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OPERATORS ON CONTROLLED K-G-FRAMES IN HILBERT SPACES

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Abstract. Controlled frames for spherical wavelets were introduced by Bogdanova et al. to get a numerically more efficient approximation algorithm. In this paper we propose some useful results about controlled K-g-frames, which are generalizations of K-g-frames. We show that if $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame and U, K are bounded linear operators in a separable Hilbert space, then, under certain conditions, the sequences of operators$\{\Lambda_i U\}_{i\in\mathbb{I}}$, $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ are also CC'-controlled K-g-frames. We also present the concept of CC'-controlled K-g-dual frame and extend some known equalities and inequalities. Finally we study the stability problem for perturbation of CC'-controlled K-g-frames.

1. Introduction

Frames which are generalization of orthonormal bases, were first introduced in the context of non-harmonic Fourier series by Duffin and Schaeffer in [7]. They provide robust, stable and usually non-unique representations of vectors in a Hilbert space. Theory of frames began to be more widely and deeply studied during the last 20 years with several new applications, e. g. signal processing, image processing, data compression, sampling theory and quantum computing. G-frames have been introduced by Sun in [18]. They are generalized frames and include ordinary frames and many recent generalizations of them, e. g., bounded quasi-projectors and fusion frames. Najati and et al. completed the concept of g-frames in [13] and proved some new results.

Frames for operators or K-frames have been introduced by Găvruţa in [9] to study the nature of atomic systems for a Hilbert space with respect to a bounded linear operator K. It is a well-known fact that K-frames are more general than the classical frames and due to higher generality, many properties of frames (such as invertibility of the frame operator, etc.) may not hold for K-frames.

Controlled frames for spherical wavelets are introduced in [2] to get a numerically more efficient approximation algorithm and the related theories are developed in [1, 15, 16]. The concept of controlled g-frames is presented in [11, 17] and also controlled K-frames are presented in [14]. Hua and Huang [10] introduced the concept of controlled K-g-frames.

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In this paper, inspired by the results of Hua and Huang [10], we obtain some new results about CC'-controlled K-g-frames.

Throughout the paper, H is a separable Hilbert space, $\mathcal{B}(H)$ is the set of all bounded linear operators from H into H, $\mathcal{GL}(H)$ is the set of all bounded linear operators which have bounded inverses and $\mathcal{GL}^+(H)$ is the set of all positive operators in $\mathcal{GL}(H)$. The operators $C, C' \in \mathcal{GL}^+(H)$, $K \in \mathcal{B}(H)$ and $\Lambda := \{\Lambda_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ is a sequence of bounded operators. Here $\mathbb{I} \subset \mathbb{Z}$.

Let us recall basic definitions and notations of controlled K-g-frames.

Lemma 1.1. *([6]). Suppose that H_1 and H_2 are Hilbert spaces and $L_1 \in \mathcal{B}(H_1, H)$, $L_2 \in \mathcal{B}(H_2, H)$. Then the following assertions are equivalent:*

(1) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;
(2) $L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ *for some* $\lambda > 0$;
(3) *There exists a mapping* $U \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2 U$.

Moreover, if above conditions are valid, then there exists a unique operator U such that

(a) $\|U\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;
(b) $\ker(L_1) = \ker(U)$;
(c) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

If an operator U has a closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following sense.

Lemma 1.2. *(see [4]). Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

Definition 1.1. [18] A family $\Lambda := \{\Lambda_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ is called a g-frame for H with respect to $\{H_i\}_{i\in\mathbb{I}}$, if there exist constants $0 < A \leq B < \infty$ such that for each $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2.$$

For this frame, the g-frame operator is defined by

$$S_\Lambda(f) = \sum_{i\in\mathbb{I}} \Lambda_i^* \Lambda_i f, \quad f \in H,$$

which is positive and invertible.

