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QUASI-KÄHLERIAN STRUCTURES CARRIED ON CODAZZI CONNECTIONS

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Abstract. Our aim is to introduce quasi-Kählerian Codazzi manifolds as some natural structures. Supposing a Kählerian structure on a manifold, we investigate its relation with quasi-Kählerian Codazzi manifolds. Furthermore, it will be proved that under some conditions, the Codazzi and Levi-Civita connections are adapted. Also, quasi-Kählerian Codazzi manifolds have non-trivial examples shown in this paper.

1. Introduction

For an open subset $\Theta \subseteq \mathbb{R}^n$, S is an statistical model, when S is a set of probability density functions on a sample space Ω with data behavior represented as the parameter $\theta = (\theta^1, \cdots, \theta^n)$ such that

$$S = \{p(x;\theta) : \int_\Omega p(x;\theta) = 1, \quad p(x;\theta) > 0, \quad \theta \in \Theta \subseteq \mathbb{R}^n\}.$$

Probability distributions are playing important roles in science encountering with presented data sets. People employ them to study the prediction and evaluation of different models of actions in any network of nodes. When Fisher exhibited a formula as a mathematical translation of information (see [7]), differential geometry joined to this contribution. Indeed, for a statistical model S, the semi-definite Fisher information matrix $g(\theta) = [g_{ij}(\theta)]$ is defined as

$$\begin{aligned} g_{ij}(\theta) &:= \int_\Omega \partial_i \ell_\theta \partial_j \ell_\theta p(x;\theta) \mathrm{d}x \\ &= E_p[\partial_i \ell_\theta \partial_j \ell_\theta], \end{aligned}$$

where $\ell_\theta = \ell(x;\theta) := \log p(x;\theta)$, $\partial_i := \frac{\partial}{\partial \theta^i}$ and $E_p[f]$ is the expectation of $f(x)$ with respect to $p(x;\theta)$. S is called an information manifold, when it equipped by such matrix. If g is positive-definite and all of its components are finite, then (S, g) will be a Riemannian manifold and g will be called a Fisher metric on S. In this situation, g reads

$$g_{ij}(\theta) = \int_\Omega \partial_i p(x;\theta) \partial_j \ell_\theta \mathrm{d}x = \int_\Omega \frac{1}{p(x;\theta)} \partial_i p(x;\theta) \partial_j p(x;\theta) \mathrm{d}x.$$

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These kinds of metrics were first studied by Rao (see [14]). For any $\alpha \in \mathbb{R}$, Amari's α-connection ∇^α with respect to $p(x;\theta)$ is defined by the Christoffel symbols

$$\Gamma^{(\alpha)}{}_{ij,k} = g(\nabla^\alpha_{\partial_i}\partial_j, \partial_k) := E_p[(\partial_i\partial_j\ell_\theta + \frac{1-\alpha}{2}\partial_i\ell_\theta\partial_j\ell_\theta)(\partial_k\ell_\theta)]. \tag{1.1}$$

Chentsov began to study α-connections in the case of finite and discrete sample spaces (see [6] and [5]). After that, Amari studied them in an independent manner and general case by formula (1.1). One can see e.g. [2] as a first collection of results in this framework. Moreover, there are detailed monographs about applications of information geometry such some chapters of [12].

Lauritzen was the first who composed a parallel framework called statistical manifolds. The difference between two α and β-connections ∇^α and ∇^β is as follow

$$\Gamma^{(\alpha)}{}_{ij,k} - \Gamma^{(\beta)}{}_{ij,k} = \frac{\beta-\alpha}{2}\mathbb{T}_{ijk}, \tag{1.2}$$

where $\mathbb{T}$ is a covariant symmetric tensor of degree 3 defined by $\mathbb{T}_{ijk} := E_\theta[\partial_i\ell_\theta\partial_j\ell_\theta\partial_k\ell_\theta]$. For the case that $\beta = 0$, (1.2) reduces to

$$\Gamma^{(\alpha)}{}_{ij,k} - \overset{g}{\Gamma}_{ij,k} = -\frac{\alpha}{2}\mathbb{T}_{ijk},$$

where $\overset{g}{\Gamma}_{ij,k}$'s are the Christoffel symbols of the Levi-Civita connection induced by a Riemannian metric g on M. A statistical manifold is a triple (M, g, ∇) where the manifold M is equipped with a statistical structure (g, ∇) containing a Riemannian metric g and an affine symmetric connection ∇ on M such that the covariant derivative ∇g is symmetric. There is a one to one correspondence between tensors $\mathbb{T}_{ijk}$ and statistical connections (for a detailed discussion, see [10]). Writing the symmetric property off from ∇, generalize the above definition to the Codazzi manifold.

