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AN APPLICATION OF THE INVERSE SCATTERING PROBLEM FOR THE DISCRETE DIRAC OPERATOR

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Abstract. The method of the inverse scattering problem is used to solve the Cauchy problem for the Langmuir lattice with initial data of step type. Formulas are obtained for transforming the scattering data with respect to the time, making it possible to obtain a solution of the problem for arbitrary moment with the aid of linear integral equations of scattering theory. The existence of a solution is proved.

1. Introduction

It is known that the method of the inverse scattering problem allows a detailed study of the Cauchy problem for some nonlinear evolution equations (see [1], [18],[19] and the literature therein). In works [2]-[7], [9]-[11], [17],[20] the Cauchy problem was studied for nonlinear difference equations with different initial data. In solving these problems, inverse spectral problems are used for the discrete Sturm-Liouville operator. On the other hand, the inverse scattering problem for the discrete Dirac operator was studied in [12]-[15]. However, as far as we know, applications of the latter problem to nonlinear equations have not been considered. In this paper, we apply the inverse scattering problem for some discrete Dirac operator to the Cauchy problem of the Langmuir lattice with a step-like initial condition.

It should be noted that the Langmuir lattice has important applications in plasma physics and in zoology. The Cauchy problems for this lattice in various classes of initial conditions were studied in [2],[11], [17],[18],[20].

For a sequence of positive functions $c_n = c_n(t), c_n \in C^{(1)}[0,\infty)$ we consider the Langmuir lattice

$$\dot{c}_n = c_n(c_{n-1} - c_{n+1}), \cdot = \frac{d}{dt}, n \in Z. \tag{1.1}$$

For (1.1) we pose the following Cauchy problem with the initial condition

$$c_n(0) = c_n^0, n \in Z, \tag{1.2}$$

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satisfying requirement

$$\sum_{n=-\infty}^{-1} |n| \{|c_n(0) - 1|\} + \sum_{n=1}^{\infty} |n| \{|c_n(0) - A|\} < \infty, \tag{1.3}$$

where $0 < A \leq 1$.

We will seek a solution to problem (1.1) and (1.2), such that

$$\left\| \sum_{n=-\infty}^{-1} (1 + |n|) |c_n(t) - 1| + \sum_{n=0}^{\infty} (1 + |n|) |c_n(t) - A| \right\|_{C[0,T]} < \infty \tag{1.4}$$

for all $T > 0$.

In this paper, based on the Gelfand-Levitan-Marchenko formalism, the problem (1.1)-(1.2) is studied. Explicit formulas are obtained that describe the evolution of scattering data. The unique solvability of problem (1.1)-(1.2) in the class of functions satisfying (1.4) is proved. We note that the asymptotic behavior of the so-called rapidly decreasing solution of problem (1.1)-(1.2) using the Riemann problem was studied in [20] .

2. THE UNIQUE SOLVABILITY OF PROBLEM (1.1) - (1.2)

Theorem 2.1. *Problem (1.1)-(1.2) has a unique solution in the class of functions satisfying (1.4), if the condition (1.3) is satisfied.*

Proof. Introduce the Banach space E of sequences $x = (x_n)_{n=-\infty}^{\infty}$ with the norm $\|x\|_E = \sum_{n=-\infty}^{\infty} (1 + |n|) |x_n|$. Then, the set $C([0,T]; E)$ of functions continuous on $[0,T]$ with the values in E is a Banach space with the norm $\|x(t)\|_{C([0,T];E)} = \max_{0 \leq t \leq T} \|x(t)\|_E$ (see, e.g.,[16]). Assume

$$x_n(t) = \begin{cases} c_n(t) - A, n \geq 0, \\ c_n(t) - 1, n < 0. \end{cases} \tag{2.1}$$

Then the problem (1.1)-(1.2) is equivalent to the problem

$$\begin{cases} \dot{x}_n = x_n(x_{n-1} - x_{n+1}) + A(x_{n-1} - x_{n+1}), n \geq 1, \\ \dot{x}_0 = x_0(x_{-1} - x_1) + A(x_{-1} - x_1) + (1 - A)x_0 + A(1 - A), \\ \dot{x}_{-1} = x_{-1}(x_{-2} - x_0) + (x_{-2} - x_{-1}) + (1 - A)x_0 + (1 - A), \\ \dot{x}_n = x_n(x_{n-1} - x_{n+1}) + (x_{n-1} - x_{n+1}), n < -1. \end{cases} \tag{2.2}$$