Definition 1.2. [1] Let $C \in \mathcal{GL}(H)$. We say that $F := \{f_i\}_{i\in\mathbb{I}}$ is a C-controlled frame for H if there exist constants $0 < A_C \leq B_C < \infty$ such that for each $f \in H$

$$A_C\|f\|^2 \leq \sum_{i\in\mathbb{I}} \langle f, f_i\rangle\langle Cf_i, f\rangle \leq B_C\|f\|^2. \tag{1.1}$$

Definition 1.3. [10] Let $C, C' \in \mathcal{GL}^+(H)$ and $K \in \mathcal{B}(H)$. We say that $\Lambda := \{\Lambda_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ is a (C, C')-controlled K-g-frame for H if there exist constants $0 < A_{CC'} \leq B_{CC'} < \infty$ such that for each $f \in H$

$$A_{CC'}\|K^* f\|^2 \leq \sum_{i\in\mathbb{I}} \langle \Lambda_i C' f, \Lambda_i C f\rangle \leq B_{CC'}\|f\|^2. \tag{1.2}$$

If the right hand of (1.2) holds, Λ is called a *(C, C')-controlled K-g-Bessel sequence* for H with bound B_C. We call Λ a *Parseval (C, C')-controlled K-g-frame* if

$$\sum_{i\in\mathbb{I}}\langle\Lambda_i C' f, \Lambda_i C f\rangle = \|K^* f\|^2.$$

If $K = I_H$ then Λ is called a (C, C')-controlled g-frame.

For simplicity, we will use a notation CC' instead of (C, C').
If Λ is a CC'-controlled g-frame for H and $C^*\Lambda_i^*\Lambda_i C'$ is positive for all $i \in \mathbb{I}$, then for each $f \in H$

$$A_{CC'}\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\|^2 \leq B_{CC'}\|f\|^2.$$

Now, let

$$\mathfrak{K}_2 := \{(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f : \ f \in H\}_{i\in\mathbb{I}} \subset (\sum_{i\in\mathbb{I}} \oplus H)_{\ell^2}.$$

It is easy to check that $\mathfrak{K}_2$ is a closed subspace of $(\sum_{i\in\mathbb{I}} \oplus H)_{\ell^2}$.
Now, we can define the synthesis and analysis operators of the CC'-controlled g-frames as

$$T_{CC'} : \mathfrak{K}_2 \longrightarrow H,$$
$$T_{CC'}(\{(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} C^*\Lambda_i^*\Lambda_i C' f$$

and

$$T^*_{CC'} : H \longrightarrow \mathfrak{K}_2,$$
$$T^*_{CC'}(f) = \{(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\}_{i\in\mathbb{I}}.$$

Thus, the CC'-controlled g-frame operator is given by

$$S_{CC'}f = T_{CC'}T^*_{CC'}f = \sum_{i\in\mathbb{I}} C^*\Lambda_i^*\Lambda_i C' f, \qquad f \in H.$$

So,

$$\langle S_{CC'}f, f\rangle = \sum_{i\in\mathbb{I}}\langle\Lambda_i C' f, \Lambda_i C f\rangle, \qquad f \in H.$$

and

$$A_{CC'} Id_H \leq S_{CC'} \leq B_{CC'} Id_H.$$

Lemma 1.3. [8] *Let $C, C' \in \mathcal{GL}^+(H)$. A sequence Λ is a CC'-controlled g-Bessel sequence for H with bound $B_{CC'}$ if and only if the operator*

$$T_{CC'} : \mathfrak{K}_2 \longrightarrow H,$$
$$T_{CC'}(\{(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} C^*\Lambda_i^*\Lambda_i C' f$$

is well-defined and bounded with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$.

Lemma 1.4. [8] *Let $C, C' \in \mathcal{GL}^+(H)$. A sequence Λ is a CC'-controlled g-frame for H if and only if the operator*

$$T_{CC'} : \mathfrak{K}_2 \longrightarrow H,$$
$$T_{CC'}(\{(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} C^*\Lambda_i^*\Lambda_i C' f$$

is well-defined, bounded and surjective.