The statistical (or more generally, Codazzi) affine connection can be studied with other existent structures and so we can generalize some aspects from Riemannian manifolds to the statistical (Codazzi) manifolds. Some of the important subject matters are Kählerian structures. These structures have important situation in other aspects of science. A series of applications of these structures in the other fields and in math can be found in the collection [12], [4] and [13].

In this paper, we define Kählerian Codazzi manifolds containing three structures (g, ∇, J) on M such that (M, g, ∇) makes a Codazzi manifold and J is a parallel almost complex structure with respect to ∇. This definition is natural and compatible with the definition of classical Kählerian manifolds. Since, when ∇ is the Levi-Civita connection of g we have a Kählerian manifold which carries a trivial Codazzi structure. Moreover, starting from a Kählerian Codazzi manifold $(M, g, \overset{g}{\nabla}, J)$ we can get a Kählerian Codazzi manifold $(M, g, \tilde{\nabla}, J)$ naturally by putting the metric

$$G(X, Y) = g(X, Y) + g(JX, JY),$$

on M where $\tilde{\nabla}$ is the Levi-Civita connection of G. Take to the account that in our case there is no Hermitian relation between g, J, essentially.

There are other similar studies also. In [17], the Codazzi condition is considered for a pair of affine connection and a Kählerian structure. In [1], they considered a structure similar to ours whereas the involved space is very similar to a statistical manifold. In [15], the authors investigate the integrable almost anti-Hermitian manifolds with the Codazzi condition on its twin anti-Hermitian metric and achieved some results on the curvature tensor. Moreover, in [16], they could get a coincidence between the Ricci curvature tensor of the anti-Kähler-Codazzi manifold and the Ricci curvature tensor of the manifold. Though, we set the Kählerian isomorphism with the Codazzi connection instead.

We will give some examples on Kählerian statistical and Codazzi manifolds when M is the 2-dimensional sphere without north pole . In the case of perforated 2-sphere, we classify all of statistical connections where they are appointed by the standard complex structure.

It is notable that researches on the vector (and specially tangent) bundles of statistical manifolds are scant until now ([3, 8, 9, 11] are examples of such researches). In this paper we lift the complex structure to the tangent bundle equipped with the Sasaki metric and characterize a class of Kählerian statistical manifolds on tangent bundles.

We prove that for the Kählerian statistical manifold $(\mathbb{R}^2, g, J, \nabla)$ where g is an arbitrary Riemannian metric and J is the matrix

$$J = \begin{pmatrix} 0 & h \\ \frac{-1}{h} & 0 \end{pmatrix},$$

with the additional condition $\nabla = \overset{g}{\nabla} + \overset{g}{\nabla} J$, the equation $\overset{g}{\nabla} = \nabla$ hold. So, it is natural to ask "is the latter statement true generally?" The following can be stated as an open problem in Riemannian geometry and statistical manifolds area.

Open problem. Let (M, g, ∇, J) be an arbitrary Kählerian statistical manifold such that $\nabla, \overset{g}{\nabla}$ satisfy $\nabla = \overset{g}{\nabla} + \overset{g}{\nabla} J$. Is it true to say that $\nabla = \overset{g}{\nabla}$?

2. Preliminaries

First, we give some preliminaries on Kählerian and Codazzi manifolds.