$$x_n(0) = x_n^0, n \in Z \tag{2.3}$$

where

$$x_n^0 = \begin{cases} c_n^0 - A, n \geq 0, \\ c_n^0 - 1, n < 0. \end{cases}$$

Rewrite problem (2.2) and (2.3) in the form of an operator equation:

$$x(t) = x(0) + \int_0^T \Psi(x(\tau))\, d\tau$$

where $x(t) = (x_n(t))_{n=-\infty}^{\infty}$ and Ψ is the operator generated by the right-hand side of system (2.2). It is obvious that, for each $T > 0$, the operator is a continuously differentiable mapping of the space $C([0, T]; E)$ into itself. Therefore, we can apply to the last equation the contraction mapping principle. As a result, we find that, in a certain interval $[0, \delta]$, problem (2.2) and (2.3) has a unique solution $x(t) = (x_n(t))_{n=-\infty}^{\infty}$ and with a finite norm $\|x(t)\|_{C([0,\delta];E)}$. Let us prove that this solution can be continued to the finite interval $[0, T]$. Assume the contrary. Then, there is a point $t_0 < T$ such that problem (2.2)-(2.3) has a solution $x(t) = (x_n(t))_{n=-\infty}^{\infty}$ continuous in the interval $(0, t_0)$, but $\overline{\lim\limits_{t\to t_0}} \|x(t)\|_E = +\infty$.

Using relations (2.1), we obtain

$$\begin{aligned}
&x_n(t) = x_n(0) + \int_0^t [x_n(\tau)(x_{n-1}(\tau) - x_{n+1}(\tau)) + \\
&+ A(x_{n-1}(\tau) - x_{n+1}(\tau))]\,d\tau, n \geq 1, \\
&x_0(t) = x_0(0) + A(1-A)t + \int_0^t [x_0(\tau)(x_{-1}(\tau) - x_1(\tau))| + \\
&+ A(x_{-1}(\tau) - x_1(\tau)) + (1-A)x_0(\tau)]\,d\tau, \\
&x_{-1}(t) = x_{-1}(0) + (1-A)t + \int_0^t [x_{-1}(\tau)(x_{-2}(\tau) - x_0(\tau)) + \\
&+ (x_{-2}(\tau) - x_{-1}(\tau)) + (1-A)x_{-1}(\tau)]\,d\tau, \\
&x_n(t) = x_n(0) + \int_0^t [x_n(\tau)(x_{n-1}(\tau) - x_{n+1}(\tau)) + \\
&+A(x_{n-1}(\tau) - x_{n+1}(\tau))]\,d\tau, n \leq -2.
\end{aligned} \tag{2.4}$$

It is known (see) that problem (1.1) and (1.2) has a unique solution bounded uniformly with respect to t in each finite interval, i.e., $|c_n(t)| < C, n \in Z, t \in [0, T]$. It follows that the problem (2.2)-(2.3) also has a unique solution $x_n(t)$ bounded uniformly with respect to t in each finite interval: $|x_n(t)| < M, n \in Z, t \in [0, T]$. Using the last fact, from relations (2.4) we have

$$\|x(t)\|_E \leq \|x(0)\|_E + (5M+1)T + (2M+2)\int_0^t \|x(\tau)\|_E\,d\tau, 0 < t < t_0.$$

By the Gronwall lemma, we find from the last inequality that

$$\|x(t)\|_E \leq [\|x(0)\|_E + (5M+1)T]\exp((2M+2)T),$$

which contradicts our assumption. Hence, problem (2.2)-(2.3) on the interval $[0, T]$ has a unique solution with a finite norm $\|x(t)\|_{C([0,T];E)}$. The theorem is proved. □

3. EVOLUTION OF THE SCATTERING DATA

Consider problem (1.1) - (1.2). Putting in equation (1.1)

$$c_{2n} = a_{1,n}^2, c_{2n-1} = a_{2,n}^2, a_{1,n} > 0, a_{2,n} < 0, n \in Z, \tag{3.1}$$

we obtain

$$\begin{aligned}
\dot{a}_{1,n} &= \tfrac{1}{2}a_{1,n}\left(a_{2,n-1}^2 - a_{2,n}^2\right) = \tfrac{d}{dt}, \\
\dot{a}_{2,n} &= \tfrac{1}{2}a_{2,n}\left(a_{1,n}^2 - a_{1,n+1}^2\right), n \in Z,
\end{aligned} \tag{3.2}$$

$$a_{1,n}(0)=\hat{a}_{1,n}, a_{2,n}(0)=\hat{a}_{2,n}, \tag{3.3}$$

where $\hat{a}_{1,n},\hat{a}_{2,n}$ satisfy the condition

$$\sum_{n=-\infty}^{-1}\{|n|\left(|\hat{a}_{1,n}-1|+|\hat{a}_{2,n}+1|\right)\}+\sum_{n=1}^{\infty}\{|n|\left(|\hat{a}_{1,n}-A|+|\hat{a}_{2,n}+A|\right)\}<\infty.$$

