

LOCAL SOLVABILITY OF THE $\bar{\partial}$ -EQUATIONS ON l^2

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Abstract. In this work, we establish sufficient conditions for the existence of solutions of the Cauchy-Riemann equations on the unit ball B of the Hilbert space l^2 for a particular class of $\bar{\partial}$ -closed C^∞ smooth $(0,1)$ -forms ω . The used method is based on the expansion in Fourier series of the indefinitely differentiable functions on the closed unit ball of \mathbb{C}^N .

1. Introduction

This paper addresses a fundamental problem that arises in infinite dimensional complex analysis that concerns the solvability of inhomogeneous Cauchy-Riemann, or $\bar{\partial}$ -equations for $(0,1)$ -forms on Banach spaces.

Up to now precious little has been known about the solvability of the infinite equation

$$\bar{\partial}f = \omega, \quad (\bar{\partial}\omega = 0) \tag{1.1}$$

when ω is a $(0,1)$ -form, even on Hilbert space. However, we must mention some important results. First, Coeuré in [3] gave an example of $(0,1)$ -form ω on l^2 of class C^1 for which (1.1) is not solvable on any open set. No other example is known with ω of class C^p ($1 < p \leq \infty$). Second, L. Lempert gets local exactness on the space l^1 and on any Banach space when the forms are real analytical [1, 2].

In this work, we study the local solvability of $\bar{\partial}$ in the closed unit ball of l^2 denoted \bar{B} for a particular class of C^∞ smooth $(0,1)$ -forms of the type

$$\omega(z) = \sum_{k=1}^{\infty} \omega_k(z^k), \quad z = (z_i) \text{ in } l^2 \tag{1.2}$$

where $\mathbb{N} = \cup I_k$ is a partition of \mathbb{N} , ($\text{card.} I_k = N_k < +\infty$) with $z^k = (z_i)_{i \in I_k}$ standing for the projection of z on \mathbb{C}^{N_k} . We assume the following assumptions (H):

- i) Each form ω_k is indefinitely differentiable on the closed unit ball of \mathbb{C}^{N_k} provided with the norm of l^2 , and of the form

$$\omega_k(z^k) = \sum_{i \in I_k} z_i \omega_k^i(z^k) d\bar{z}_i.$$

- ii) The series (1.2) is supposed to be absolutely convergent.

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The used method is based on the expansion in Fourier series of the indefinitely differentiable functions on the closed unit ball of \mathbb{C}^N . According to ([5], Theorem 2.1) each function ω_k^i admits necessarily a Fourier expansion of the form

$$\omega_k^i(z^k) = \sum_{(\alpha, \beta) \in \mathbb{N}^{2N_k}} (z^k)^\alpha (\bar{z}^k)^\beta \omega_{k,(\alpha, \beta)}^i(|z^k|^2), \text{ for all } i \text{ and } k,$$

where $|z^k|^2 = (|z_i|^2)_{i \in I_k}$ and $(z^k)^\alpha = \prod_{i \in I_k} z_i^{\alpha_i}$.

In ([5, 6, 7]) the second author studied the local exactness of $\bar{\partial}$ for a restricted class of forms ω which respond moreover to the additional assumption (\tilde{H}) :

$$\omega_k^i(z^k) = \sum_{\alpha \in \mathbb{N}^{N_k}} (z^k)^\alpha \omega_{k,\alpha}^i(|z^k|^2), \text{ for all } i \text{ and } k,$$

and gets positive results in some particular cases, especially when the sequence (N_k) is bounded.

In this paper, in order to solve the $\bar{\partial}$ -equation for a large class of forms ω , we drop the hypothesis (\tilde{H}) , and we prove another positive result more general than the earlier ones. The main result in this work is

Theorem 1.1 (Main Theorem). *Let ω be a closed $(0, 1)$ -form of class C^∞ on \bar{B} according to the type $\omega = \sum_k \omega_k$ satisfying (H) and one of the following assumptions.*

- (1) *There exists a positive integer n such that the coefficients $\omega_{k,(\alpha, \beta)}^i$ are null if $(|\alpha| - |\beta|) > n$, for all k and all i in I_k , and the derivatives $D^m \omega_k^i$ are uniformly bounded in i and k on B for $0 \leq m \leq n$.*
- (2) *There exists a real number $\lambda > 0$ such that the coefficients $\omega_{k,(\alpha, \beta)}^i$ are null if $(|\alpha| - |\beta|) < \lambda N_k$, for all k and all i in I_k , and the derivatives $D^m \omega_k^i$ are uniformly bounded in i and k on B for $0 \leq m \leq 2$.*

Then there exists a real number $r > 0$, a positive integer M and a function F of class C^∞ on the ball of radius r that satisfies the equation $\bar{\partial}F = \omega$ such that

$$|F(z)| \leq C \sup_{\substack{i, k \\ 0 \leq m \leq M}} \|D^m \omega_k^i\|_\infty \text{ for } \|z\| < r,$$

where C is a constant and D designates the differentiation operator.