A tangent bundle isomorphism $J : TM \to TM$ is known as an *almost para-complex structure* if $J^2 = -\mathrm{I}$. Moreover, if (M, g) is a Riemannian manifold, we say that g is *compatible* with J if J is orthogonal, i.e.,

$$g(JX, JY) = g(X, Y),$$

for vector fields X, Y on M. In this case (M, g, J) is called an *almost Hermitian manifold*, and *Hermitian manifold* if J is integrable. In the almost Hermitian realm, one can define a 2-form $\Omega(X, Y) = g(JX, Y)$ called fundamental 2-form where

$$d\Omega(JX, Y, JZ) - d\Omega(JY, X, JZ) = g(N_J(X, Y), Z) + 2g(J(\overset{g}{\nabla}_{JZ} J)Y, X), \tag{2.1}$$

and

$$d\Omega(X, Y, Z) = g((\overset{g}{\nabla}_X J)Y, Z) - g((\overset{g}{\nabla}_Y J)X, Z) + g((\overset{g}{\nabla}_Z J)X, Y), \tag{2.2}$$

where $\overset{g}{\nabla}$ is the Levi-Civita connection of g and N_J is the Nijenhuis tensor of J given by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad \forall X, Y \in \chi(M).$$

An (almost) Hermitian manifold (M, g, J) is called an *(almost) Kählerian manifold* if Ω is closed, i.e., $\mathrm{d}^2\Omega = 0$. From (2.1) and (2.2) one deduce that (M, g, J) is a Kählerian manifold if and only if the structure J is parallel with respect to the Levi-Civita connection of g, i.e., $\overset{g}{\nabla}J = 0$.

A *Codazzi manifold* is a triplet (M, g, ∇) where g is a Riemannian metric on M and ∇ is a (not necessarily torsion-free) affine connection that the cubic tensor field $\mathcal{C} = \nabla g$ is totally symmetric, namely the Codazzi equations hold:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(Z, X) \qquad (= (\nabla_Z g)(X, Y)), \forall X, Y, Z \in \mathfrak{X}(M).$$

In a local coordinate $(\mathcal{U}, x^1, \cdots, x^n)$ on M, $\mathcal{C}$ has the following form

$$\mathcal{C}(\partial_i, \partial_j, \partial_k) = \partial_i g(\partial_j, \partial_k) - g(\nabla_{\partial_i}\partial_j, \partial_k) - g(\partial_j, \nabla_{\partial_i}\partial_k),$$

and so

$$\mathcal{C}_{ijk} = \partial_k(g_{ij}) - \Gamma^h{}_{ik}g_{jh} - \Gamma^h{}_{jk}g_{ih}, \qquad \mathcal{C}_{ijk} = \mathcal{C}_{jki} = \mathcal{C}_{kij},$$

where $\partial_i := \frac{\partial}{\partial x^i}$, $g_{ij} := g(\partial_i, \partial_j)$ and $\Gamma^i{}_{jk}$'s be the Christoffel symbols of ∇. Moreover, the contraction

$$\mathcal{C}^k_{ij} = g^{rk}\mathcal{C}_{ijr},$$

can be applying.

3. Quasi-Kählerian statistical manifolds

We begin this section by fixing the definition of a quasi-Kählerian Codazzi manifold as follow.

Definition 3.1. A quadruplet (M, g, ∇, J) is called a quasi-Kählerian Codazzi manifold if (M, g, ∇) is a Codazzi structure and J is integrable such that $\nabla J = 0$.

We have the attention that when ∇ is arising from the Riemannian metric g, then definition 3.1 is as the same as the usual definition of a Kählerian structure, whenever J preserves the length. Indeed, for $X, Y \in \mathrm{T}M$ we have

$$g(X, X) + 2g(X, Y) + g(Y, Y) = g(JX, JX) + 2g(JX, JY) + g(JY, JY),$$

giving the compatibility condition $g(X, Y) = g(JX, JY)$. It is remarkable that the length-preserving property of J is independent and completely essential. For instance, let's make an example that shows this necessity.

Example 3.1. Let $(\mathbb{E}^2, g)$ be the standard Euclidean space and ∇ be its covariant derivative. Define the complex structure J by

$$J(\partial_1) = 2\partial_2, \quad J(\partial_2) = \frac{-1}{2}\partial_1.$$

As $\mathrm{d}x^i, \partial_i$ are parallel and the multiplications of $\mathrm{d}x^i \otimes \partial_j$ are constants, then J is parallel. But it is not length preserver.