According to the results of the previous paragraph, problem (3.2)-(3.3) has a solution $a_{1,n}=a_{1,n}(t)$, $a_{2,n}=a_{2,n}(t)$ satisfying the condition

$$\begin{aligned}&\left\|\sum_{n=-\infty}^{-1}(1+|n|)\left\{|a_{1,n}(t)-1|+|a_{2,n}(t)+1|\right\}+\right.\\&\left.+\sum_{n=0}^{\infty}(1+|n|)\left\{|a_{1,n}(t)-A|+|a_{2,n}(t)+A|\right\}\right\|_{C[0,T]}<\infty\end{aligned} \tag{3.4}$$

for all $T>0$.

Now we consider a system of equations

$$\left\{\begin{array}{l}a_{1,n}y_{2,n}+a_{2,n}y_{2,n+1}=\lambda y_{1,n},\\ a_{1,n-1}y_{1,n-1}+a_{2,n}y_{1,n}=\lambda y_{2,n}, n\in Z,\end{array}\right. \tag{3.5}$$

in which the coefficients $a_{1,2},a_{2,n}$ depend on t and are a solution to problem (3.2)-(3.3) in class of functions satisfying (3.4). For definiteness, we assume that $A\leq 1$. For solving problem (3.2)-(3.3), we use the scattering theory developed in [13],[14] for the system of equations (3.5).

Let Γ_j be the complex λ- plane with the cut along segment $\left[-A^{2-j},A^{2-j}\right]$ and denote its boundary $j=1,2$. Consider the function

$$z_j=z_j(\lambda)=-\frac{\lambda^2-2A^{2(2-j)}}{2A^{2(2-j)}}+\frac{\lambda}{2A^{2-j}}\sqrt{\lambda^2-4A^{2(2-j)}},$$

where the regular branch of the radical is selected from the condition $\sqrt{\lambda^2-4A^{2(2-j)}}>0$ for $\lambda>2A^{2-j}, j=1,2$. As follows from [13],[14] the system of equations has solutions $\left\{f_{j,n}^{+}(\lambda,t)\right\}$ and $\left\{f_{j,n}^{-}(\lambda,t)\right\}$, $j=1,2$, representable in the form

$$\left.\begin{array}{l}f_{j,n}^{+}(\lambda,t)(\lambda,t)=\alpha_j^{+}(n,t)\left(\frac{Az_1-A}{\lambda}\right)^{2-j}z_1^n\left(1+\sum\limits_{m\geq 1}K_j^{+}(n,m,t)z_1^m\right),\\ f_{j,n}^{-}(\lambda,t)=\alpha_j^{-}(n,t)\left(\frac{z_2^{-1}-1}{\lambda}\right)^{2-j}z_2^{-n}\left(1+\sum\limits_{m\leq -1}K_j^{-}(n,m,t)z_2^{-m}\right), n\in Z,\end{array}\right\} \tag{3.6}$$

where the quantities $\alpha_J^{\pm}(n,t), K_j^{\pm}(n,m,t), j=1,2$ satisfy the relations

$$\left.\begin{array}{l}\alpha_j^{\pm}(n,t)=1+o(1)\, n\to\pm\infty, j=1,2,\\ K_j^{\pm}(n,m,t)=O\left(\sigma^{\pm}\left(n+\left[\frac{m}{2}\right]+\frac{1\mp 1}{2}\right)\right), n+m\to\pm\infty\end{array}\right\}$$

here $\sigma^{\pm}(n)=\sum\limits_{\pm m\geq\pm n}\left\{\left|a_{1,m}-A^{\frac{1\pm 1}{2}}\right|+\left|a_{2,m}+A^{\frac{1\pm 1}{2}}\right|\right\}$ and $[x]$ is the integer part of x. Substituting the representation (3.6) of the Jost solution in the system of equations (3.5), we have