2. Preliminaries

2.1. Notations. In this work, unless indicated otherwise, $\| \cdot \|$ will denote the l^2 -norm on l^2 or on \mathbb{C}^N : if $z = (z_i) \in l^2$ or \mathbb{C}^N , $\|z\| = \sum |z_i|^2$. $B(r)$ resp. $B_N(r)$ will denote the ball $\|z\| < r$ in l^2 resp. \mathbb{C}^N . When $r = 1$, we simply write B resp. B_N for $B(1)$ resp. $B_N(1)$.

In the sequel α denote a multiindex. A multiindex $\alpha = (\alpha_i)_{i=1}^\infty$ for us is a sequence of integers $\alpha_i \geq 0$ with $\alpha_i = 0$ for i large enough. The length of α is $|\alpha| = \sum_{i=1}^\infty \alpha_i$. We let $\alpha! = \prod_{i=1}^\infty \alpha_i!$, where the usual convention $0! = 1$ is observed. For a sequence of complex numbers $z = (z_i)_{i=1}^\infty$, we put $z^\alpha = \prod_{i=1}^\infty z_i^{\alpha_i}$, where 0^0 is defined to be 1.

If z and w are in \mathbb{C}^N , we denote:

$$z'_i = (z_1, \dots, z_i); \quad z''_i = (z_i, \dots, z_N) \quad (i = 1, \dots, N)$$

$$|z|^2 = (|z_1|^2, \dots, |z_N|^2), \quad zw = (z_1 w_1, \dots, z_N w_N).$$

If f is in $C^\infty(\bar{B}_N)$, then for each $m \in \mathbb{N}$, we put

$$\|D^m f\|_\infty = \sup_{z \in \bar{B}_N} \|D^m f(z)\|,$$

where $\|D^m f(z)\|$ is the operator norm of the m th Fréchet derivative $D^m f$ of f . The norm $\|D^m \omega\|_\infty$ of a C^∞ smooth $(0,1)$ -form ω on \bar{B}_N is defined by

$$\|D^m \omega\|_\infty = \sup_{(z,h) \in \bar{B}_N^2} \|D^m f(z, h)\|,$$

where $f(z, h) = \omega(z)h$ for $z, h \in \bar{B}_N$.

2.2. Auxiliary results. To prove Theorem 1.1, we require some preliminary results.

If $z = (z_i)_{i=1}^\infty$ is in the unit ball of l^2 , we put

$$P_n(z) = \sum_{|\alpha| \geq n} \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} z^\alpha, \quad n \in \mathbb{N},$$

where $\alpha^{\frac{\alpha}{2}} = \prod_{i=1}^\infty \alpha_i^{\frac{\alpha_i}{2}}$.

Lemma 2.1. *There is a constant $Q > 1$ such that if $z \in B_N(r)$ with $r < \frac{1}{Q}$, then*

$$|P_n(z)| \leq (Qr)^n 2^N.$$

for every $n \in \mathbb{N}$.

Proof. Let us consider in \mathbb{C} the entire function $g(z) = \sum_{\alpha \geq 0} \frac{z^\alpha}{\alpha^{\alpha/2}}$. For every $\epsilon \in]0, \frac{1}{2}[$, we have

$$|g(z)| \leq \sum_{\alpha \geq 0} \frac{|z|^\alpha}{\epsilon^{\alpha/2} \sqrt{\alpha!}} \epsilon^{\alpha/2}.$$

Using Cauchy-Schwarz inequality, we obtain

$$|g(z)| \leq \left(\sum_{\alpha \geq 0} \frac{|z|^{2\alpha}}{\epsilon^\alpha \alpha!} \right)^{1/2} \left(\sum_{\alpha \geq 0} \epsilon^\alpha \right)^{1/2}.$$