It is worth paying attention to the case that $\nabla = \overset{g}{\nabla}$. Since, if we have a quasi-Kählerian Codazzi manifold (M, g, ∇, J) such that $\nabla = \overset{g}{\nabla}$, then we can define a Kählerian structure on M (we establish Proposition 3.1 on this fact). Indeed, we know that if J is an almost complex structure on M and g is a Riemannian metric, then the metric

$$G(X, Y) = g(X, Y) + g(JX, JY), \tag{3.1}$$

with J define an almost Hermitian structure on M. We devote $\widetilde{\nabla}$ for the Levi-Civita connection of G. Now, we can make the following lemma.

Lemma 3.1. *Let (M, g) be a Riemannian manifold and J be an almost complex structure on M such that $\overset{g}{\nabla} J = 0$. Then we have*

$$G(\widetilde{\nabla}_X Y, Z) = g(\overset{g}{\nabla}_X Y, Z) + g(\overset{g}{\nabla}_X JY, JZ). \tag{3.2}$$

Proof. Using the Koszul equation we have

$$\begin{aligned} 2G(\widetilde{\nabla}_X Y, Z) &= XG(Y, Z) + YG(X, Z) - ZG(X, Y) \\ &\quad + G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X), \end{aligned}$$

and by equation (3.1), we get

$$\begin{aligned} 2G(\widetilde{\nabla}_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \\ &\quad + Xg(JY, JZ) + Yg(JX, JZ) - Zg(JX, JY) \\ &\quad + g(J[X, Y], JZ) - g(J[X, Z], JY) - g(J[Y, Z], JX). \end{aligned}$$

The first two lines are the $2g(\widetilde{\nabla}_X Y, Z)$ and if we use the equations

$$Ag(B, C) = g(\overset{g}{\nabla}_A B, C) + g(\overset{g}{\nabla}_A C, B),$$

and

$$J[A, B] = J\overset{g}{\nabla}_A B - J\overset{g}{\nabla}_B A = \overset{g}{\nabla}_A JB - \overset{g}{\nabla}_B JA,$$

we get the following (take note to the attention that $\overset{g}{\nabla}_A JB = J\overset{g}{\nabla}_A B$)

$$2G(\widetilde{\nabla}_X Y, Z) = 2g(\overset{g}{\nabla}_X Y, Z) + 2g(\overset{g}{\nabla}_X JY, JZ),$$

which gives us the result. □

Using Lemma 3.1, we can make a proposition to give a sufficient condition for which (M, G, J) be a Kählerian structure on M.

Proposition 3.1. *If $(M, g, \overset{g}{\nabla}, J)$ is a quasi-Kählerian Codazzi manifold, then the triple (M, G, J) is a Kählerian structure on M.*

Proof. Let X, Y, Z be three arbitrary vector fields then we will conclude that $G((\widetilde{\nabla}_X J)Y, Z) = 0$. Using the equation

$$G(\widetilde{\nabla}_A B, C) = g(\overset{g}{\nabla}_A B, C) + g(\overset{g}{\nabla}_A JB, JC),$$

and the fact that $G(JA,B)=-G(A,JB)$ we get the following

$$\begin{aligned} G((\widetilde{\nabla}_X J)Y,Z) &= G(\widetilde{\nabla}_X JY,Z)+G(\widetilde{\nabla}_X Y,JZ)\\ &= g(\overset{g}{\nabla}_X JY,Z)+g(\overset{g}{\nabla}_X J^2Y,JZ)\\ &+ g(\overset{g}{\nabla}_X Y,JZ)+g(\overset{g}{\nabla}_X JY,J^2Z)=0. \end{aligned}$$

So, $G((\widetilde{\nabla}_X J)Y,Z)=0$ and this yields $\widetilde{\nabla}J=0$. On the other hand, J is integrable. Then, (M,G,J) is a Kählerian structure on M. □

Now, we will use Proposition 3.1 to give a natural quasi-Kählerian Codazzi statistical manifold as follow.