$$\left.\begin{array}{l}\frac{a_{1,n}}{A^{\frac{1\pm1}{2}}}=\left(\frac{\alpha_2^{\pm}(n+1,t)}{\alpha_1^{\pm}(n,t)}\right)^{\pm1},\frac{a_{2,n}}{A^{\frac{1\pm1}{2}}}=-\left(\frac{\alpha_1^{\pm}(n,t)}{\alpha_2^{\pm}(n,t)}\right)^{\pm1},\\ \frac{a_{1,n}^2-A^{1\pm1}}{A^{1\pm1}}=\pm\left(K_2^{\pm}\left(n+\frac{1\mp1}{2},\pm1,t\right)-K_1^{\pm}\left(n+\frac{1\mp1}{2},\pm1,t\right)\right),\\ \frac{a_{2,n}^2-A^{1\pm1}}{A^{1\pm1}}=\pm\left(K_1^{\pm}\left(n-\frac{1\pm1}{2},\pm1,t\right)-K_2^{\pm}\left(n+\frac{1\mp1}{2},\pm1,t\right)\right),n\in Z\end{array}\right\}\quad(3.7)$$

These solutions are coupled on the continuous spectrum by the equalities

$$\begin{array}{l}f_{j,n}^{-}(\lambda,t)=a_1(\lambda,t)\overline{f_{j,n}^{+}(\lambda,t)}+b_1(\lambda,t)f_{j,n}^{+}(\lambda,t),\lambda\in\partial\Gamma_1,\lambda^2\neq4A^2,\\ f_{j,n}^{+}(\lambda,t)=a_2(\lambda,t)\overline{f_{j,n}^{-}(\lambda,t)}+b_2(\lambda,t)f_{j,n}^{-}(\lambda,t),\lambda\in\partial\Gamma_2,\lambda^2\neq4,\end{array}\quad(3.8)$$

where the coefficients $a_j(\lambda,t), b_j(\lambda,t), j=1,2$ are determined by the formulas

$$\left.\begin{array}{l}a_1(\lambda,t)=\frac{\lambda W\left[f_{j,n}^{+}(\lambda,t),f_{j,n}^{-}(\lambda,t)\right]}{A^2\left(z_1-z_1^{-1}\right)}\\ b_1(\lambda,t)=\frac{\lambda W\left[\overline{f_{j,n}^{+}(\lambda,t)},f_{j,n}^{-}(\lambda,t)\right]}{A^2\left(z_1^{-1}-z_1\right)}\\ a_2(\lambda,t)=\frac{\lambda W\left[f_{j,n}^{+}(\lambda,t),f_{j,n}^{-}(\lambda,t)\right]}{\left(z_2-z_2^{-1}\right)}\\ b_2(\lambda,t)=\frac{\lambda W\left[f_{j,n}^{+}(\lambda,t),\overline{f_{j,n}^{-}(\lambda,t)}\right]}{\left(z_2^{-1}-z_2\right)}\end{array}\right\}$$

The coefficients $a_1(\lambda,t), a_2(\lambda,t)$ are limiting values of functions analytic in the plane Γ_2, wherein they have a finite number of simple zeros $\lambda_k=\pm\mu_k=\pm\mu_k(t), \mu_k>0, k=1,...,N$. Assume

$$\left(m_k^{+}(t)\right)^{-2}=\sum_{n\in Z}\left\{\left|f_{1,n}^{+}(\pm\mu_k,t)\right|^2+\left|f_{2,n}^{+}(\pm\mu_k,t)\right|^2\right\},r^{+}(\lambda,t)=\frac{b_1(\lambda,t)}{a_1(\lambda,t)}$$

$$\left(m_k^{-}(t)\right)^{-2}=\sum_{n\in Z}\left\{\left|f_{1,n}^{-}(\pm\mu_k,t)\right|^2+\left|f_{2,n}^{-}(\pm\mu_k,t)\right|^2\right\},r^{-}(\lambda,t)=\frac{b_2(\lambda,t)}{a_2(\lambda,t)}$$

where we took into account that (see [14]), to symmetric eigenvalues $\pm\mu_k$ correspond equal normalization coefficients $m_k^{+}(t)$ and $m_k^{-}(t)$.