Proposition 1.1. *Let Λ be a CC'-controlled K-g-frame for H and K has a dense range. Suppose that $C^*\Lambda_i^*\Lambda_i C'$ is positive and also $V_i := (C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}$ for each $i \in \mathbb{I}$. Then $(\bigcap_{i\in\mathbb{I}} \ker V_i)^\perp = H$.*

Proof. Assume that $A_{CC'}$ and $B_{CC'}$ are the frame bounds of Λ. Hence,

$$A_{CC'}\|K^*f\|^2 \le \sum_{i\in\mathbb{I}} \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}} f\|^2 \le B_{CC'}\|f\|^2. \tag{1.3}$$

Since $\ker K^* = (\mathcal{R}(K))^\perp$ and K has a dense range, K^* is injective. Then from (1.3), for each $i \in I$, we get

$$\bigcap_{i\in\mathbb{I}} \ker V_i \subseteq \ker K^* = \{0\}.$$

So we get the proof. □

Remark 1.1. Suppose that Λ is a CC'-controlled K-g-frame for H with lower bound $A_{CC'}$. Then we have $S_{CC'} \ge A_{CC'}KK^*$. So by Lemma 1.1, there exists an operator $U \in \mathcal{B}\big(H, \mathfrak{K}_2\big)$ such that

$$T_{CC'}U = K. \tag{1.4}$$

Now, we can obtain optimal frame bounds of Λ by the operator U. Indeed, it is obvious that

$$B_{op} = \|S_{CC'}\| = \|T_{CC'}\|^2.$$

By Lemma 1.1, the equation (1.4) has a unique solution as U_0 such that

$$\begin{aligned}\|U_0\|^2 &= \inf\{\alpha > 0 \mid KK^* \le \alpha T_{CC'}T_{CC'}^*\} \\ &= \inf\{\alpha > 0 \mid \|K^*f\|^2 \le \alpha\|T_{CC'}^* f\|^2 \ , \ f \in H\}.\end{aligned}$$

Now, we have

$$\begin{aligned}A_{op} &= \sup\{A > 0 \mid A\|K^*f\|^2 \le \|T_{CC'}^* f\|^2 \ , \ f \in H\} \\ &= \Big(\inf\{\alpha > 0 \mid \|K^*f\|^2 \le \alpha\|T_{CC'}^* f\|^2 \ , \ f \in H\}\Big)^{-1} \\ &= \|U_0\|^{-2}.\end{aligned}$$

2. Operators preserving controlled K-g-frames

In this section, for the CC'-controlled K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$, we consider some proper relations between the operators $U, K \in B(H)$ and $C, C' \in GL^+(H)$ and investigate the cases that $\{\Lambda_i U\}_{i\in\mathbb{I}}$, $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ can also be CC'-controlled K-g-frames. Next, by putting connections between the operators S_Λ, K,C and C', we reach to necessary and sufficient conditions that $\{\Lambda_i\}_{i\in\mathbb{I}}$ can be a Parseval CC'-controlled K-g-frame.

Theorem 2.1. *Let Λ be a CC'-controlled K-g-frame for H and $U \in \mathcal{B}(H)$ such that $\mathcal{R}(U) \subset \mathcal{R}(K)$. Then Λ is a CC'-controlled U-g-frame for H.*

Proof. Suppose that $A_{CC'}$ is a lower frame bound of Λ. By Lemma 1.1, there exists $\alpha > 0$ such that $UU^* \leq \alpha^2 KK^*$. Now, for each $f \in H$ we have

$$\frac{A_{CC'}}{\alpha^2}\|U^*f\|^2 \leq A_{CC'}\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}}\langle \Lambda_i C'f, \Lambda_i Cf\rangle.$$

So the proof is completed. □

Theorem 2.2. *Let Λ be a CC'-controlled K-g-frame for H. Assume that K has a closed range and $U \in \mathcal{B}(H)$ such that $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Also suppose that U^* commutes with C and C'. Then $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame for $\mathcal{R}(U)$ if and only if there exists $\delta > 0$ such that for each $f \in \mathcal{R}(U)$,*

$$\|U^*f\| \geq \delta\|K^*f\|.$$

Proof. Suppose that $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame for H with a lower frame bound $E_{CC'}$. If $B_{CC'}$ is an upper frame bound of Λ, then for each $f \in \mathcal{R}(U)$

$$E_{CC'}\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle \leq B_{CC'}\|U^*f\|^2.$$