Theorem 3.1. *Let $(M,g,\overset{g}{\nabla},J)$ be a quasi-Kählerian Codazzi manifold. Then $(M,g,\tilde{\nabla},J)$ is a quasi-Kählerian Codazzi manifold.*

Proof. First, using Proposition 3.1, we know that $\tilde{\nabla}J=0$. So, it is remained to prove the Codazzi equation for $(\tilde{\nabla},g)$. Suppose $(x_1,\cdots,x_n)$ is the normal coordinate system with respect to g around $p\in M$. We know that in this coordinate system we have

$$\overset{g}{\Gamma}{}^k_{ij}(p)=0,\qquad g_{ij}(p)=\delta_{ij}.$$

Using equation (3.2), one can show that $\tilde{\Gamma}^k_{ij}(p)=0$ where $\tilde{\Gamma}^k_{ij}$ are the Christoffel symbols of $\widetilde{\nabla}$. Indeed, translating (3.2) to the local form, read

$$\tilde{\Gamma}^l_{ij}G_{lk}=\overset{g}{\Gamma}{}^l_{ij}g_{lk}+\overset{g}{\Gamma}{}^l_{ij}J^t_lJ^s_kg_{ts}. \tag{3.3}$$

Looking at (3.3) when is evaluated at p, easily yields the vanishing of $\tilde{\Gamma}^k_{ij}$ at p. Now, one can check that the couple $(\tilde{\nabla},g)$ satisfies the Codazzi equation at p. But p was an arbitrary point and so the proof is completed. □

In the following theorem, we want to specify statistical manifolds (M,G,∇) such that (M,g,∇,J) be a quasi-Kählerian statistical manifold and $\nabla=\overset{g}{\nabla}+\overset{g}{\nabla}J$. Indeed, the latter condition is a canonical criterion to see how many ∇ and $\overset{g}{\nabla}$ are far from together in the sense of J.

Theorem 3.2. *Let (M,g,∇,J) be a quasi-Kählerian statistical manifold such that $\nabla=\overset{g}{\nabla}+\overset{g}{\nabla}J$. Then (M,G,∇) is a statistical manifold if and only if the tensor field*

$$\Omega(X,Y)=g((\overset{g}{\nabla}_X J)Z-J(\overset{g}{\nabla}_X J)Z,JY),$$

is symmetric for all $Z\in \mathrm{T}M$.

Proof. If (M,G,∇) is a statistical manifold then

$$(\nabla_X G)(Y,Z)=(\nabla_Y G)(X,Z),$$

that is equivalent to

$$Xg(Y,Z)+Xg(JY,JZ)-g(\nabla_X Y,Z)-g(J\nabla_X Y,JZ)-g(Y,\nabla_X Z)$$
$$-g(JY,J\nabla_X Z)=Yg(X,Z)+Yg(JX,JZ)-g(\nabla_Y X,Z)$$
$$-g(J\nabla_Y X,JZ)-g(X,\nabla_Y Z)-g(JX,J\nabla_Y Z).$$

Since (M,g,∇) is statistical and $\nabla=\overset{g}{\nabla}+\overset{g}{\nabla}J$ then using the compatibility of $(g,\overset{g}{\nabla})$ the above equality holds if and only if

$$g((\overset{g}{\nabla}_X J)Y-J(\overset{g}{\nabla}_X J)Y,JZ)+g((\overset{g}{\nabla}_X J)Z-J(\overset{g}{\nabla}_X J)Z,JY)$$
$$=g((\overset{g}{\nabla}_Y J)X-J(\overset{g}{\nabla}_Y J)X,JZ)+g((\overset{g}{\nabla}_Y J)Z-J(\overset{g}{\nabla}_Y J)Z,JX).$$

But $\nabla,\overset{g}{\nabla}$ are torsion free and the equation $\nabla=\overset{g}{\nabla}+\overset{g}{\nabla}J$ gives that

$$g((\overset{g}{\nabla}_X J)Y-J(\overset{g}{\nabla}_X J)Y,JZ)=g((\overset{g}{\nabla}_Y J)X-J(\overset{g}{\nabla}_Y J)X,JZ),$$

proving the theorem. □

3.1. A natural lift to the tangent bundle. The following conversation is on a class of quasi-Kählerian statistical manifolds on tangent bundles.

