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ON LOCAL UNIFORM EQUICONVERGENCE RATE FOR THE DIRAC OPERATOR

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Abstract. We consider one-dimensional Dirac operator with a summable potential on the interval $G = (0, 2\pi)$. Componentwise uniform equiconvergence rate of spectral expansion for absolutely continuous function and trigonometric Fourier series is estimated on a compact.

1. Introduction and statement of results

In [2-3], Il'in developed a method for studying uniform equiconvergence of spectral expansions for differential operators.

Il'in modified his method in [4] to establish a componentwise uniform equiconvergence in case of Schrodinger operator with a matrix potential. Further, componentwise equiconvergence for an arbitrary order differential operator was established in [5], while componentwise equiconvergence rate was studied in [6].

Componentwise uniform equiconvergence of spectral expansions for Dirac operator was considered in [7], [8]. In [1], componentwise equiconvergence of spectral expansion in the metrics L_s, $s \geq 1$, was studied in the compact and a sufficient condition providing equiconvergence in this metrics was found.

In this work, we study componentwise uniform equiconvergence rate of spectral expansion for absolutely continuous vector function with respect to eigenvector functions of Dirac operator and trigonometric Fourier series expansion of this function. We establish estimates for uniform equiconvergence rate on any compact $K \subset G = (0, 2\pi)$.

Consider one-dimensional Dirac operator

$$Dy = By' + P(x)y, \qquad y(x) = (y_1(x),\, y_2(x))^T,$$

on the interval $G = (0, 2\pi)$, where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $P(x) = \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}$, $p(x)$ and $q(x)$ are real-valued functions from the class $L_\alpha(G),\ \alpha \geq 1$.

Following Il'in [2-4], by the eigenvector function of the operator D, corresponding to the real eigenvalue λ, we mean any not identically zero vector function $y(x)$ which is absolutely continuous on $\bar{G} = [0, 2\pi]$ and satisfies the equation $Dy = \lambda y$ almost everywhere in G.

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Let $L_p^2(G)$, $p \geq 1$, be a space of two-component vector functions $f(x) = (f_1(x),\, f_2(x))^T$ with the norm

$$\|f\|_{p,2,G} = \|f\|_{p,2} = \left(\int_G |f(x)|^p \, dx \right)^{1/p}, \quad \left(\|f\|_{\infty,2} = \sup_{x \in \bar{G}} vrai \, |f(x)| \right).$$

Obviously, for $f(x) \in L_p^2(G)$, $g(x) \in L_q^2(G)$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, there exists a "scalar product"

$$(f, g) = \int_G \langle f(x), g(x) \rangle \, dx = \int_G \sum_{j=1}^{2} f_j(x) \overline{g_j(x)} dx\,.$$

Let $\{u_n(x)\}_{n=1}^{\infty}$ be a complete orthonormal system of eigenvector functions of the operator D in $L_2^2(G)$ and $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \in R$, be a corresponding system of eigenvalues.

By $W_{p,2}^1(G)$, $p \geq 1$ we denote a space of absolutely continuous two-component vector functions $f(x)$ on $\bar{G}$ with $f'(x) \in L_p^2(G)$.

Introduce partial sum of spectral expansion of the vector function $f(x) \in W_{1,2}^1(G)$ with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$:

$$\sigma_\nu(x, f) = \left(\sigma_\nu^1(x, f),\, \sigma_\nu^2(x, f)\right)^T, \quad \sigma_\nu^j(x, f) = \sum_{|\lambda_n| \leq \nu} (f, u_n)\, u_n^j(x), \; j = 1, 2;$$

$$\left(u_n^1(x),\, u_n^2(x)\right)^T = u_n(x), \quad f(x) = (f_1(x),\, f_2(x))^T.$$

In addition to the partial sum $\sigma_\nu^j(x, f)$ we also introduce a modified partial sum of trigonometric Fourier series of the function $f_j(x)$, i.e.

$$S_\nu(x, f_j) = \frac{1}{\pi} \int_G \frac{\sin \nu (x - y)}{x - y} f_j(y) dy, \quad j = 1, 2;$$

$$S_\nu(x, f) = (S_\nu(x, f_1),\; S_\nu(x, f_2))^T.$$

Definition 1.1. If the difference $\left\| \sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j) \right\|_{C(K)}$ tends to zero as $\nu \to +\infty$ on the compact $K \subset G$, then we say that the $j-$th component of spectral expansion of the vector function $f(x)$ with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$ uniformly equiconverges on the compact $K \subset G$ with expansion into trigonometric Fourier series corresponding to the $j-$th component $f_j(x)$ of the vector function $f(x)$.

The following theorems are the main results of this work.

Theorem 1.1. *Let $f(x) \in W_{1,2}^1(G)$, the coefficients $p(x)$ and $q(x)$ belong to $L_\alpha(G)$, $\alpha > 1$. Then the $j-$th component of spectral expansion of the vector function $f(x)$ with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$ uniformly equiconverges on any compact $K \subset G$ with expansion into trigonometric Fourier series corresponding to the $j-$th component $f_j(x)$ of the vector function $f(x)$, and the estimate*

$$\left\| \sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j) \right\|_{C(K)} = \begin{cases} O\left(\nu^{1/\alpha - 1} \ln \nu\right) & for \ \alpha \in (1, \infty), \\ O\left(\nu^{-1} \ln^2 \nu\right) & for \ \alpha = \infty, \end{cases} \tag{1.1}$$

is true as $\nu \to +\infty$.

Theorem 1.2. *Let the condition*

$$\langle Bu_n(x),\ f(x)\rangle\,\big|_0^{2\pi} = 0, \quad n = 1, 2, ..., \tag{1.2}$$

be fulfilled for the vector function $f(x) \in W^1_{p,2}(G)$, $p > 1$, *and for the system* $\{u_n(x)\}_{n=1}^{\infty}$ *and the coefficients* $p(x)$ *and* $q(x)$ *belong to* $L_\alpha(G)$, $\alpha > 1$. *Then the estimation*

$$\left\|\sigma_\nu^j(\cdot, f) - S_\nu(\cdot, f_j)\right\|_{C(K)} = \begin{cases} O\left(\nu^{1/\alpha - 1}\right) & for\ \ \alpha \in (1, \infty), \\ O\left(\nu^{-1}\ln\nu\right) & for\ \ \alpha = \infty, \end{cases} \tag{1.3}$$

is true as $\nu \to +\infty$.

2. Some auxiliary facts

Lemma 2.1. *([9]). If the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(G)$ *and the points* $x - t, x, x + t$ *lie on the domain* $\bar{G}$, *then the following formulas are valid:*

$$u_n(x \pm t) = \left(\cos\lambda_n t\cdot I \mp \sin\lambda_n t\cdot B\right)u_n(x) \pm \int_x^{x\pm t}\{\sin\lambda_n\left(t - |x-\xi|\right)I +$$

$$+ sign\,(\xi - x)\cos\lambda_n\left(t - |x - \xi|\right)B\}\,P(\xi)u_n(\xi)d\xi; \tag{2.1}$$

$$\frac{u_n(x-t) + u_n(x-t)}{2} = u_n(x)\cos\lambda_n t + \frac{1}{2}\int_{x-t}^{x+t}\{\sin\lambda_n\left(t - |x-\xi|\right)I +$$

$$+ sign\,(\xi - x)\cos\lambda_n\left(t - |x - \xi|\right)B\}\,P(\xi)u_n(\xi)d\xi, \tag{2.2}$$

where I *is a unit operator in* E^2.