Thus, $\|U^*f\| \geq \sqrt{\frac{E_{CC'}}{B_{CC'}}}\|K^*f\|$. For the opposite implication, for each $f \in H$ we have

$$\|U^*f\| = \|(K^\dagger)^*K^*U^*f\| \leq \|K^\dagger\|\|K^*U^*f\|.$$

Therefore, if $A_{CC'}$ is a lower frame bound of Λ, we have

$$\begin{aligned} A_{CC'}\delta^2\|K^\dagger\|^{-2}\|K^*f\|^2 &\leq A_{CC'}\|K^\dagger\|^{-2}\|U^*f\|^2 \\ &\leq A_{CC'}\|K^*U^*f\|^2 \\ &\leq \sum_{i\in\mathbb{I}}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle. \end{aligned}$$

For the upper bound, it is clear that

$$\sum_{i\in\mathbb{I}}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle \leq B_{CC'}\|U\|^2\|f\|^2.$$

So, $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame for H with frame bounds $A_{CC'}\delta^2\|K^\dagger\|^{-2}$ and $B_{CC'}\|U\|^2$. □

Theorem 2.3. *Let Λ be a CC'-controlled K-g-frame for H and the operator K has a dense range. Assume that $U \in \mathcal{B}(H)$ has a closed range and U and U^* commute with C and C'. If $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ and $\{\Lambda_i U\}_{i\in\mathbb{I}}$ are CC'-controlled K-g-frame for H, then U is invertible.*

Proof. Assume that $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame for H with frame bounds A_1 and B_1. Then for each $f \in H$

$$A_1\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}}\langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle \leq B_1\|f\|^2. \tag{2.1}$$

Since K has a dense range, K^* is injective. Then by (2.1), U^* is injective. Moreover, $\mathcal{R}(U) = (\ker U^*)^\perp = H$. So U is surjective.
Now, suppose that $\{\Lambda_i U\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame for H with frame bounds A_2 and B_2. Then, for each $f \in H$

$$A_2\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} \langle \Lambda_i UC'f, \Lambda_i UCf\rangle \leq B_2\|f\|^2.$$

Therefore U is injective, since $\ker U \subseteq \ker K^*$. Thus, U is an invertible operator and the proof is completed. □

Theorem 2.4. *Let Λ be a CC'-controlled K-g-frame for H and $U \in \mathcal{B}(H)$ be a co-isometry (i.e. $UU^* = Id_H$) such that $UK = KU$ and U^* commutes with C and C'. Then $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ is a CC'-controlled K-g-frame for H.*

Proof. Suppose that Λ is a CC'-controlled K-g-frame for H with frame bounds $A_{CC'}$ and $B_{CC'}$. For each $f \in H$, we have

$$\sum_{i\in\mathbb{I}} \langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle = \sum_{i\in\mathbb{I}} \langle \Lambda_i C'U^*f, \Lambda_i CU^*f\rangle \leq B_{CC'}\|U\|^2\|f\|^2.$$

So, $\{\Lambda_i U^*\}_{i\in\mathbb{I}}$ is a controlled g-Bessel sequence. For the lower bound, we can write

$$\begin{aligned}\sum_{i\in\mathbb{I}} \langle \Lambda_i U^*C'f, \Lambda_i U^*Cf\rangle &= \sum_{i\in\mathbb{I}} \langle \Lambda_i C'U^*f, \Lambda_i CU^*f\rangle \\ &\geq A_{CC'}\|K^*U^*f\|^2 \\ &= A_{CC'}\|U^*K^*f\|^2 \\ &= A_{CC'}\|K^*f\|^2.\end{aligned}$$

□

Theorem 2.5. *Let $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ and $\Theta := \{\Theta_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ be two CC'-controlled K-g-Bessel sequences for H with bounds B_Λ and B_Θ, respectively. Suppose that $T_{\Lambda,C,C'}$ and $T_{\Theta,C,C'}$ are their synthesis operators such that $T_{\Theta,C,C'}T^*_{\Lambda,C,C'} = K^*$. Then Λ and Θ are CC'-controlled K and K^*-g-frames, respectively.*