Let $q \in \mathbb{N}$, and $z \in \mathbb{C}^N$, we observe that the series $\sum_{|\alpha|=q} \frac{z^\alpha}{\alpha^{\alpha/2}}$ is the homogeneous component of degree q of the product $g(z_1) \cdots g(z_N)$. It follows, when $z \in B_N(\sqrt{q})$, the majorization

$$\left| \sum_{|\alpha|=q} \frac{z^\alpha}{\alpha^{\alpha/2}} \right| \leq 2^{N/2} \exp\left(\frac{q}{2\epsilon}\right).$$

By homothety on the ball of radius r , we get

$$\left| \sum_{|\alpha|=q} \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} z^\alpha \right| \leq r^q \exp\left(\frac{q}{2\epsilon}\right) 2^{N/2}.$$

Therefore, if r is sufficiently small, we get

$$|P_n(z)| \leq \left(2e^{1/2\epsilon_r}\right)^n 2^N.$$

□

Theorem 2.1. *If f is in $\mathcal{C}^\infty(\overline{B}_N)$, then it admits the Fourier series expansion*

$$f(z) = \sum_{(\alpha, \beta) \in \mathbb{N}^{2N}} z^\alpha \bar{z}^\beta f_{(\alpha, \beta)}(|z|^2), \quad (2.1)$$

The series (2.1) is normally convergent with its derivatives on \overline{B}_N ; the coefficients $f_{(\alpha, \beta)}$ are \mathcal{C}^∞ on the closed unit ball of \mathbb{R}^N provided with the l^1 -norm, that satisfy

$$\forall (\alpha, \beta) \in \mathbb{N}^{2N}; z^\alpha \bar{z}^\beta f_{(\alpha, \beta)}(|z|^2) = \int_{[0, 2\pi]^N} f(z e^{i\theta}) e^{-i(\alpha \cdot \theta)} e^{i(\beta \cdot \theta)} \frac{d\theta}{(2\pi)^N} \quad (2.2)$$

where $(\alpha \cdot \theta) = \sum_{i=1}^N \alpha_i \theta_i$, $e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_N})$, and $\frac{d\theta}{(2\pi)^N} = \frac{d\theta_1}{2\pi} \dots \frac{d\theta_N}{2\pi}$.

For the proof see ([5], Theorem 2.1).

3. Closure characterization

Let ω be a \mathcal{C}^∞ smooth $(0, 1)$ -form on the closed unit ball of \mathbb{C}^N according to the type

$$\omega(z) = \sum_{i=1}^N z_i \omega^i(z) d\bar{z}_i, \quad z = (z_i)_{i=1}^N \text{ in } \overline{B}_N$$

According to Theorem 2.1, for all i the function ω^i admits a Fourier series expansion in the form $\omega^i(z) = \sum_{(\alpha, \beta) \in \mathbb{N}^{2N}} z^\alpha \bar{z}^\beta \omega_{(\alpha, \beta)}^i(|z|^2)$, where the coefficients $\omega_{(\alpha, \beta)}^i$ are functions of class \mathcal{C}^∞ on the closed unit ball of \mathbb{R}^N , and the series is normally convergent with its derivatives on \overline{B}_N .

Put

$$\omega_{(\alpha, \beta)}(z) = \sum_{i=1}^N z_i z^\alpha \bar{z}^\beta \omega_{(\alpha, \beta)}^i(|z|^2) d\bar{z}_i$$

It is easy to check that ω is closed if and only if $\omega_{(\alpha, \beta)}$ is closed for every $(\alpha, \beta) \in \mathbb{N}^{2N}$.

Proposition 3.1. *Let ω be a \mathcal{C}^∞ smooth $(0, 1)$ -form on the closed unit ball of \mathbb{C}^N of the type $\omega(z) = \sum_{i=1}^N z_i \omega^i(z) d\bar{z}_i$. Then ω is $\bar{\partial}$ -closed on \overline{B}_N if and only if $\Phi_{(\alpha, \beta)}(t) = \sum_{i=1}^N t^\beta \omega_{(\alpha, \beta)}^i(t) dt_i$ is d -closed on the closed unit ball of \mathbb{R}^N for every $(\alpha, \beta) \in \mathbb{N}^{2N}$.*

Proof. ω is closed on \overline{B}_N if and only if

$$z_i z^\alpha \frac{\partial}{\partial \bar{z}_j} \left(\bar{z}^\beta \omega_{(\alpha, \beta)}^i(|z|^2) \right) = z_j z^\alpha \frac{\partial}{\partial \bar{z}_i} \left(\bar{z}^\beta \omega_{(\alpha, \beta)}^j(|z|^2) \right)$$

for every $(\alpha, \beta) \in \mathbb{N}^{2N}$ and $(i, j) \in \mathbb{N}^2$.