Lemma 2.2. *Let the functions* $p(x)$ *and* $q(x)$ *belong to the class* $L_1(G)$. *Then there exist the constants* C_1 *and* C_2 *such that the inequalities*

$$\|u_n\|_{\infty,2} \le C_1, \quad n = 1, 2, ..., \tag{2.3}$$

$$\sum_{|t-\lambda_n|\le 1} 1 \le C_2, \quad \forall t \in R. \tag{2.4}$$

are valid.

The estimate (2.3) follows from Theorem 2 of [10], and the estimate (2.4) was proved in [11] (see Theorem 1.4 [11]).

Denote

$$T_n^1(r, R, \nu) = \int_r^R \frac{\sin\nu t}{t}\sin\lambda_n(t - r)dr;$$

$$T_n^2(r, R, \nu) = \int_r^R \frac{\sin\nu t}{t}\cos\lambda_n(t - r)dr;$$

$$\left\|T_n^j(\cdot, R, \nu)\right\|_{p,[0,R]} = \left\{\int_0^R \left|T_n^j(r, R, \nu)\right|^p dr\right\}^{1/p},$$

where $R_0/2 \le R \le R_0,\ 0 < r < R,\ \nu > 0,\ n \in N,\ j = 1,2;\ p \in [1,\infty],\ R_0$ is a sufficiently small positive number.

The following estimates are valid for the integrals $T_n^j(r,R,\nu),\ j = 1,2.$

Lemma 2.3. *The estimates*

$$\left|T_n^j(r,R,\nu)\right| \le C(\beta)\begin{cases} \left|\nu - |\lambda_n|\right|^{-\beta} r^{-\beta} & for\ \ \left|\nu - |\lambda_n|\right| \ge 1, \\ \max\{|\ln r|\,,\ |\ln R|\}\,, & for\ \ \left|\nu - |\lambda_n|\right| < 1, \end{cases} \tag{2.5}$$

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{\gamma,[0,R]} \le C(R_0)\begin{cases} \left|\nu - |\lambda_n|\right|^{-\frac{1}{\gamma}} & for\ \ \left|\nu - |\lambda_n|\right| \ge 1, \\ 1, & for\ \ \left|\nu - |\lambda_n|\right| < 1, \end{cases} \tag{2.6}$$

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{1,[0,R]} \le$$

$$\le C(R_0)\begin{cases} \left|\nu - |\lambda_n|\right|^{-1}\left(1 + \ln\left|\nu - |\lambda_n|\right|\right) & for\ \ |\nu - \lambda_n| \ge 1, \\ 1 & for\ \ \left|\nu - |\lambda_n|\right| < 1, \end{cases} \tag{2.7}$$

are valid for the integrals $T_n^j(r,R,\nu),\ j = 1,2;\ n \in N,$ *with any* $\beta \in (0,1]$ *where* $\gamma \in (1,\infty)$.

Proof. The estimates (2.5) and (2.6) were proved in [1]. Let us prove (2.7). Fix the number $R_0,\ 0 < R_0 \le \frac{1}{2}$ and consider the case $R_0\left|\nu - |\lambda_n|\right| \ge 2$. Then $\left|\nu - |\lambda_n|\right|^{-1} \le R_0/2 \le R$. By the triangle inequality, we have

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{1,[0,R]} \le \left\|T_n^j(\cdot,R,\nu)\right\|_{1,\left[0,\left|\nu-|\lambda_n|\right|^{-1}\right]} + \left\|T_n^j(\cdot,R,\nu)\right\|_{1,\left[\left|\nu-|\lambda_n|\right|^{-1},R\right]}\,.$$

Apply the estimate (2.5) to the first term on the right-hand side of this inequality with $\left|\nu - |\lambda_n|\right| \ge \frac{2}{R_0} \ge 1,\ \beta = \frac{1}{2}$, and to the second term with $\beta = 1$. Then we obtain

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{1,\left[0,\left|\nu-|\lambda_n|\right|^{-1}\right]} = O\left(\int\limits_0^{\left|\nu-|\lambda_n|\right|^{-1}} \left|\nu - |\lambda_n|\right|^{-\frac{1}{2}} r^{-\frac{1}{2}} dr\right) =$$

$$= O\left(\left|\nu - |\lambda_n|\right|^{-\frac{1}{2}}\right)\left(\left|\nu - |\lambda_n|\right|^{-\frac{1}{2}}\right) = O\left(\left|\nu - |\lambda_n|\right|^{-1}\right)\,;$$

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{1,\left[\left|\nu-|\lambda_n|\right|^{-1},R\right]} = O\left(\int\limits_{\left|\nu-|\lambda_n|\right|^{-1}}^{R} \left|\nu - |\lambda_n|\right|^{-1} r^{-1} dr\right) =$$

$$= O\left(\left|\nu - |\lambda_n|\right|^{-1}\right)\left(\int\limits_{\left|\nu-|\lambda_n|\right|^{-1}}^{R} r^{-1} dr\right) =$$

$$= O\left(\left|\nu - |\lambda_n|\right|^{-1}\right)\left(\ln R_0 + \ln\left|\nu - |\lambda_n|\right|\right) =$$

$$= O\left(\left|\nu - |\lambda_n|\right|^{-1}\right)\left(1 + \ln\left|\nu - |\lambda_n|\right|\right)\,.$$

If $1 \le \left|\nu - |\lambda_n|\right| < \frac{2}{R_0}$, then, applying inequality (2.5) with $\beta = \frac{1}{2}$, we get

$$\left\|T_n^j(\cdot,R,\nu)\right\|_{1,[0,R]} = O\left(\left|\nu - |\lambda_n|\right|^{-\frac{1}{2}}\right) R_0^{1/2} =$$

$$= O\left(\left|\nu - |\lambda_n|\right|^{-\frac{1}{2}}\right)\left(\frac{2}{\left|\nu - |\lambda_n|\right|}\right)^{1/2} = O\left(\left|\nu - |\lambda_n|\right|^{-1}\right)\,.$$

For $|\nu - |\lambda_n|| < 1$, the estimate (2.7) follows from (2.5) taking into account the integrability of the function $|\ln r|$ on $[0, R]$. So we get the validity of (2.7). □

Lemma 2.4. *If* $|\lambda_n| \geq 1$ *then the estimate*

$$|f_n| \leq \frac{C(f)}{|\lambda_n|} + \frac{1}{|\lambda_n|} \left|\left(B^* f', u_n\right)\right| + \frac{1}{|\lambda_n|} \left|(Pf, u_n)\right|, \tag{2.8}$$

is valid for the Fourier coefficients of the arbitrary vector function $f(x) \in W_{1,2}^1(G)$ *with respect to the system* $\{u_n(x)\}_{n=1}^{\infty}$, *where* $C(f)$ *is a positive constant.*