Let (M,g) be a Riemannian manifold with the unique Levi-Civita connection $\overset{g}{\nabla}$. Considering the splitting

$$\mathrm{T}_{(x,y)}\mathrm{T}M=\mathcal{H}_{(x,y)}\oplus\mathcal{V}_{(x,y)},$$

it can be verified that if $X=X^i\frac{\partial}{\partial x^i}$, then

$$X^v=X^i\frac{\partial}{\partial y^i},\qquad X^h=X^i\frac{\partial}{\partial x^i}-X^jy^k\,\overset{g}{\Gamma^i}_{jk}\,\frac{\partial}{\partial y^i},$$

where $\overset{g}{\Gamma^i}_{jk}$'s are Christoffel symbols of the Levi-Civita connection $\overset{g}{\nabla}$. If $\overset{g}{R}$ denotes the Riemann curvature tensor of $\overset{g}{\nabla}$, then

$$\begin{cases}[X^v,Y^v]=0,\\ [X^h,Y^v]=(\overset{g}{\nabla}_X\,Y)^v,\\ [X^h,Y^h]=[X,Y]^h-(\overset{g}{R}\,(X,Y)y)^v,\end{cases}$$

for any $X,Y,Z\in\mathfrak{X}(M)$ and any point $(x,y)\in\mathrm{T}M$. The Sasaki metric g_s on the tangent bundle $\mathrm{T}M$ is a natural lift of the metric g given by

$$\begin{cases}g_s(X^h,Y^h)_{(x,y)}=g_x(X,Y),\\ g_s(X^v,Y^h)_{(x,y)}=0,\\ g_s(X^v,Y^v)_{(x,y)}=g_x(X,Y).\end{cases}$$

Now, suppose that (M,g,J) is a Kählerian manifold, then lift J to an almost complex structure on $\mathrm{T}M$ and equip it with the Sasaki metric. The following proposition classifies quasi-Kählerian statistical manifolds $(\mathrm{T}M,g_s,\bar{\nabla},\bar{J})$ where g_s is the Sasaki metric of g and $\bar{J}$ is the lift of J defined in the following and $\bar{\nabla}$ is a connection such that $(g_s,\bar{\nabla})$ provide a statistical manifold where $\bar{J}$ is parallel with respect to the $\bar{\nabla}$. Note that we use the notation $A^{\bar{i}}B_i:=\sum_{i=1}^n A^{n+i}B_i$

where n is the dimension of M. Moreover, when we say $\bar{\Gamma}^i_{\bar{k}\bar{j}}$, it is the coefficient of δ_i of $\bar{\nabla}_{\partial_{\bar{k}}}\partial_{\bar{j}}$ and the other Christoffel symbols can be defined similarly.

Proposition 3.2. *Let $\bar{J}$ be the natural lift of J defined by*

$$\bar{J}X^h = (JX)^h, \qquad \bar{J}X^v = (JX)^v.$$

Then $(\mathrm{T}M, g_s, \bar{\nabla}, \bar{J})$ is a quasi-Kählerian statistical manifold if and only if the equations

$$\begin{cases} (\bar{\Gamma}^r{}_{ik} - \Gamma^r{}_{ik})g_{rj} = (\bar{\Gamma}^r{}_{jk} - \Gamma^r{}_{jk})g_{ri}, & (3.4)\\ \bar{\Gamma}^{\bar{r}}{}_{ij}g_{rk} = \bar{\Gamma}^r{}_{\bar{k}j}g_{ri}, \\ \bar{\Gamma}^r{}_{i\bar{k}}g_{jr} - y^m \overset{g}{R}_{ijmk} = \bar{\Gamma}^r{}_{j\bar{k}}g_{ri}, \\ (\bar{\Gamma}^{\bar{r}}{}_{i\bar{k}} - \overset{g}{\Gamma}{}^r{}_{ik})g_{jr} = \bar{\Gamma}^r{}_{\bar{j}\bar{k}}g_{ri}, \\ \bar{\Gamma}^{\bar{r}}{}_{\bar{j}i}g_{rk} = \bar{\Gamma}^{\bar{r}}{}_{\bar{k}i}g_{rj}, \\ \bar{\Gamma}^{\bar{r}}{}_{\bar{i}\bar{k}}g_{rj} = \bar{\Gamma}^{\bar{r}}{}_{\bar{j}\bar{k}}g_{ri}, \end{cases}$$