By multiplying the above equality by z^β , one obtains

$$z_i z^\alpha \frac{\partial}{\partial \bar{z}_j} \left(|z|^{2\beta} \omega_{(\alpha, \beta)}^i(|z|^2) \right) = z_j z^\alpha \frac{\partial}{\partial \bar{z}_i} \left(|z|^{2\beta} \omega_{(\alpha, \beta)}^j(|z|^2) \right)$$

If we put $t = |z|^2$, this implies

$$z_i z_j z^\alpha \frac{\partial}{\partial t_j} \left(|z|^{2\beta} \omega_{(\alpha, \beta)}^i(|z|^2) \right) = z_j z_i z^\alpha \frac{\partial}{\partial t_i} \left(|z|^{2\beta} \omega_{(\alpha, \beta)}^j(|z|^2) \right)$$

Thus, ω is $\bar{\partial}$ -closed on \bar{B}_N if and only if $\Phi_{(\alpha, \beta)}$ is d -closed on the closed unit ball of \mathbb{R}^N . \square

Now, we consider a closed $(0, 1)$ -form ω of class C^∞ on the closed unit ball of l^2 according to the type $\omega = \sum_k \omega_k$ and satisfying the assumption (H). For all $z \in \bar{B}$, all t in the closed unit ball of \mathbb{R}^{N_k} , and all $(\alpha, \beta) \in \mathbb{N}^{2N_k}$ we put

$$\begin{aligned} \omega_{k, (\alpha, \beta)}(z^k) &= \sum_{i \in I_k} z_i (z^k)^\alpha (\bar{z}^k)^\beta \omega_{k, (\alpha, \beta)}^i(|z^k|^2) d\bar{z}_i \\ \Phi_{k, (\alpha, \beta)}(t) &= \sum_{i \in I_k} t^\beta \omega_{k, (\alpha, \beta)}^i(t) dt_i \end{aligned} \quad (3.1)$$

Theorem 3.1. ω is closed on \bar{B} if and only if for all $k \geq 1$, and $(\alpha, \beta) \in \mathbb{N}^{2N_k}$

$$(z^k)^\beta \omega_{k, (\alpha, \beta)}(z^k) = \sum_{i \in I_k} z_i (z^k)^\alpha \frac{\partial \Omega_{k, (\alpha, \beta)}}{\partial t_i} (|z^k|^2) d\bar{z}_i, \quad (3.2)$$

where $\Omega_{k, (\alpha, \beta)}$ is an indefinitely differentiable function on the closed unit ball of \mathbb{R}^{N_k} .

Proof. ω_k is the restriction to \mathbb{C}^{N_k} of ω , thus ω is closed on \bar{B} if and only if ω_k is closed on \bar{B}_{N_k} for every k . By using Proposition 3.1, and Poincaré's lemma, there exists for all k and $(\alpha, \beta) \in \mathbb{N}^{2N_k}$, a function $\Omega_{k, (\alpha, \beta)}$ indefinitely differentiable on the closed unit ball of \mathbb{R}^{N_k} provided with the norm of l^1 such that

$$d\Omega_{k, (\alpha, \beta)} = \Phi_{k, (\alpha, \beta)}. \quad (3.3)$$

Now, (3.2) follows immediately from (3.1) and (3.3). \square

4. Solvability

Let ω be a $\bar{\partial}$ -closed C^∞ smooth $(0, 1)$ -form on the closed unit ball of l^2 according to the type $\omega = \sum_k \omega_k$ such that $\omega_k = \sum_{(\alpha, \beta) \in \mathbb{N}^{2N_k}} \omega_{k, (\alpha, \beta)}$, where

$$\omega_{k, (\alpha, \beta)}(z^k) = \sum_{i \in I_k} z_i (z^k)^\alpha (\bar{z}^k)^\beta \omega_{k, (\alpha, \beta)}^i(|z^k|^2) d\bar{z}_i,$$

and $\omega_{k, (\alpha, \beta)}$ is $\bar{\partial}$ -closed $(0, 1)$ -form of class C^∞ on the closed unit ball of \mathbb{C}^{N_k} .