Proof. From the equation $Du_n = \lambda_n u_n$ for (u_n, f) we get

$$(u_n, f) = \int_0^{2\pi} \langle u_n(x), f(x)\rangle\, dx = \frac{1}{\lambda_n} \int_0^{2\pi} \langle Du_n(x), f(x)\rangle\, dx =$$

$$= \frac{1}{\lambda_n} \int_0^{2\pi} \left\langle Bu_n'(x), f(x)\right\rangle dx + \frac{1}{\lambda_n} \int_0^{2\pi} \langle P(x)u_n(x), f(x)\rangle\, dx\,.$$

Integrating by parts, from the last relation we obtain

$$(u_n, f) = \frac{1}{\lambda_n} \langle Bu_n(x), f(x)\rangle \,|_0^{2\pi} - \frac{1}{\lambda_n} \int_0^{2\pi} \left\langle Bu_n(x), f'(x)\right\rangle dx +$$

$$+\frac{1}{\lambda_n} \int_0^{2\pi} \langle P(x)u_n(x), f(x)\rangle\, dx = \frac{1}{\lambda_n} \left(\overline{f_1(x)} u_{n2}(x) - \overline{f_2(x)} u_{n1}(x)\right) |_0^{2\pi} -$$

$$-\frac{1}{\lambda_n} \int_0^{2\pi} \left\langle u_n(x), B^* f'(x)\right\rangle dx + \frac{1}{\lambda_n} \int_0^{2\pi} \langle u_n(x), P^*(x) f(x)\rangle\, dx, \tag{2.9}$$

where $B^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $P^*(x) = P(x)$, $u_n(x) = (u_{n1}(x), u_{n2}(x))^T$.

By (2.3), we have

$$\left|\left(\overline{f_1(x)} u_{n2}(x) - \overline{f_2(x)} u_{n1}(x)\right) |_0^{2\pi} \right| \leq$$

$$\leq C_1 \left(|f_1(2\pi)| + |f_1(0)| + |f_2(2\pi)| + |f_2(0)|\right) \equiv C(f)\,.$$

Using this inequality, from (2.9) we get the estimate (2.8). Lemma 2.4 is proved. □

Corollary 2.1. *If the vector function* $f(x) \in W_{1,2}^1(G)$ *and the system* $\{u_n(x)\}_{n=1}^{\infty}$ *satisfy the condition* (1.2), *then the following estimate is valid for the coefficients* $f_n = (f, u_n)$ *with* $|\lambda_n| \geq 1$:

$$|f_n| \leq |\lambda_n|^{-1} \left\{ \left|\left(B^* f', u_n\right)\right| + |(Pf, u_n)| \right\}. \tag{2.10}$$

In particular, (2.10) *is valid for the Fourier coefficients* f_n *of the vector function* $f(x) \in W_{1,2}^1(G)$ *satisfying the condition* $f(0) = f(2\pi) = 0$.

Lemma 2.5. $\{u_n(x)\}_{n=1}^{\infty}$ *is a Riesz system, i.e. the Riesz inequality*

$$\left(\sum_{n=1}^{\infty} |(f, u_n)|^q\right)^{1/q} \leq M_p \, \|f\|_{p,2} \, ,$$

holds for every vector function $f(x) \in L_p^2(G)$, $1 < p \leq 2$, *where* $M_p > 0$ *is a constant independent of* $f(x)$, $p^{-1} + q^{-1} = 1$.

The proof of Lemma 2.5 follows from the orthonormality and uniform boundedness of the system $\{u_n(x)\}_{n=1}^{\infty}$ (see (2.3)) and from the Riesz theorem [13. p. 154].

3. Proof of results

Proof of theorem 1.1. Let $f(x)$ be an arbitrary function from $W_{1,2}^1(G)$ and $p(x), q(x) \in L_\alpha(G)$. Fixing an arbitrary connected compact $K \subset G$, we choose the number R_0 satisfying the condition $0 \leq 4R_0 < \min\{1, dist(K, \partial G)\}$. Let us introduce the following vector function:

$$\tilde{S}_\nu(x, f) = \left(\tilde{S}_\nu(x, f_1), \, \tilde{S}_\nu(x, f_2)\right)^T ,$$

where $\tilde{S}_\nu(x, f_j)$, $j = 1, 2$, is is defined by the formula

$$\tilde{S}_\nu(x, f_j) = \frac{1}{\pi} \int\limits_{|x-y| \leq R} \frac{\sin \nu(x-y)}{x-y} f_j(y) dy, \quad j = 1, 2,$$

$$x \in K, \quad R \in [R_0/2, R_0] \, , \quad f(x) = (f_1(x), \, f_2(x))^T \, .$$

As the difference $S_\nu(x, f_j) - \tilde{S}_\nu(x, f_j)$ tends to zero uniformly with respect to $x \in K$ with $\nu \to +\infty$ and is of order $O\left(\nu^{-1}\right)$ (see [12]), it suffices to prove Theorem 1.1 for the partial sum $\tilde{S}_\nu(x, f)$.

As $W_{1,2}^1(G) \subset L_2^2(G)$ and the system $\{u_n(x)\}_{n=1}^{\infty}$ forms a basis for $L_2^2(G)$, every vector function $f(x) \in W_{1,2}^1(G)$ can be represented in the form

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x), \quad f_n = (f, u_n) \, ,$$

where the convergence of the series is to be understood in the norm of the space $L_2^2(G)$. Therefore, the partial sum $\tilde{S}_\nu(x, f)$ can be represented as

$$\tilde{S}_\nu(x, f) = \frac{2}{\pi} \sum_{n=1}^{\infty} (f, u_n) \int\limits_0^R \frac{\sin \nu t}{t} \frac{u_n(x-t) + u_n(x+t)}{2} dt \, .$$

Applying the mean value formula (2.2) and carrying out some calculations (see [1] for details), we get

$$\tilde{S}_\nu(x, f) - \sigma_\nu(x, f) = -\frac{1}{2} \sum_{|\lambda_n| = \nu} (f, u_n) u_n(x) + \sum_{n=1}^{\infty} (f, u_n) \, u_n(x) I(\nu, \lambda_n) +$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}(f,u_n)\int_0^R \{P(x+r)u_n(x+r)+P(x-r)u_n(x-r)\}\times$$

$$\times T_n^1(r,R,\nu)dr+\frac{1}{\pi}\sum_{n=1}^{\infty}(f,u_n)\times$$

$$\times\int_0^R \{P(x+r)u_n(x+r)-P(x-r)u_n(x-r)\}\,T_n^2(r,R,\nu)dr=$$

$$=\Phi_1(x,f)+\Phi_2(x,f)+\Phi_3(x,f)+\Phi_4(x,f)\,, \tag{3.1}$$

where $x\in K$, and the estimate

$$|I(\nu,\lambda_n)|\le C(R)\;(1+|\nu-|\lambda_n||)^{-1}\;. \tag{3.2}$$

is valid for the factors $I(\nu,\lambda_n)$ (see [2-3], [1])