We call the set $\left\{r^{+}(\lambda,t),\lambda\in\partial\Gamma_1;\mu_k(t);m_k^{+}(t),k=1,2,...,N\right\}$ and $\left\{r^{-}(\lambda,t),\lambda\in\partial\Gamma_2;\mu_k(t);m_k^{-}(t),k=1,2,...,N\right\}$ the right and left scattering data for the system of equations (3.5) respectively. It is appropriate to emphasize that the right scattering data is uniquely determined by the left scattering data. The inverse scattering problem for the system (3.5) is recovering the coefficients $a_{1,2}, a_{2,n}$ by the left scattering data. From the left scattering data we can recover the coefficients $a_{1,2}, a_{2,n}$ in (3.5). Let

$$\begin{array}{l}F_j^{-}(n,t)=\sum_{k=1}^{N}\left(m_k^{-}(t)\right)^2\frac{\lambda}{\left(z_2^{-1}-z_2\right)}\left(\frac{z_2^{-1}-1}{\lambda}\right)^{2(2-j)}z_2^{-n}\Big|_{\lambda=\pm\nu_k}+\\ +\frac{1}{2\pi i}\int\limits_{\partial\Gamma_2}\frac{\lambda r^{-}(\lambda,t)}{\left(z_2^{-1}-z_2\right)}\left(\frac{z_2^{-1}-1}{\lambda}\right)^{2(2-j)}z_2^{-n}d\lambda.\end{array}\quad(3.9)$$

The kernels $K_1^{-}(n,m,t), K_2^{-}(n,m,t)$ are obtained from the main equations of the Gelfand-Levitan-Marchenko type

$$K_j^{-}(n,m,t)+F_j^{-}(2n+m,t)+$$

$$+\sum_{r\leq -1} K_j^-(n,r,t)\,F_j^-(2n+m+r,t)=0, m\leq -1, j=1,2, \tag{3.10}$$

which have unique solutions for arbitrary n. Following this, we determine the coefficients $a_{1,2}, a_{2,n}$ from either one of the formulas (3.7).

Theorem 3.1. *Consider the system of equations (3.5) with coefficients* $a_{1,n} = a_{1,n}(t), a_{2,n} = a_{2,n}(t)$ *. If these coefficients* $a_{1,2}, a_{2,n}$ *make up a solution of equation (3.2) in the class of functions satisfying (3.4), then the evolution of the left scattering data is described by the following formulae:*

$$\begin{array}{l} r^-(\lambda,t)=r^-(\lambda,0)\exp\left\{\left(z_2^{-1}-z_2\right)t\right\} \\ \mu_k(t)=\mu_k(0)=\mu_k, \\ \left(m_k^-(t)\right)^{-2}=\left(m_k^-(0)\right)^{-2}\exp\left\{\left(z_2^{-1}(\mu_k)-z_2(\mu_k)\right)t\right\}, k=1,...,N \end{array} \tag{3.11}$$

Proof. We consider the operator $L = L(t)$ generated by the right-hand side of the system of equations (3.5). We also introduce the operator $A = A(t)$, setting

$$\begin{array}{l} (Ay)_{1.n}=\frac{1}{2}a_{1,n-1}a_{2,n}y_{1,n-1}-\frac{1}{2}a_{1,n}a_{2,n+1}y_{1,n+1}, \\ (Ay)_{2.n}=\frac{1}{2}a_{1,n-1}a_{2,n-1}y_{2,n-1}-\frac{1}{2}a_{1,n}a_{2,n}y_{2,n+1}. \end{array}$$

It is easy to verify that the operators L and A form a Lax pair, and therefore the system of equations (3.2) is equivalent to the operator equation

$$\dot{L}=AL-LA.$$

By virtue of the last relation, the operator $B=\frac{d}{dt}-A$ takes solutions of the system equation with parameter t to solutions of the same system. Using (3.6), we can show by standard methods (see) that, as $n\to\infty$

$$\left[Bf^+(\lambda,t)\right]_{j,n}=\frac{A^2}{2}\left(z_1^{-1}-z_1\right)\left(\frac{Az_1-A}{\lambda}\right)^{2-j}z_1^n+o(1), n\to\infty$$

A solution with such asymptotic behavior is unique; hence,

$$\left[Bf^+(\lambda,t)\right]_{j,n}=\frac{A^2}{2}\left(z_1^{-1}-z_1\right)f_{j,n}^+(\lambda,t), j=1,2$$

On the other hand, from (3.8) we obtain

$$\begin{array}{l} [Bf^+(\lambda,t)]_{j,n}=\left[\dot{a}_2(\lambda,t)+\frac{1}{2}\left(z_2^{-1}-z_2\right)a_2(\lambda,t)\right]\overline{f_{j,n}^-(\lambda,t)}+ \\ \left[\dot{b}_2(\lambda,t)-\frac{1}{2}\left(z_2^{-1}-z_2\right)b_2(\lambda,t)\right]f_{j,n}^-(\lambda,t), j=1,2. \end{array}$$