and

$$\begin{cases} \delta_i(J^k_j) + J^l_j\bar{\Gamma}^k_{il} - \bar{\Gamma}^l_{ij}J^k_l = 0, \\ \delta_i(J^b_a) + J^c_a\bar{\Gamma}^{\bar{b}}_{i\bar{c}} - \bar{\Gamma}^{\bar{c}}_{i\bar{a}}J^b_c = 0, \\ J^b_a\bar{\Gamma}^j_{i\bar{b}} - \bar{\Gamma}^k_{i\bar{a}}J^j_k = 0, \\ J^k_j\bar{\Gamma}^{\bar{a}}_{ik} - \bar{\Gamma}^{\bar{b}}_{ij}J^a_b = 0, \\ J^j_i\bar{\Gamma}^k_{\bar{a}j} - \bar{\Gamma}^j_{\bar{a}i}J^k_j = 0, \\ J^j_i\bar{\Gamma}^{\bar{b}}_{\bar{a}j} - \bar{\Gamma}^{\bar{c}}_{\bar{a}i}J^b_c = 0, \\ J^c_b\bar{\Gamma}^i_{\bar{a}\bar{c}} - \bar{\Gamma}^j_{\bar{a}\bar{b}}J^i_j = 0, & (3.5)\\ J^c_b\bar{\Gamma}^{\bar{d}}_{\bar{a}\bar{c}} - \bar{\Gamma}^{\bar{c}}_{\bar{a}\bar{b}}J^d_c = 0, & (3.6) \end{cases}$$

hold.

Proof. Let (M, g, ∇) be a statistical manifold. Then, $\bar{\nabla}$ is a statistical connection on $(\mathrm{T}M, G)$ if and only if the first set of equations hold. We just prove (3.4) where the others are similar (for a detailed proof, see [3]). Using

$$(\bar{\nabla}_{\delta_i}G)(\delta_j, \delta_k) = \partial_i g_{jk} - \bar{\Gamma}^r{}_{ij}g_{rk} - \bar{\Gamma}^r{}_{ik}g_{rj},$$

and Codazzi equation

$$(\bar{\nabla}_{\delta_i}G)(\delta_j, \delta_k) = (\bar{\nabla}_{\delta_j}G)(\delta_k, \delta_i) = (\bar{\nabla}_{\delta_k}G)(\delta_i, \delta_j),$$

we get

$$\partial_i g_{jk} - \bar{\Gamma}^r{}_{ij}g_{rk} - \bar{\Gamma}^r{}_{ik}g_{rj} = \partial_j g_{ki} - \bar{\Gamma}^r{}_{jk}g_{ri} - \bar{\Gamma}^r{}_{ji}g_{rk}.$$

So, thanks to torsion-freeness of $\bar{\nabla}$ we have

$$\partial_i g_{jk} - \partial_j g_{ki} = \bar{\Gamma}^r{}_{ik}g_{rj} - \bar{\Gamma}^r{}_{jk}g_{ri}. \qquad (3.7)$$

Moreover, applying the Codazzi equations for (g, ∇), yields

$$\partial_i g_{jk} - \partial_j g_{ki} = \Gamma^r{}_{ik}g_{rj} - \Gamma^r{}_{jk}g_{ri}. \qquad (3.8)$$

From (3.7) and (3.8), we have (3.4).

The second set of equations are coming from $\bar{\nabla}\bar{J} = 0$ and we only prove the last two equations. From $0 = \bar{\nabla}_{\partial_{\bar{i}}}\bar{J}\partial_{\bar{j}} - \bar{J}\bar{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}}$ we have

$$J_j^l(\bar{\Gamma}_{\bar{i}l}^a\delta_a + \bar{\Gamma}_{\bar{i}l}^{\bar{a}}\partial_{\bar{a}}) - \bar{\Gamma}_{\bar{i}\bar{j}}^k J_k^a\delta_a - \bar{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} J_k^b\partial_{\bar{b}} = 0,$$

giving (3.5) and (3.6). □

4. Some two dimensional statistical Kählerian manifolds

We denote $\mathbb{S}^2$ without its north pole by $\mathbb{S}^2_O$. The following is the classification of quasi-Kählerian statistical manifolds on $\mathbb{S}^2_O$ by the standard complex structure. Computations are straightforward and we omit them for a clear look at the result.