Let us estimate the series $\Phi_j(x,f),\;\; j=\overline{1,4}$ in the metrics $C(K)$. To estimate the series $\Phi_1(x,f)$, we use Lemma 2.2 and the estimate (2.8):

$$\|\Phi_1(\cdot,f)\|_{C(K)}=\frac{1}{2}\left\|\sum_{|\lambda_n|=\nu}(f,u_n)u_n\right\|_{C(K)}\le$$

$$\le C_1\sum_{|\lambda_n|=\nu}|(f,u_n)|\le C_1C(f)\sum_{|\lambda_n|=\nu}|\lambda_n|^{-1}+$$

$$+C_1\left\|B^*f'\right\|_{1,2}\sum_{|\lambda_n|=\nu}|\lambda_n|^{-1}\|u_n\|_{\infty,2}+C_1\|Pf\|_{1,2}\sum_{|\lambda_n|=\nu}|\lambda_n|^{-1}\|u_n\|_{\infty,2}\le$$

$$\le C_1C(f)\nu^{-1}\left(\sum_{|\lambda_n|=\nu}1\right)+C_1^2\left\|B^*f'\right\|_{1,2}\nu^{-1}\left(\sum_{|\lambda_n|=\nu}1\right)+$$

$$+C_1^2\|Pf\|_{1,2}\,\nu^{-1}\left(\sum_{|\lambda_n|=\nu}1\right)\le$$

$$\le\left(C_1C_2C(f)+C_2C_1^2\left\|B^*f'\right\|_{1,2}+C_2C_1^2\|Pf\|_{1,2}\right)\nu^{-1}\le C_1(f)\nu^{-1}.$$

Consequently, the following estimate is true for the sum $\Phi_1(x,f)$

$$\|\Phi_1(\cdot,f)\|_{C(K)}=O\left(\nu^{-1}\right)\;. \tag{3.3}$$

Let us estimate the series $\Phi_2(x,f)$. To do so, we apply the inequalities (2.3), (3.2) and make use of the estimate (2.8) of Lemma 2.4. Then we have

$$\|\Phi_2(\cdot,f)\|_{C(K)}=\left\|\sum_{n=1}^{\infty}(f,u_n)u_n(x)I(\nu,\lambda_n)\right\|_{C(K)}\le$$

$$\le C(R)C_1(f)\sum_{|\lambda_n|\ge 1}^{\infty}|\lambda_n|^{-1}(1+|\nu-|\lambda_n||)^{-1}+$$

$$+C_1^2C(R)\|f\|_{1,2}\sum_{|\lambda_n|\le 1}(1+|\nu-|\lambda_n||)^{-1}\le$$

$$\leq C(R)C_1(f)\left\{\sum_{||\lambda_n|-\nu|\leq 1,\,|\lambda_n|\geq 1}|\lambda_n|^{-1}\left(1+|\nu-|\lambda_n||\right)^{-1}+\right.$$
$$+\sum_{\substack{1\leq||\lambda_n|-\nu|\leq\nu/2\\|\lambda_n|\geq 1}}|\lambda_n|^{-1}\left(1+|\nu-|\lambda_n||\right)^{-1}+$$
$$\left.+\sum_{\substack{||\lambda_n|-\nu|\geq\nu/2\\|\lambda_n|\geq 1}}|\lambda_n|^{-1}\left(1+||\lambda_n|-\nu|\right)^{-1}\right\}+C_1^2C(R)\nu^{-1}\|f\|_{1,2}\leq$$
$$\leq C(R)C_1(f)\left\{\sum_{||\lambda_n|-\nu|\leq 1,\,|\lambda_n|\geq 1}|\lambda_n|^{-1}+\sum_{1\leq||\lambda_n|-\nu|\leq\nu/2,|\lambda_n|\geq 1}|\lambda_n|^{-1}|\nu-|\lambda_n||^{-1}+\right.$$
$$\left.+\sum_{||\lambda_n|-\nu|\geq\nu/2,\,|\lambda_n|\geq 1}|\lambda_n|^{-1}\left(1+|\nu-|\lambda_n||\right)^{-1}\right\}+O\left(\nu^{-1}\right).$$

The first term inside curly braces has been already estimated above and is of order $O\left(\nu^{-1}\right)$. The two remaining series, by virtue of inequality (2.4), can be estimated as follows:

$$\sum_{1\leq||\lambda_n|-\nu|\leq\nu/2,|\lambda_n|\geq 1}|\lambda_n|^{-1}|\nu-|\lambda_n||^{-1}\leq\frac{2}{\nu}\sum_{1\leq||\lambda_n|-\nu|\leq\nu/2}||\lambda_n|-\nu|^{-1}\leq$$
$$\leq\frac{2}{\nu}\sum_{k=1}^{[\nu/2]}k^{-1}\left(\sum_{k\leq||\lambda_n|-\nu|\leq k+1}1\right)\leq\frac{2C_1}{\nu}\sum_{k=1}^{[\nu/2]}k^{-1}=O\left(\frac{\ln\nu}{\nu}\right);$$
$$\sum_{||\lambda_n|-\nu|\geq\nu/2,|\lambda_n|\geq 1}|\lambda_n|^{-1}\left(1+||\lambda_n|-\nu|\right)^{-1}=\sum_{1\leq|\lambda_n|\leq\nu/2}|\lambda_n|^{-1}\left(||\lambda_n|-\nu|+1\right)^{-1}+$$
$$+\sum_{|\lambda_n|\geq 3\nu/2}|\lambda_n|^{-1}\left(1+||\lambda_n|-\nu|\right)^{-1}\leq(1+\nu/2)^{-1}\sum_{1\leq|\lambda_n|\leq\nu/2}|\lambda_n|^{-1}+$$
$$+\sum_{|\lambda_n|\geq 3\nu/2}||\lambda_n|-\nu|^{-2}=O\left(\frac{\ln\nu}{\nu}\right)+O\left(\nu^{-1}\right)=O\left(\frac{\ln\nu}{\nu}\right).$$

Thus, the following estimate is true for the sum $\Phi_2(x,f)$

$$\|\Phi_2(\cdot,f)\|_{C(K)}=O\left(\frac{\ln\nu}{\nu}\right). \tag{3.4}$$

Now let us estimate the series $\Phi_3(x,f)$. By virtue of (2.3)

$$|\Phi_3(x,f)|=\frac{1}{\pi}\left|\sum_{n=1}^{\infty}(f,u_n)\int_0^R\left\{P(x+r)u_n(x+r)+\right.\right.$$
$$\left.\left.+\;P(x-r)u(_n(x-r)\right\}T_n^1(r,R,\nu)dr\right|\leq$$
$$\leq C\sum_{n=1}^{\infty}|(f,u_n)|\int_0^R Q(x,r)\left|T_n^1(r,R,\nu)\right|dr\,, \tag{3.5}$$

where $Q(x,r)=|p(x+r)|+|p(x-r)|+|q(x-r)|+|q(x+r)|$.