For each $(\alpha, \beta) \in \mathbb{N}^{2N_k}$ we can solve the equation

$$\bar{\partial} U_{k, (\alpha, \beta)} = (z^k)^\beta \omega_{k, (\alpha, \beta)} \quad (4.1)$$

with $U_{k, (\alpha, \beta)} \in C^\infty(\bar{B}_{N_k})$, by putting

$$U_{k, (\alpha, \beta)}(z^k) = (z^k)^\alpha \Omega_{k, (\alpha, \beta)}(|z^k|^2).$$

This determines $U_{k,(\alpha,\beta)}$ up to a holomorphic term. To determine $U_{k,(\alpha,\beta)}$ unambiguously we assign to each $\alpha \in \mathbb{N}^{N_k}$ a fixed point $M_\alpha = (M_\alpha^i)_{i \in I_k}$ in the closed unit ball of \mathbb{R}^{N_k} such that $M_\alpha^i = 0$ if $\alpha_i = 0$ and require that $\Omega_{k,(\alpha,\beta)}(M_\alpha) = 0$.

Lemma 4.1. *Let $\Omega_{k,(\alpha,\beta)}$ be an antiderivative of $\Phi_{k,(\alpha,\beta)}$ that vanishes at the point M_α , then*

$$F_{k,(\alpha,\beta)}(z^k) = \frac{(z^k)^\alpha (\overline{z^k})^\beta}{|z^k|^{2\beta}} \Omega_{k,(\alpha,\beta)}(|z^k|^2)$$

define \mathcal{C}^∞ smooth function on \overline{B}_{N_k} that solves the equation $\overline{\partial} F_{k,(\alpha,\beta)} = \omega_{k,(\alpha,\beta)}$.

Proof. By applying Taylor formula to the function $\Omega_{k,(\alpha,\beta)}$ at the point M_α , and by using (3.3), it follows that

$$\Omega_{k,(\alpha,\beta)}(t) = t^\beta H_{k,(\alpha,\beta)}(t)$$

where $H_{k,(\alpha,\beta)}$ is \mathcal{C}^∞ smooth on the closed unit ball of \mathbb{R}^{N_k} . Therefore $F_{k,(\alpha,\beta)}$ is of class \mathcal{C}^∞ on \overline{B}_{N_k} .

Since $U_{k,(\alpha,\beta)}(z^k) = (z^k)^\beta F_{k,(\alpha,\beta)}(z^k)$, it follows from (4.1) that

$$\overline{\partial} F_{k,(\alpha,\beta)} = \omega_{k,(\alpha,\beta)}.$$

□

Remark 4.1. Suppose that for some $M \geq 1$, the derivatives $D^m \omega_k^i$ are uniformly bounded in i and k on the unit ball of l^2 for $0 \leq m \leq M$, and put $F_k = \sum_{(\alpha,\beta) \in \mathbb{N}^{2N_k}} F_{k,(\alpha,\beta)}$. Then, by applying ([3], Appendix 3, Lemma 5), for some $0 < r < 1$, the function $F = \sum_{k=1}^\infty F_k$ define a \mathcal{C}^∞ -smooth solution of the equation $\overline{\partial} F = \omega$ on $B(r)$ if there exists $C > 0$, such that

$$|F_k(z^k)| \leq C \sup_{\substack{i,k \\ 0 \leq m \leq M}} \|D^m \omega_k^i\|_\infty < +\infty$$

for every $z \in B(r)$ and $k \geq 1$.

From this observation the problem is reduced to the existence of an antiderivative $\Omega_{k,(\alpha,\beta)}$ such that the series F_k converge and satisfies an estimate independent of the dimension N_k . We give a positive response for two particular cases.

The polynomial case. Let us choose for $\Omega_{k,(\alpha,\beta)}$ the integral of $\Phi_{k,(\alpha,\beta)}$ along a path connecting the origin to $|z^k|^2$ inside the unit ball of \mathbb{R}^{N_k} provided with the l^1 -norm, then we get the following result

Proposition 4.1. *If there exists a positive integer M such that the coefficients $\omega_{k,(\alpha,\beta)}^i$ are null if $(|\alpha| - |\beta|) > M$, for all k and all i in I_k , and if the derivatives $D^m \omega_k^i$ are uniformly bounded in i and k on B for $0 \leq m \leq M$. then the series F_k converge and define indefinitely differentiable functions on \overline{B}_{N_k} that satisfies*

$$|F_k(z^k)| \leq C \|z^k\|^2 \sup_{m \leq M} \|D^m \omega\|_\infty < +\infty,$$

where C is a constant independent of k .