Proof. For each $f \in H$ we have

$$\begin{aligned}\|K^*f\|^4 &= \langle K^*f, K^*f\rangle^2 \\ &= \langle T^*_{\Lambda,C,C'}f, T^*_{\Theta,C,C'}K^*f\rangle^2 \\ &\leq \|T^*_{\Lambda,C,C'}f\|^2\|T^*_{\Theta,C,C'}K^*f\|^2 \\ &= \Big(\sum_{i\in\mathbb{I}} \langle \Lambda_i C'f, \Lambda_i Cf\rangle\Big)\Big(\sum_{i\in\mathbb{I}} \langle \Theta_i C'K^*f, \Theta_i CK^*f\rangle\Big) \\ &\leq \Big(\sum_{i\in\mathbb{I}} \langle \Lambda_i C'f, \Lambda_i Cf\rangle\Big) B_\Theta\|K^*f\|^2.\end{aligned}$$

Thus

$$B_\Theta^{-1}\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}} \langle \Lambda_i C'f, \Lambda_i Cf\rangle.$$

This means that Λ is a CC'-controlled K-g-frame for H. Similarly, since $T_{\Lambda,C,C'}T^*_{\Theta,C,C'} = K$, Θ is a CC'-controlled K^*-g-frame with lower bound B_Λ^{-1}. □

Theorem 2.6. *Let Λ be a g-frame for H with frame operator S_Λ. Also assume that Λ is a CC'-controlled g-Bessel sequence with frame operator $S_{CC'}$. Then Λ is a Parseval CC'-controlled K-g-frame for H if and only if $C = (S_\Lambda^{-p})^*\Phi$ and $C' = S_\Lambda^{-q}\Psi$ where Φ, Ψ are two operators on H such that $\Phi^*\Psi = KK^*$ and $p + q = 1$ where $p, q \in \mathbb{R}$.*

Proof. Assume that Λ is a Parseval CC'-controlled K-g-frame for H. It is clear that $S_{CC'} = C^*S_\Lambda C'$ and $S_{CC'} = KK^*$. Therefore, for each $p, q \in \mathbb{R}$ such that $p + q = 1$, we obtain

$$KK^* = C^*S_\Lambda^p S_\Lambda^q C'.$$

We define $\Phi := (S_\Lambda^p)^*C$ and $\Psi := S_\Lambda^q C'$. So

$$\Phi^*\Psi = C^*S_\Lambda^p S_\Lambda^q C' = KK^*.$$

Conversely, Let Φ, Ψ be two operators on H such that $\Phi^*\Psi = KK^*$. Suppose that $C = (S_\Lambda^{-p})^*\Phi$ and $C' = S_\Lambda^{-q}\Psi$ are two operators on H where $p, q \in R$ and $p + q = 1$. Since

$$KK^* = \Phi^*\Psi = C^*S_\Lambda^p S_\Lambda^q C' = C^*S_\Lambda C' = S_{CC'},$$

so, for each $f \in H$

$$\|K^*f\|^2 = \langle KK^*f, f\rangle = \sum_{i\in\mathbb{I}} \langle C^*\Lambda_i^*\Lambda_i C'f, f\rangle.$$

Thus Λ is a Parseval CC'-controlled K-g-frame for H. □

3. Duals of controlled K-g-frames and some equalities

In this section, by the concept of K-g-dual pair, we present a bounded operator called dual operator and propose some known equalities and inequalities between dual operator and CC'-controlled K-g-frames.

Definition 3.1. Suppose that Λ is a CC'-controlled K-g-frame for H with synthesis operator $T_{\Lambda,C,C'}$. Then $\widetilde{\Lambda} := \{\widetilde{\Lambda}_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ is called a CC'-controlled K-g-dual frame (or brevity CC'-K-g-dual) for Λ if

$$T_{\Lambda,C,C'}T^*_{\widetilde{\Lambda},C,C'} = K, \tag{3.1}$$

and $\widetilde{\Lambda}$ is a CC'-controlled K-g-Bessel sequence.

In this case, $(\Lambda, \widetilde{\Lambda})$ is called a CC'-controlled K-g-dual pair. The following result presents equivalent conditions of the CC'-K-g-dual.