Introduce the following integrals

$$L^{\pm}(x)=\int\limits_0^R |p(x\pm r)|\,\left|T_n^1(r,R,\nu)\right|\,dr;\;\; M^{\pm}(x)=\int\limits_0^R |q(x\pm r)|\,\left|T_n^1(r,R,\nu)\right|\,dr\,.$$

By Hölder's inequality, we have

$$\left\|L^{\pm}\right\|_{C(K)}\le\|p\|_{\alpha}\left\|T_n^1\left(\cdot,R,\nu\right)\right\|_{\alpha',[0,R]},$$

where $1/\alpha+1/\alpha'=1$.

Using here lemma 2.3 (see estimations (2.6), (2.7)), we get

$$\left\|L^{\pm}\right\|_{C(K)}\le C(R_0)\,\|p\|_{\alpha}\times$$

$$\times\begin{cases}\left|\nu-|\lambda_n|\right|^{-1/\alpha'} \; for\;\; \left|\nu-|\lambda_n|\right|\ge1,\;\alpha\in(1,\infty)\,,\\ 1 \qquad\qquad for\;\; \left|\nu-|\lambda_n|\right|<1,\\ \left|\nu-|\lambda_n|\right|^{-1}\left(\ln\left|\nu-|\lambda_n|\right|+1\right),\; for\; \left|\nu-|\lambda_n|\right|\ge1,\;\alpha=\infty.\end{cases} \qquad (3.6)$$

Obviously, the same estimate holds for the integrals $M^{\pm}(x)$, i.e.

$$\left\|M^{\pm}\right\|_{C(K)}\le C(R_0)\,\|q\|_{\alpha}\times$$

$$\times\begin{cases}1\;\; for\;\; \left|\nu-|\lambda_n|\right|<1,\\ \left|\nu-|\lambda_n|\right|^{-1/\alpha'} \;\; for\;\; \left|\nu-|\lambda_n|\right|\ge1,\;\alpha\in(1,\infty)\,,\\ \left|\nu-|\lambda_n|\right|^{-1}\left(\ln\left|\nu-|\lambda_n|\right|+1\right)\; for\; \left|\nu-|\lambda_n|\right|\ge1,\;\alpha=\infty.\end{cases}\quad . \qquad (3.7)$$

Let $\alpha\in(1,\infty)$. Then from the inequality (3.5) taking into account the estimates (3.6) and (3.7) we get

$$\|\Phi_3(\cdot,f)\|_{C(K)}\le C\sum_{n=1}^{\infty}|(f,u_n)|\left\|\int\limits_0^R Q(\cdot,r)\,\left|T_n^1(r,R,\nu)\right|dr\right\|_{C(K)}\le$$

$$\le C(R_0)\left(\|p\|_{\alpha}+\|q\|_{\alpha}\right)\times$$

$$\times\left\{\sum_{\left|\nu-|\lambda_n|\right|<1}|(f,u_n)|+\sum_{\left|\nu-|\lambda_n|\right|\ge1}|(f,u_n)|\,\left|\nu-|\lambda_n|\right|^{-1/\alpha'}\right\} \qquad (3.8)$$

As $\nu>2$, the inequality $|\lambda_n|\ge1$ is true for λ_n satisfying $\left|\nu-|\lambda_n|\right|<1$. Therefore, Lemma 2.4 can be applied to estimate the first term inside curly braces. Then,

$$\sum_{\left|\nu-|\lambda_n|\right|<1}|(f,u_n)|\le C(f)\sum_{\left|\nu-|\lambda_n|\right|<1}|\lambda_n|^{-1}+C_1\left\|B^*f'\right\|_{1,2}\sum_{\left|\nu-|\lambda_n|\right|<1}|\lambda_n|^{-1}+$$

$$+C_1\left\|Pf\right\|_{1,2}\sum_{\left|\nu-|\lambda_n|\right|\le1}|\lambda_n|^{-1}\le$$

$$\le C_2\left(C(f)+C_1\left\|B^*f'\right\|_{1,2}+C_1\left\|Pf\right\|_{1,2}\right)\times$$

$$\times\left(\nu-1\right)^{-1}=O\left(\nu^{-1}\right),\;\;\nu\to+\infty\,. \qquad (3.9)$$

Let us divide the second term in curly braces on the right-hand side of (3.8)) into two sums, the first of them corresponding to the case $\left|\nu-|\lambda_n|\right|\ge1,\;\;|\lambda_n|<1$, and the second one corresponding to the case $\left|\nu-|\lambda_n|\right|\ge1,\;\;|\lambda_n|\ge1$.

In the first case, we apply the inequality $|(f, u_n)| \leq \|f\|_{2,2}$ to the coefficients (f, u_n), and in the second case we apply lemma 2.4 and use the estimate (2.4). Then we obtain the following inequalities:

$$\sum_{|\nu-|\lambda_n||\geq 1} |(f,u_n)|\, |\nu-|\lambda_n||^{-1/\alpha'} \leq \sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|<1}} |\nu-|\lambda_n||^{-1/\alpha'} \|f\|_{2,2} +$$

$$+\sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|\geq 1}} \left\{C(f)+C_1\left\|B^* f'\right\|_{1,2}+C_1\|Pf\|_{1,2}\right\} |\lambda_n|^{-1}|\nu-|\lambda_n||^{-1/\alpha'} \leq$$

$$\leq \sum_{|\lambda_n|<1} |\nu-|\lambda_n||^{-1/\alpha'} \|f\|_{2,2} + C_2(f) \sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|\geq 1}} |\lambda_n|^{-1}|\nu-|\lambda_n||^{-1/\alpha'} \leq$$

$$\leq (\nu-1)^{-1/\alpha'} \|f\|_{2,2} \left(\sum_{|\lambda_n|<1} 1\right) + C_2(f) \sum_{\substack{1\leq|\nu-|\lambda_n||\leq \nu/2\\ |\lambda_n|\geq 1}} |\lambda_n|^{-1}|\nu-|\lambda_n||^{-1/\alpha'} +$$

$$+C_2(f) \sum_{\substack{|\nu-|\lambda_n||\geq \nu/2\\ |\lambda_n|\geq 1}} |\lambda_n|^{-1}|\nu-|\lambda_n||^{-1/\alpha'} \leq C_2\|f\|_{2,2}\left(1-\nu^{-1}\right)^{-1/\alpha'}\nu^{-1/\alpha'}+$$

$$+2C_2(f)\nu^{-1} \sum_{\substack{1\leq|\nu-|\lambda_n||\leq \nu/2\\ |\lambda_n|\geq 1}} |\nu-|\lambda_n||^{-1/\alpha'} +$$

$$+C_2(f)\left\{\sum_{1\leq|\lambda_n|\leq \nu/2} |\lambda_n|^{-1}|\nu-|\lambda_n||^{-1/\alpha'} + \sum_{|\lambda_n|\geq 3\nu/2} |\lambda_n|^{-1}|\nu-|\lambda_n||^{-1/\alpha'}\right\} \leq$$

$$\leq O\left(\nu^{-1/\alpha'}\right)+O\left(\nu^{-1/\alpha'}\right)+O\left(\nu^{-1/\alpha'}\ln\nu\right)+O\left(\nu^{-1/\alpha'}\right)=O\left(\nu^{-1/\alpha'}\ln\nu\right).$$

Taking into account this estimate and the estimate (3.9), from (3.8) we obtain

$$\|\Phi_3(\cdot,f)\|_{C(K)} = O\left(\nu^{-1/\alpha'}\ln\nu\right), \quad \nu\to\infty. \tag{3.10}$$

The series $\Phi_4(x, f)$ is estimated in the same way, and the estimate (3.10) is valid for it. Thus, the estimate (3.10) is valid for the series $\Phi_3(x, f)$ and $\Phi_4(x, f)$ when $\alpha \in (1, \infty)$.