Example 4.1. Equip $\mathbb{S}^2_O$ by the Riemannian metric

$$g = \frac{4}{({x_1}^2 + {x_2}^2 + 1)^2}(\mathrm{d}x_1^2 \otimes \mathrm{d}x_1^2 + \mathrm{d}x_2^2 \otimes \mathrm{d}x_2^2),$$

where (x_1, x_2) is the stereographic coordinate system on $\mathbb{S}^2_O$ and suppose the complex structure

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

on that. Then $(\mathbb{S}^2_O, g, \nabla, J)$ is a quasi-Kählerian Codazzi structure if and only if the Christoffel symbols of ∇ are given by

$$\begin{cases} \Gamma_{21}^1 = -\Gamma_{11}^2, \quad \Gamma_{22}^2 = -\Gamma_{11}^2, \quad \Gamma_{22}^1 = -\Gamma_{12}^2, \quad \Gamma_{22}^1 = -\Gamma_{12}^2, \quad \Gamma_{11}^1 = \Gamma_{12}^2, \\ \Gamma_{11}^2 = \dfrac{2x_2}{{x_1}^2 + {x_2}^2 + 1}, \quad \Gamma_{12}^2 = \dfrac{-2x_1}{{x_1}^2 + {x_2}^2 + 1}. \end{cases}$$

Next, we want to use an additional condition on a class of quasi-Kählerian statistical manifolds to adapt the Levi-Civita and statistical connections. Note to the fact that there exists a coordinate system on $\mathbb{R}^2$ such that any Riemannian metric $g = [g_{ij}]$ has the expression $g = f\mathrm{I}_2$ and $f \in C^\infty(M)$.

Theorem 4.1. *Let* $g = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$ *and* $J = \begin{pmatrix} 0 & h \\ \frac{-1}{h} & 0 \end{pmatrix}$ *be Riemannian metric and almost complex structure on* $M = \mathbb{R}^2$ *with respect to the standard coordinate system* (x_1, x_2) *and its associated vector fields* ∂_1, ∂_2. *If* (M, ∇, g, J) *be a quasi-Kählerian statistical manifold with condition* $\nabla = \overset{g}{\nabla} + \overset{g}{\nabla} J$ *then* $\nabla = \overset{g}{\nabla}$.

Proof. First, using the equation $\nabla = \overset{g}{\nabla} + \overset{g}{\nabla} J$ we will give the Christoffel symbols of ∇ as follows.

$$\begin{cases} \Gamma^1_{12} = \dfrac{\partial_1 J_{12}}{(J_{12})^2} + \dfrac{1}{2}\dfrac{\partial_2 f}{f}, \\ \Gamma^2_{12} = \dfrac{1}{2fJ_{12}}(J_{12}\partial_1 f + (1-(J_{12})^2)\partial_2 f), \\ \Gamma^1_{11} = \dfrac{1}{2fJ_{12}}(J_{12}\partial_1 f - (1-(J_{12})^2)\partial_2 f), \\ \Gamma^2_{12} = \partial_1 J_{12} - \dfrac{\partial_2 f}{2f}, \\ \Gamma^1_{22} = \dfrac{\partial_2 J_{12}}{(J_{12})^2} + \dfrac{1}{2}\dfrac{\partial_1 f}{f}, \\ \Gamma^2_{22} = \dfrac{1}{2fJ_{12}}(J_{12}\partial_2 f - (1-(J_{12})^2)\partial_1 f), \\ \Gamma^1_{21} = \dfrac{1}{2fJ_{12}}(J_{12}\partial_2 f + (1-(J_{12})^2)\partial_1 f), \\ \Gamma^1_{21} = \partial_2 J_{12} - \dfrac{\partial_1 f}{2f}. \end{cases}$$

Now $\nabla J = 0$ and the statistical conditions give us the following equations.

$$\begin{cases} (h^2-1)\partial_2 f + 4f\partial_1 h = 0, & (4.1) \\ (1-h^2)\partial_2 f + f\partial_1 h = 0, & (4.2) \\ (1-h^2)\partial_1 f + 4f\partial_2 h = 0, & (4.3) \\ (h^2-1)\partial_1 f + f\partial_2 h = 0, & (4.4) \\ (1-h^2)\partial_1 f - 2fh\partial_1 h = 0, \\ (h^3-h)\partial_2 f + 2f\partial_2 h = 0. \end{cases}$$

If we sum the equations (4.1) and (4.2) we get $\partial_1 h = 0$. Adding the equations (4.3) and (4.4) gives us $\partial_2 h = 0$. So, h is a constant that proves the theorem. □

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