Proposition 3.1. *Let $\widetilde{\Lambda}$ be a CC'-K-g-dual for Λ. Then the following conditions are equivalent:*

(1) $T_{\Lambda,C,C'}T^*_{\widetilde{\Lambda},C,C'} = K$;
(2) $T_{\widetilde{\Lambda},C,C'}T^*_{\Lambda,C,C'} = K^*$;
(3) *for each $f, f' \in H$, we have*

$$\langle Kf, f'\rangle = \langle T^*_{\widetilde{\Lambda},C,C'}f, T^*_{\Lambda,C,C'}f'\rangle.$$

Proof. Straightforward. □

Theorem 3.1. *If $\widetilde{\Lambda}$ is a CC'-K-g-dual for Λ, then $\widetilde{\Lambda}$ is a CC'-controlled K^*-g-frame for H.*

Proof. Suppose that $f \in H$ and B_C is an upper bound of Λ. Therefore,

$$\begin{aligned}\|Kf\|^4 &= |\langle Kf, Kf\rangle|^2\\ &= |\langle T_{\Lambda,C,C'}T^*_{\widetilde{\Lambda},C,C'}f, Kf\rangle|^2\\ &= |\langle T^*_{\widetilde{\Lambda},C,C'}f, T^*_{\Lambda,C,C'}Kf\rangle|^2\\ &\leq \|T^*_{\widetilde{\Lambda},C,C'}f\|^2 B_C\|Kf\|^2\\ &= B_C\|Kf\|^2\sum_{i\in I}\langle\widetilde{\Lambda}_iC'f, \widetilde{\Lambda}_iCf\rangle,\end{aligned}$$

and this completes the proof. □

Corollary 3.1. *Assume that C_{op} and D_{op} are the optimal bounds of $\widetilde{\Lambda}$, respectively. Then*

$$C_{op} \geq B_{op}^{-1} \quad and \quad D_{op} \geq A_{op}^{-1},$$

for which A_{op} and B_{op} are the optimal bounds of Λ, respectively.

Assume that $(\Lambda, \widetilde{\Lambda})$ is a CC'-controlled K-g-dual pair and $\mathbb{J} \subset \mathbb{I}$. We define

$$S_{\mathbb{J}}f := \sum_{i\in\mathbb{J}}(C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}(C^*\widetilde{\Lambda}_i^*\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, f \in H,$$

and we call it a dual operator.
It is easy to check that $S_{\mathbb{J}} \in \mathcal{B}(H)$ and $S_{\mathbb{J}} + S_{\mathbb{J}^c} = K$, where $\mathbb{J}^c$ is the complement of $\mathbb{J}$.
Now, by that operator $S_{\mathbb{J}}$ we extend some known equalities and inequalities for controlled K-g-frames in the following theorems.

Theorem 3.2. *If $f \in H$ then*

$$\begin{aligned}&\textstyle\sum_{i\in\mathbb{J}}\langle(C^*\widetilde{\Lambda}_i^*\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}Kf\rangle - \|S_{\mathbb{J}}f\|^2\\ &= \textstyle\sum_{i\in\mathbb{J}^c}\overline{\langle(C^*\widetilde{\Lambda}_i^*\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}Kf\rangle} - \|S_{\mathbb{J}^c}f\|^2\end{aligned}$$

Proof. Let $f \in H$. We can write

$$\begin{aligned}\sum_{i\in\mathbb{J}}\langle(C^*\widetilde{\Lambda}_i^*\widetilde{\Lambda}_iC')^{\frac{1}{2}}f, (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}Kf\rangle - \|S_{\mathbb{J}}f\|^2 &= \langle K^*S_{\mathbb{J}}f, f\rangle - \|S_{\mathbb{J}}f\|^2\\ &= \langle K^*S_{\mathbb{J}}f, f\rangle - \langle S_{\mathbb{J}}^*S_{\mathbb{J}}f, f\rangle\\ &= \langle (K - S_{\mathbb{J}})^*S_{\mathbb{J}}f, f\rangle\\ &= \langle S_{\mathbb{J}^c}^*(K - S_{\mathbb{J}^c})f, f\rangle\\ &= \langle S_{\mathbb{J}^c}^*Kf, f\rangle - \langle S_{\mathbb{J}^c}^*S_{\mathbb{J}^c}f, f\rangle\\ &= \langle Kf, S_{\mathbb{J}^c}f\rangle - \langle S_{\mathbb{J}^c}f, S_{\mathbb{J}^c}f\rangle\\ &= \overline{\langle S_{\mathbb{J}^c}f, Kf\rangle} - \|S_{\mathbb{J}^c}f\|,\end{aligned}$$