Consider the case $\alpha = \infty$. Applying the estimates (3.6) and (3.7) for $\alpha = \infty$ we get (see (3.8), (3.9))

$$\|\Phi_3(\cdot,f)\|_{C(K)} \leq C(R_0)\,(\|p\|_\infty+\|q\|_\infty)\left\{\sum_{|\nu-|\lambda_n||<1} |(f,u_n)| + \right.$$

$$\left.+\sum_{|\nu-|\lambda_n||\geq 1} |(f,u_n)|\,|\nu-|\lambda_n||^{-1}\ln(1+|\nu-|\lambda_n||)\right\} \leq$$

$$\leq C(R_0)\,(\|p\|_\infty+\|q\|_\infty)\,O\left(\nu^{-1}\right)+C(R_0)\,(\|p\|_\infty+\|q\|_\infty)$$

$$\sum_{|\nu-|\lambda_n||\geq 1} |(f,u_n)|\,|\nu-|\lambda_n||^{-1}\ln(1+|\nu-|\lambda_n||). \tag{3.11}$$

Let us prove that the second term on the right-hand side of (3.11) is of order $O\left(\nu^{-1}\ln^2\nu\right),\ \ \nu\to+\infty$. For this aim, we divide this sum into two sums, apply the inequality $|\,(f,u_n)\,|\le\|f\|_{2,2}$ in the case $|\lambda_n|<1,\ |\nu-|\lambda_n||\ge 1$ and lemma 2.4 in the case $|\lambda_n|\ge 1,\ \ |\nu-|\lambda_n||\ge 1:$

$$\sum_{|\nu-|\lambda_n||\ge 1}|(f,u_n)|\;|\nu-|\lambda_n||^{-1}\ln\left(1+|\nu-|\lambda_n||\right)\le$$

$$\le\sum_{\substack{|\lambda_n|<1\\ |\nu-|\lambda_n||\ge 1}}|\nu-|\lambda_n||^{-1}\ln\left(1+|\nu-|\lambda_n||\right)\|f\|_{2,2}+$$

$$+C_2(f)\sum_{|\nu-|\lambda_n||\ge 1,\,|\lambda_n|\ge 1}|\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1}\ln\left(1+|\nu-|\lambda_n||\right)\le$$

$$\le(\nu-1)^{-1}\ln\nu\,\|f\|_{2,2}\left(\sum_{|\nu-|\lambda_n||<1}1\right)+C_2(f)\sum_{\substack{1\le|\nu-|\lambda_n||\le\nu/2\\ |\lambda_n|\ge 1}}(\cdot)+$$

$$+C_2(f)\sum_{\substack{|\nu-|\lambda_n||\ge\nu/2\\ |\lambda_n|\ge 1}}(\cdot)\le O\left(\frac{\ln\nu}{\nu}\right)+2C_2(f)\nu^{-1}\sum_{1\le|\nu-|\lambda_n||\le\nu/2}|\nu-|\lambda_n||^{-1}$$

$$\ln\left(1+|\nu-|\lambda_n||\right)+C_2(f)\left\{\sum_{1\le|\lambda_n|\le\nu/2}|\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1}\ln\left(1+|\nu-|\lambda_n||\right)+\right.$$

$$\left.+\sum_{|\lambda_n|\ge 3\nu/2}|\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1}\ln\left(1+|\nu-|\lambda_n||\right)\right\}\le$$

$$\le O\left(\frac{\ln\nu}{\nu}\right)+O\left(\frac{\ln^2\nu}{\nu}\right)+O\left(\frac{\ln\nu}{\nu}\right)\left(\sum_{1\le|\lambda_n|\le\nu/2}1\right)+$$

$$+C_2(f)\sum_{||\lambda_n|-\nu|\ge\nu/2}|\nu-|\lambda_n||^{-2}\ln\left(1+|\nu-|\lambda_n||\right)=O\left(\frac{\ln^2\nu}{\nu}\right).$$

Consequently, the following estimate is true for the series $\Phi_3(x,f)$ (as well as for the series $\Phi_4(x,f)$) when $\alpha=\infty$ (see (3.11)):

$$\|\Phi_3(\cdot,f)\|_{C(K)}=O\left(\frac{\ln^2\nu}{\nu}\right),\ \ \nu\to+\infty\,. \tag{3.12}$$

Now, taking into account the estimates (3.3), (3.4), (3.10) and (3.12) in (3.1), we get

$$\max_{x\in K}\left|\sigma_\nu(x,f)-\tilde S_\nu(x,f)\right|=\begin{cases}O\left(\nu^{\frac{1}{\alpha}-1}\ln\nu\right) & for\ \alpha\in(1,\infty),\\ O\left(\nu^{-1}\ln^2\nu\right) & for\ \ \alpha=\infty,\end{cases}$$

as $\nu\to+\infty$. Theorem 1.1 is completely proved.

Proof of theorem 1.2. Let the vector function $f(x)\in W^1_{p,2}(G)$, $p>1$ and the system $\{u_n(x)\}_{n=1}^{\infty}$ satisfy the condition (1.2), i.e.$\langle Bu_n(x),\ f(x)\rangle\,\big|_0^{2\pi}=0$. Let us estimate the difference $\tilde S_\nu(x,f)-\sigma_\nu(x,f),\ \ x\in K$ as $\nu\to+\infty$ (see (3.1)).

The series $\Phi_i(x,f),\ i=1,2$ have already been estimated in the proof of theorem 1.1, and the estimates (3.3) and (3.4), respectively, are true for them. It remains to estimate the series $\Phi_3(x,f)$ and $\Phi_4(x,f)$ under the conditions of theorem 1.2. Let $\alpha\in(1,\infty)$. Let us estimate the series $\Phi_3(x,f)$. By virtue of (3.8) and (3.9), to estimate $\Phi_3(x,f)$ it suffices to estimate the series

$$\sum_{|\nu-|\lambda_n||\geq 1,|\lambda_n|\geq 1} |(f,u_n)|\ |\nu-|\lambda_n||^{-1/\alpha'}\ ,\ \ \frac{1}{\alpha}+\frac{1}{\alpha'}=1. \tag{3.13}$$