Comparing the last equalities with (3.8), we obtain

$$\begin{array}{l} \dot{a}_2(\lambda,t)+\frac{1}{2}\left(z_2^{-1}-z_2\right)a_2(\lambda,t)=\frac{A^2}{2}\left(z_1^{-1}-z_1\right)a_2(\lambda,t), \\ \dot{b}_2(\lambda,t)-\frac{1}{2}\left(z_2^{-1}-z_2\right)b_2(\lambda,t)=\frac{A^2}{2}\left(z_1^{-1}-z_1\right)b_2(\lambda,t). \end{array}$$

Whence it follows that

$$\begin{array}{l} a_2(\lambda,t)=a_2(\lambda,0)\exp\left\{\frac{A^2}{2}\left(z_1^{-1}-z_1\right)t-\frac{1}{2}\left(z_2^{-1}-z_2\right)t\right\}, \\ b_2(\lambda,t)=b_2(\lambda,0)\exp\left\{\frac{A^2}{2}\left(z_1^{-1}-z_1\right)t+\frac{1}{2}\left(z_2^{-1}-z_2\right)t\right\}. \end{array}$$

As follows from last formulas, the zeros of the function $a_2(\lambda,t)$ do not depend on t and the first two relations in (3.7) are valid.

We now find the law for transforming the normalization $m_k^-(t)$.

Let $\varphi_{j,n}(\mu_k,t)$ be a normalized eigenfunction of the operator L corresponding to the eigenvalue μ_k:

$$\sum_{n=-\infty}^{\infty}\left\{\varphi_{1,n}^2(\mu_k,t)+\varphi_{2,n}^2(\mu_k,t)\right\}=1,$$

$$\varphi_{j,n}(\mu_k,t)\sim c(t)\left(\tfrac{z_2^{-1}(\mu_k)-1}{\mu_k}\right)^{2-j}z_2^{-n}(\mu_k),n\to-\infty$$

Since

$$f_{j,n}^-(\mu_k,t)\sim\left(\frac{z_2^{-1}(\mu_k)-1}{\mu_k}\right)^{2-j}z_2^{-n}(\mu_k),n\to-\infty$$

it follows that

$$f_{j,n}^-(\mu_k,t)=\frac{1}{c(t)}\varphi_{j,n}(\mu_k,t),$$

and consequently

$$\left(m_k^-(t)\right)^{-2}=\sum_{n\in Z}\left\{\left|f_{1,n}^-(\mu_k,t)\right|^2+\left|f_{2,n}^-(\mu_k,t)\right|^2\right\}=c^2(t).$$

It is easy to verify that the normalized eigenfunctions $\varphi_{j,n}(\mu_k,t)$ satisfy the equation

$$\dot{\varphi}_{j,n}(\mu_k,t)-\left(A\varphi(\mu_k,t)\right)_{j,n}=0,j=1,2.$$

Therefore, by virtue of (3.12), we have

$$\dot{c}(t)-\frac{1}{2}\left(z_2^{-1}(\mu_k)-z_2(\mu_k)\right)c(t)=0$$

whence we find that

$$c(t)=c(0)\exp\left\{\frac{1}{2}\left(z_2^{-1}(\mu_k)-z_2(\mu_k)\right)t\right\}$$

$$\left(m_k^-(t)\right)^{-2}=\left(m_k^-(0)\right)^{-2}\exp\left\{\left(z_2^{-1}(\mu_k)-z_2(\mu_k)\right)t\right\}$$

This completes the proof of the theorem. □

The theorems proved above give an algorithm for finding a solution to problem (1.1) - (1.2). For constructing the solution of problem (1.1)-(1.2) by the initial data $a_{j,n}(0), j=1,2$, we calculate the scattering data $\{r^-(\lambda,t),\lambda\in\partial\Gamma_2;$ $\mu_k(t);m_k^-(t),k=1,2,...,N\}$, the of the system of equations (3.5) for $t=0$. Then the set $\{r^-(\lambda,t),\lambda\in\partial\Gamma_2;\ \mu_k(t);m_k^-(t),k=1,2,...,N\}$ may be found by means of (3.11). Construct the function $F_j^-(n,t)$ by the formula (3.9). Find the solution of the equation (3.10). Calculate $c_n=c_n(t)$ by the formula (3.1), (3.7).

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