Proof. By applying Lemma 4.1, we have

$$\begin{aligned} F_k(z^k) &= \sum_{q=-\infty}^M \sum_{|\alpha|-|\beta|=q} \int_0^1 (z^k)^\alpha (\bar{z}^k)^\beta u^{|\beta|} \sum_{i \in I_k} \omega_{k,(\alpha,\beta)}^i(u|z^k|^2) \cdot |z_i|^2 du \\ &= \sum_{q=-\infty}^M \int_0^1 \int_0^{2\pi} \frac{1}{(\sqrt{u})^{q+1}} \langle \omega_k(\sqrt{u}z^k e^{i\theta}), z^k \rangle e^{-iq\theta} \frac{d\theta}{2\pi} du, \end{aligned}$$

where $\langle \omega_k(\sqrt{u}z^k e^{i\theta}), z^k \rangle = \omega_k(\sqrt{u}z^k e^{i\theta})z^k = \sum_{i \in I_k} |z_i|^2 \omega_k^i(\sqrt{u}z^k e^{i\theta})$.

By making q integrations by parts relatively to θ in each term of the above sum when $1 \leq q \leq M$, we obtain

$$|F_k(z^k)| \leq C \|z^k\|^2 \sup_{m \leq M} \|D^m \omega\|_\infty,$$

where C is a constant independent of k , and hence

$$|F(z)| \leq C \|z\|^2 \sup_{m \leq M} \|D^m \omega\|_\infty.$$

□

Thus we have proved the first part of Theorem 1.1.

Non-polynomial case. Let ω be a $\bar{\partial}$ -closed C^∞ smooth $(0,1)$ - form on the closed unit ball of \mathbb{C}^N according to the type

$$\omega(z) = \sum_{i=1}^N z_i \omega^i(z) d\bar{z}_i, \quad z = (z_i)_{i=1}^N \text{ in } \bar{B}_N$$

We recall that ω^i admits a Fourier series expansion of the type

$$\omega^i(z) = \sum_{(\alpha,\beta) \in \mathbb{N}^{2N}} z^\alpha \bar{z}^\beta \omega_{(\alpha,\beta)}^i(|z|^2).$$

We also recall that $\Phi_{(\alpha,\beta)}$ designates the closed form in \mathbb{R}^N defined by

$$\Phi_{(\alpha,\beta)}(t) = \sum_{i=1}^N t^\beta \omega_{(\alpha,\beta)}^i(t) dt_i.$$

The anti-derivates of $\Phi_{(\alpha,\beta)}$ is given by $\Omega_{(\alpha,\beta)}(|z|^2) = \int_\gamma \Phi_{(\alpha,\beta)}$, where the path γ defined below joins the point $|z|^2$ to a fixed point of the closed unit ball of \mathbb{R}_+^N . We can take the function

$$F(z) = \sum_{(\alpha,\beta) \in \mathbb{N}^{2N}} \frac{z^\alpha \bar{z}^\beta}{|z|^{2\beta}} \Omega_{(\alpha,\beta)}(|z|^2)$$

for a $\bar{\partial}$ - anti-derivate of ω conditioned by its series convergence.

If we suppose that there exists a real number $\lambda > 0$ such that the coefficients $\omega_{(\alpha,\beta)}^i$ are null if $(|\alpha| - |\beta|) < \lambda N$, then we shall prove that for some $0 < r < 1$ and for a suitable choice of the path γ , the series F converge and define C^∞ -functions

on $B_N(r)$ with estimates independent of N .

Put for $\alpha \in \mathbb{N}^N$

$$M_\alpha = \begin{cases} 0 & \text{if } \alpha = 0 \\ \alpha/|\alpha| & \text{otherwise,} \end{cases}$$

Given $\|z\| < r < 1$, and $(\alpha, \beta) \in \mathbb{N}^{2N}$, we denote by γ the union of the adjacent segments $[M^m, M^{m+1}]$ ($m = 0, 1, 2$), connecting $M^0 = |z|^2$ to the point $M^3 = M_\alpha$ with $M^1 = \frac{1}{r^2}|z|^2$ and

$$M_i^2 = \begin{cases} \frac{1}{r^2}|z_i|^2 & \text{if } \alpha_i \neq 0 \\ r^2|z_i|^2 & \text{if } \alpha_i = 0 \end{cases}$$

In the sequel, we shall use the majorization of the following lemma.