Let us show that the series (3.13) is of order $O\left(\nu^{-\frac{1}{\alpha'}}\right)$. Due to Corollary 2.1 (see (2.10)), for the series (3.13) we have

$$\begin{aligned}
&\sum_{|\nu-|\lambda_n||\geq 1,|\lambda_n|\geq 1} |(f,u_n)|\ |\nu-|\lambda_n||^{-1/\alpha'}\leq\\
&\leq \sum_{|\nu-|\lambda_n||\geq 1,|\lambda_n|\geq 1} \left|(B^*f',u_n)\right|\ |\lambda_n|^{\ -1}\,|\nu-|\lambda_n||^{-1/\alpha'}+\\
&+\sum_{|\nu-|\lambda_n||\geq 1,|\lambda_n|\geq 1} |(Pf,u_n)|\ |\lambda_n|^{\ -1}\,|\nu-|\lambda_n||^{-1/\alpha'}\leq\\
&\leq \sum_{\substack{1\leq|\nu-|\lambda_n||\leq\nu/2\\ |\lambda_n|\geq 1}} \left|(B^*f',u_n)\right|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}+\\
&+\sum_{1\leq|\lambda_n|\leq\nu/2} \left|(B^*f',u_n)\right|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}+\\
&+\sum_{|\lambda_n|\geq 3\nu/2} \left|(B^*f',u_n)\right|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}+\\
&+\sum_{\substack{1\leq|\nu-|\lambda_n||\leq\nu/2\\ |\lambda_n|\geq 1}} |(Pf,u_n)|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}+\\
&+\sum_{1\leq|\lambda_n|\leq\nu/2} |(Pf,u_n)|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}\\
&+\sum_{|\lambda_n|\geq 3\nu/2} |(Pf,u_n)|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}\ .
\end{aligned} \tag{3.14}$$

As $f'(x)\in L_p^2(G)$, $p>1$, we have $B^*f'\in L_p^2(G)\subset L_\beta^2(G)$, where $\beta=\min\{p,2\}$. Therefore, we can apply the Riesz inequality to the function B^*f' (see Lemma 2.5). Thus, for $\beta'=\beta/(\beta-1)$ we have

$$\left(\sum_{n=1}^{\infty}\left|\left(B^*f',u_n\right)\right|^{\beta'}\right)^{1/\beta'}\leq M\left\|B^*f'\right\|_{\beta,2}. \tag{3.15}$$

Applying Hölder's inequality and (3.15) to the first three sums on the right-hand side of the inequality (3.14), we obtain

$$\sum_{\substack{1\leq|\nu-|\lambda_n||\leq\nu/2\\ |\lambda_n|\geq 1}} \left|(B^*f',u_n)\right|\ |\lambda_n|^{-1}\,|\nu-|\lambda_n||^{-1/\alpha'}\leq$$

$$\leq \left(\sum_{\substack{1\leq|\nu-|\lambda_n||\leq\nu/2 \\ |\lambda_n|\geq 1}} \left|(B^* f', u_n)\right|^{\beta'} \right)^{1/\beta'} \left(\sum_{\substack{1\leq|\nu-|\lambda_n||\leq\nu/2 \\ |\lambda_n|\geq 1}} |\lambda_n|^{-\beta} \left|\nu - |\lambda_n|\right|^{-\beta/\alpha'} \right)^{1/\beta} \leq$$

$$\leq 2M \left\|B^* f'\right\|_{\beta,2} \left(\sum_{1\leq|\nu-|\lambda_n||\leq\nu/2} \left|\nu - |\lambda_n|\right|^{-\beta/\alpha'} \right)^{1/\beta} \nu^{-1} =$$

$$= O\left(\nu^{-1}\right) \left(\sum_{k=1}^{[\nu/2]} k^{-\beta/\alpha'} \left(\sum_{k\leq|\nu-|\lambda_n||\leq k+1} 1 \right) \right)^{1/\beta} = O\left(\nu^{-1}\right) \left(\sum_{k=1}^{[\nu/2]} k^{-\beta/\alpha'} \right)^{1/\beta} =$$

$$= O\left(\nu^{-1}\right) \begin{cases} \ln^{1/\beta}\nu & for\ \ \beta(1-1/\alpha) = 1, \\ \nu^{1/\beta - 1/\alpha'} & for\ \ \beta(1-1/\alpha) < 1, \\ 1 & for\ \ \beta(1-1/\alpha) > 1; \end{cases}$$

$$\sum_{1\leq|\lambda_n|\leq\nu/2} \left|(B^* f', u_n)\right| |\lambda_n|^{-1} \left|\nu - |\lambda_n|\right|^{-1/\alpha'} \leq M \left\|B^* f'\right\|_{\beta,2} \times$$

$$\times \left(\sum_{1\leq|\lambda_n|\leq\nu/2} |\lambda_n|^{-\beta} \left|\nu - |\lambda_n|\right|^{-\beta/\alpha'} \right)^{1/\beta} \leq M \left\|B^* f'\right\|_{\beta,2} \nu^{-1/\alpha'} \times$$

$$\times \left(\sum_{1\leq|\lambda_n|\leq\nu/2} |\lambda_n|^{-\beta} \right)^{1/\beta} = O\left(\nu^{-1/\alpha'}\right);$$

$$\sum_{|\lambda_n|\geq 3\nu/2} \left|(B^* f', u_n)\right| |\lambda_n|^{-1} \left|\nu - |\lambda_n|\right|^{-1/\alpha'} \leq M \left\|B^* f'\right\|_{\beta,2} \times$$

$$\times \left(\sum_{|\lambda_n|\geq 3\nu/2} |\lambda_n|^{-\beta} \left|\nu - |\lambda_n|\right|^{-\beta/\alpha'} \right)^{1/\beta} = O\left(\nu^{-\beta(1+1/\alpha')+1}\right)^{1/\beta} =$$

$$= O\left(\nu^{-1-1/\alpha'+1/\beta}\right) = O\left(\nu^{-1/\alpha'}\right).$$

Consequently, the sum of the first three series on the right-hand side of (3.14) is of order $O\left(\nu^{-1/\alpha'}\right)$.

It can be similarly proved that the sum of the last three series in (3.14) is also of order $O\left(\nu^{-1/\alpha'}\right)$. In this case, as β we need to take the number $\min\{2, \alpha\}$ and take into account that the vector function Pf belongs to $L^2_\alpha(G)$.

Thus, $\|\Phi_3(\cdot, f)\|_{C(K)} = O\left(\nu^{-1/\alpha'}\right)$ as $\nu \to +\infty$. This estimate is also true for $\Phi_4(x, f)$.

