now the relations (3.6)-(3.8) imply that

$$E_n\left(f^{(k)}\right)_{p(\cdot)} \leq c(p,k) \left\{ n^k E_n(f)_{p(\cdot)} + \sum_{\nu=n+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot)} \right\}.$$

□

Proof of Theorem 1.6*.* Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$ and let $T_n^0(f)$ be the best approximation polynomial to f. Applying *Theorems 1.1* and *1.5*, we have

$$\left\| f^{(k)} - \left(T_n^0\right)^{(k)} \right\|_{L^{p(\cdot)}} \leq c(p,k) \left\{ n^k E_n(f)_{p(\cdot)} + \sum_{\nu=n+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot)} \right\}. \tag{3.9}$$

By *Lemma 2.2*

$$\left\| \left(T_n^0\right)^{(k+r)} \right\|_{L^{p(\cdot)}} \leq c(p,k,r) \sum_{\nu=1}^{n} \nu^{k+r-1} E_\nu(f)_{p(\cdot)} \qquad n = 1, 2, ... \tag{3.10}$$

Choosing $\delta := 1/n$ and using the relations (3.9) and (3.10) in *Lemma 2.1*, we conclude that

$$\begin{aligned} \Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)} \leq\ & c_{22}(p,k,r) \left\{ n^k E_n(f)_{p(\cdot)} + \sum_{\nu=n+1}^{\infty} \nu^{k-1} E_\nu(f)_{p(\cdot)} \right. \\ & \left. + \frac{1}{n^r} \sum_{\nu=1}^{n} \nu^{k+r-1} E_\nu(f)_{p(\cdot)} \right\}. \end{aligned} \tag{3.11}$$

Since $n^k E_n(f)_{p(\cdot)} \leq \frac{c}{n^r} \sum\limits_{\nu=1}^{n} \nu^{k+r-1} E_\nu(f)_{p(\cdot)}$ with some constant c, from (3.11) we get the desired inequality

$$\begin{aligned} &\Omega_r\left(f^{(k)}, 1/n\right)_{p(\cdot)} \\ \leq\ & c_{23}(p,k,r)\left\{\frac{1}{n^r}\sum_{\nu=1}^{n}\nu^{k+r-1}E_\nu(f)_{p(\cdot)} + \sum_{\nu=n+1}^{\infty}\nu^{k-1}E_\nu(f)_{p(\cdot)}\right\}. \end{aligned}$$

□

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