Lemma 4.2. *Let n be a natural number, then for any $z \in B_N(r)$, we have*

$$\sum_{|\alpha|=n} \int_{M^2}^{M^3} \frac{|z|^\alpha}{t^{\frac{\alpha}{2}}} dt \leq (Qr)^n 2^N. \quad (4.2)$$

where Q is the constant of Lemma 2.1.

Proof. Let $[0, 1] \ni u \mapsto t(u) = (1-u)M^2 + uM^3$ be the parametrization of the segment $[M^2, M^3]$, we can write

$$\int_{M^2}^{M^3} \frac{|z|^\alpha}{t^{\frac{\alpha}{2}}} dt = \int_0^{\frac{1}{2}} \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du + \int_{\frac{1}{2}}^1 \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du.$$

When u describes $[0, \frac{1}{2}]$, we observe that $(t(u))^{\frac{\alpha}{2}} \geq \frac{|z|^\alpha}{(\sqrt{2r})^{|\alpha|}}$. Therefore

$$\int_0^{\frac{1}{2}} \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du \leq \frac{1}{2} (\sqrt{2r})^{|\alpha|} \leq (\sqrt{2r})^{|\alpha|}. \quad (4.3)$$

On the other hand, we have the inequality $t(u) \geq \frac{\alpha}{2|\alpha|}$ for every $u \in [\frac{1}{2}, 1]$. Hence

$$\int_{\frac{1}{2}}^1 \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du \leq \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} |2z|^\alpha. \quad (4.4)$$

From (4.3) and (4.4) we get the majorization

$$\sum_{|\alpha|=n} \int_{M^2}^{M^3} \frac{|z|^\alpha}{t^{\frac{\alpha}{2}}} dt \leq \sum_{|\alpha|=n} \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} \left(|2z|^\alpha + \left(2r^2 \frac{\alpha}{|\alpha|} \right)^{\frac{\alpha}{2}} \right).$$

If we choose r sufficiently small then an application of Lemma 2.1 implies the required estimate (4.2). \square

Remark 4.2. Following Proposition 4.1, we may suppose $|\alpha| - |\beta| \geq 2$.

The proof of the second part of Theorem 1.1 is a direct consequence of the forthcoming proposition.

Proposition 4.2. *Let ω be a $\bar{\partial}$ -closed C^∞ smooth $(0, 1)$ -form on \bar{B}_N according to the type $\omega(z) = \sum_{i=1}^N z_i \omega^i(z) d\bar{z}_i$ such that there exists a real number $\lambda > 0$ such that the coefficients $\omega_{(\alpha, \beta)}^i$ are null if $(|\alpha| - |\beta|) < \lambda N$. We assume furthermore that the derivatives $D^m \omega^i$ are uniformly bounded in i and k on B for $0 \leq m \leq 2$. Then there exists $0 < r < R < 1$ such that the series F converge and defines a C^∞ -smooth $\bar{\partial}$ -antiderivative of ω on $\bar{B}_N(r)$ for which*

$$|F(z)| \leq C (\|z\|^2 + R^N) \sup_{\substack{i \\ 0 \leq m \leq 2}} \|D^m \omega^i\|_\infty \quad (4.5)$$

where C is a constant independent of N .

Proof. For every $z \in B_N$, we put

$$F^m(z) = \sum_{(|\alpha| - |\beta|) \geq \lambda N} \frac{z^\alpha \bar{z}^\beta}{|z|^{2\beta}} \int_{M^m} \Phi_{(\alpha, \beta)}(t) dt, \quad m = 0, 1, 2.$$

According to Lemma 4.1, we have $F = F^0 + F^1 + F^2$, so it will be enough to prove that, for $m = 0, 1, 2$, the series F^m converges and satisfies an estimate independent of N .

Let us start with the case $m = 0$.

$$\begin{aligned} F^0(z) &= \sum_{(|\alpha| - |\beta|) \geq \lambda N} \int_1^{\frac{1}{r^2}} z^\alpha \bar{z}^\beta u^{|\beta|} \sum_{i=1}^N \omega_{(\alpha, \beta)}^i(u|z|^2) \cdot |z_i|^2 du \\ &= \sum_{q=\lambda N}^{+\infty} \int_1^{\frac{1}{r^2}} \int_0^{2\pi} \frac{1}{(\sqrt{u})^{q+1}} \langle \omega(\sqrt{u}ze^{i\theta}), z \rangle e^{-iq\theta} \frac{d\theta}{2\pi} du. \end{aligned}$$

By making two integrations by parts relatively to θ in each term of the above sum, we obtain

$$|F^0(z)| \leq C \|z\|^2 \sup_{m \leq 2} \|D^m \omega\|_\infty. \quad (4.6)$$

where C is a constant independent of N .