Consequently, for $\alpha \in (1, \infty)$ we have the estimate

$$\left\|\sigma_\nu(\cdot, f) - \tilde{S}_\nu(\cdot, f)\right\|_{C(K)} = O\left(\nu^{-1/\alpha'}\right), \ \ \nu \to +\infty.$$

Now we consider the case $\alpha = \infty$ and estimate the series $\Phi_3(x, f)$ in the metrics $C(K)$ (the series $\Phi_4(x, f)$ is estimated in a similar way). For $\Phi_3(x, f)$, when $\alpha=\infty$, we already have the inequality (3.11). Therefore, it suffices to estimate the series

$$\sum_{|\nu-|\lambda_n||\geq 1} |(f,u_n)|\ |\nu-|\lambda_n||^{-1}\ln(1+|\nu-|\lambda_n||)\ .$$

Let us prove that this series is of order $O\left(\nu^{-1}\ln\nu\right)$. The sum of terms corresponding to the eigenvalues λ_n with $|\lambda_n|<1$ is of order $O\left(\nu^{-1}\ln\nu\right)$ (see the proof of theorem 1.1 for $\alpha=\infty$). For $|\lambda_n|\geq 1$ we apply Corollary 2.1 (see (2.10)). Then we have

$$\sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|\geq 1}} |(f,u_n)|\ |\nu-|\lambda_n||^{-1}\ln(1+|\nu-|\lambda_n||)\leq$$

$$\leq \sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|\geq 1}} \left|(B^*f',u_n)\right|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}+$$

$$+\sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|\geq 1}} |(Pf,u_n)|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}\leq$$

$$\leq \sum_{1\leq|\nu-|\lambda_n||\leq\nu/2,\,|\lambda_n|\geq 1} \left|(B^*f',u_n)\right|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}+$$

$$+\sum_{1\leq|\lambda_n|\leq\nu/2} \left|(B^*f',u_n)\right|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}+$$

$$+\sum_{|\lambda_n|\geq 3\nu/2} \left|(B^*f',u_n)\right|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}+$$

$$+\sum_{1\leq|\nu-|\lambda_n||\leq\nu/2,\,|\lambda_n|\geq 1} |(Pf,u_n)|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}+$$

$$+\sum_{1\leq|\lambda_n|\leq\nu/2} |(Pf,u_n)|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}+$$

$$+\sum_{|\lambda_n|\geq 3\nu/2} |(Pf,u_n)|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\lambda_n|\ |\nu-|\lambda_n||}\,. \tag{3.16}$$

The vector functions B^*f' and Pf belong to the spaces $L_p^2(G)$ and $L_2^2(G)$, respectively, because $f(x)\in W_{p,2}^1(G)$, $p>1$, and $p(x)$, $q(x)\in L_\infty(G)$. Therefore, for the vector function B^*f' we can apply the Riesz inequality with $\gamma=\max\{2,q\}$, $q=\frac{p}{(p-1)}$, and for the vector function Pf we can apply Bessel's inequality, i.e. the Riesz inequality with $\gamma=2$ (see lemma 2.5). Then, by virtue of the Hölder's, Riesz inequalities and (2.4), from (3.16) we obtain $\left(\gamma^{-1}+(\gamma')^{-1}=1\right)$:

$$\sum_{\substack{|\nu-|\lambda_n||\geq 1\\ |\lambda_n|\geq 1}} |(f,u_n)|\ \frac{\ln(1+|\nu-|\lambda_n||)}{|\nu-|\lambda_n||}\leq$$

$$\le M_{\gamma'}\left\|B^*f'\right\|_{\gamma',2}\left\{\left(\sum_{\substack{1\le|\nu-|\lambda_n||\le\nu/2\\|\lambda_n|\ge1}}\frac{\ln^{\gamma'}(1+|\nu-|\lambda_n||)}{|\lambda_n|^{\gamma'}|\nu-|\lambda_n||^{\gamma'}}\right)^{1/\gamma'}+\right.$$

$$\left.+\left(\sum_{1\le|\lambda_n|\le\nu/2}\frac{\ln^{\gamma'}(1+|\nu-|\lambda_n||)}{|\lambda_n|^{\gamma'}|\nu-|\lambda_n||^{\gamma'}}\right)^{1/\gamma'}+\left(\sum_{|\lambda_n|\ge3\nu/2}\frac{\ln^{\gamma'}(1+|\nu-|\lambda_n||)}{|\lambda_n|^{\gamma'}|\nu-|\lambda_n||^{\gamma'}}\right)^{1/\gamma'}\right\}+$$

$$+M_2\left\|Pf\right\|_{2,2}\left\{\left(\sum_{\substack{1\le|\nu-|\lambda_n||\le\nu/2\\|\lambda_n|\ge1}}\frac{\ln^{2}(1+|\nu-|\lambda_n||)}{|\lambda_n|^{2}|\nu-|\lambda_n||^{2}}\right)^{1/2}+\right.$$

$$\left.+\left(\sum_{1\le|\lambda_n|\le\nu/2}\frac{\ln^{2}(1+|\nu-|\lambda_n||)}{|\lambda_n|^{2}|\nu-|\lambda_n||^{2}}\right)^{1/2}+\left(\sum_{|\lambda_n|\ge3\nu/2}\frac{\ln^{2}(1+|\nu-|\lambda_n||)}{|\lambda_n|^{2}|\nu-|\lambda_n||^{1/2}}\right)^{1/2}\right\}\le$$

$$\le M_{\gamma'}\left\|B^*f'\right\|_{\gamma',2}\left\{\frac{2\ln(1+\nu)}{\nu}\left(\sum_{k=1}^{[\nu/2]}\frac{1}{k^{\gamma'}}\left(\sum_{k\le|\nu-|\lambda_n||\le k+1}1\right)\right)^{1/\gamma'}+\right.$$

$$+\frac{2\ln\nu}{\nu}\left(\sum_{k=1}^{[\nu/2]}\frac{1}{k^{\gamma'}}\left(\sum_{k\le|\lambda_n|\le k+1}1\right)\right)^{1/\gamma'}+$$

$$\left.+\left(\sum_{k=[\nu/2]}^{\infty}\frac{\ln^{\gamma'}(2+k)}{k^{2\gamma'}}\left(\sum_{k\le|\nu-|\lambda_n||\le k+1}1\right)\right)^{1/\gamma'}\right\}+$$

$$+M_2\left\|Pf\right\|_{2,2}\left\{\frac{2\ln(1+\nu)}{\nu}\left(\sum_{k=1}^{[\nu/2]}\frac{1}{k^{2}}\left(\sum_{k\le|\nu-|\lambda_n||\le k+1}1\right)\right)^{1/2}+\right.$$

$$+\frac{2\ln\nu}{\nu}\left(\sum_{k=1}^{[\nu/2]}\frac{1}{k^{2}}\left(\sum_{k\le|\lambda_n|\le k+1}1\right)\right)^{1/2}+$$

$$\left.+\left(\sum_{k=[\nu/2]}^{\infty}\frac{\ln^{2}(2+k)}{k^{4}}\left(\sum_{k\le|\nu-|\lambda_n||\le k+1}1\right)\right)^{1/2}\right\}=O\left(\frac{\ln\nu}{\nu}\right).$$

Consequently, the estimate $\|\Phi_i(\cdot,f)\|_{C(K)}=O\left(\nu^{-1}\ln\nu\right),i=3,4,\ \nu\to+\infty$ holds for $\alpha=\infty$. Combining this with the estimates (3.3) and (3.4), from (3.1) we get

$$\left\|\tilde{S}_\nu(\cdot,f)-\sigma_\nu(\cdot,f)\right\|_{C(K)}=O\left(\nu^{-1}\ln\nu\right),\ \nu\to+\infty.$$

Theorem 1.2 is completely proved.

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