Now, let us consider the series F^1 .

Let J be a (possibly empty) subset of $\{1, 2, \dots, N\}$, we denote by A_J the set of multi-indices $(\alpha, \beta) \in \mathbb{N}^{2N}$ such that $\alpha_i = 0$ if and only if $i \in J$.

Note that A_J constitute a partition of \mathbb{N}^{2N} . Let

$$\omega_J = \sum_{i \in J} z_i \left[\sum_{(\alpha, \beta) \in A_J} z^\alpha \bar{z}^\beta \omega_{(\alpha, \beta)}^i(|z^k|^2) \right] d\bar{z}_i.$$

We have $F^1 = \sum_J F_J^1$, with

$$F_J^1(z) = \sum_{(\alpha, \beta) \in A_J} \int_{\frac{1}{r^2}}^{r^2} z^\alpha \bar{z}^\beta u^{|\beta|} \sum_{i \in J} \omega_{(\alpha, \beta)}^i(t(u)) \cdot |z_i|^2 du.$$

Note that if $J = \emptyset$ we have $F_J^1 = 0$, hence it is enough to consider $J \neq \emptyset$.

By a change of numerotation we may suppose that $J = \{\nu, \dots, N\}$ ($\nu \leq N$), thus

we can write

$$\begin{aligned}
F_J^1(z) &= \sum_{q-p \geq \lambda N} \sum_{|\alpha|=q} \sum_{|\beta|=p} \int_{\frac{1}{r^2}}^{r^2} z^\alpha \bar{z}^\beta u^p \sum_{i=\nu}^N \omega_{(\alpha,\beta)}^i \left(\frac{1}{r^2} |z'_{\nu-1}|^2, u |z'_\nu|^2 \right) \cdot |z_i|^2 du \\
&= \sum_{p=0}^{+\infty} \sum_{q=p+\lambda N}^{+\infty} \int_{\frac{1}{r^2}}^{r^2} \int_0^{2\pi} \int_0^{2\pi} r^q \sqrt{u}^p < \omega_J \left(\frac{1}{r} z'_{\nu-1} e^{i\theta}, \sqrt{u} z''_\nu e^{i\varphi} \right), z''_{\nu+1} > \\
&\quad \times e^{-i(q\theta-p\varphi)} \frac{dud\theta d\varphi}{4\pi^2}.
\end{aligned}$$

An easy computation shows that

$$r^p \int_{r^2}^{1/r^2} \sqrt{u}^p du \leq \frac{1}{r^2}.$$

By making two integrations by parts relatively to φ in each term of the above sum we are led to the majorization

$$|F_J^1(z)| \leq Cr^{\lambda N} \|z\|^2 \sup_{m \leq 2} \|D^m \omega\|_\infty$$

Hence

$$|F^1(z)| \leq C \|z\|^2 \sup_{m \leq 2} \|D^m \omega\|_\infty \quad (4.7)$$

where C is a constant independent of N .

It remains to consider the series F^2 , we have

$$\begin{aligned}
F^2(z) &= \sum_{q-p \geq \lambda N} \sum_{|\alpha|=q} \sum_{|\beta|=p} \int_0^1 z^\alpha \bar{z}^\beta (1-u)^p r^{2p} \sum_{i=1}^N \omega_{(\alpha,\beta)}^i(t(u)) t'_i(u) du \\
&= \sum_{p=0}^{+\infty} \sum_{q=p+\lambda N}^{+\infty} \int_0^1 \int_0^{2\pi} \frac{z^\alpha (1-u)^{\frac{p}{2}} r^p}{(t(u))^{\frac{\alpha}{2}}} \sum_{i=1}^N \omega^i(\sqrt{t(u)} e^{i\theta}) e^{-i(q-p)\theta} t'_i(u) \frac{dud\theta}{2\pi}
\end{aligned}$$

By applying Lemma 4.2 and by choosing r sufficiently small, we get

$$|F^2(z)| \leq CR^N \sup_i \|\omega^i\|_\infty \quad (4.8)$$

where $0 < R < 1$ and C are independent of N .

Now, the required estimate (4.5) follows immediately from (4.6), (4.7) and (4.8). \square

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