and this completes the proof. □

Theorem 3.3. *Let Λ be a Parseval CC'-controlled-K-g-frame for H. If $\mathbb{J} \subseteq \mathbb{I}$ and $E \subseteq \mathbb{J}^c$, then for each $f \in H$*

$$\begin{aligned}&\|\sum_{i\in\mathbb{J}\cup E} C^*\Lambda_i^*\Lambda_i C'f\|^2 - \|\sum_{i\in\mathbb{J}^c\setminus E} C^*\Lambda_i^*\Lambda_i C'f\|^2\\ =&\|\sum_{i\in\mathbb{J}} C^*\Lambda_i^*\Lambda_i C'f\|^2 - \|\sum_{i\in\mathbb{J}^c} C^*\Lambda_i^*\Lambda_i C'f\|^2 + 2\operatorname{Re}\Big(\sum_{i\in E}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle\Big).\end{aligned}$$

Proof. Let

$$S_{\Lambda,\mathbb{J}}f := \sum_{i\in\mathbb{J}} C^*\Lambda_i^*\Lambda_i C'f.$$

Therefore, $S_{\Lambda,\mathbb{I}} + S_{\Lambda,\mathbb{I}^c} = KK^*$. Hence

$$\begin{aligned}S^2_{\Lambda,\mathbb{J}} - S^2_{\Lambda,\mathbb{J}^c} &= S^2_{\Lambda,\mathbb{J}} - (KK^* - S_{\Lambda,\mathbb{J}})^2\\ &= KK^*S_{\Lambda,\mathbb{J}} + S_{\Lambda,\mathbb{J}}KK^* - (KK^*)^2\\ &= KK^*S_{\Lambda,\mathbb{J}} - S_{\Lambda,\mathbb{J}^c}KK^*.\end{aligned}$$

Now, for each $f \in H$ we obtain

$$\|S^2_{\Lambda,\mathbb{J}}f\|^2 - \|S^2_{\Lambda,\mathbb{J}^c}f\|^2 = \langle KK^*S_{\Lambda,\mathbb{J}}f, f\rangle - \langle S_{\Lambda,\mathbb{J}^c}KK^*f, f\rangle.$$

Consequently, for $\mathbb{J} \cup E$ instead of $\mathbb{J}$:

$$\begin{aligned}&\|\sum_{i\in\mathbb{J}\cup E} C^*\Lambda_i^*\Lambda_i C'f\|^2 - \|\sum_{i\in\mathbb{J}^c\setminus E} C^*\Lambda_i^*\Lambda_i C'f\|^2\\ =&\sum_{i\in\mathbb{J}\cup E}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle - \sum_{i\in\mathbb{J}^c\setminus E}\overline{\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle}\\ =&\sum_{i\in\mathbb{J}}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle - \sum_{i\in\mathbb{J}^c}\overline{\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle}\\ &\quad + 2\operatorname{Re}\Big(\sum_{i\in E}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle\Big)\\ =&\|\sum_{i\in\mathbb{J}} C^*\Lambda_i^*\Lambda_i C'f\|^2 - \|\sum_{i\in\mathbb{J}^c} C^*\Lambda_i^*\Lambda_i C'f\|^2 + 2\operatorname{Re}\Big(\sum_{i\in E}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle\Big).\end{aligned}$$

□

Theorem 3.4. *Let Λ be a Parseval CC'-controlled-K-g-frame for H. If $\mathbb{J} \subseteq \mathbb{I}$, then for each $f \in H$*

$$\begin{aligned}&\|\sum_{i\in\mathbb{J}} C^*\Lambda_i^*\Lambda_i C'f\|^2 + \operatorname{Re}\Big(\sum_{i\in J^c}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle\Big)\\ =&\|\sum_{i\in\mathbb{J}^c} C^*\Lambda_i^*\Lambda_i C'f\|^2 + \operatorname{Re}\Big(\sum_{j\in\mathbb{J}}\langle \Lambda_i C'f, \Lambda_i C^*KK^*f\rangle\Big) \geq \frac{3}{4}\|KK^*f\|^2.\end{aligned}$$

Proof. By the proof of Theorem 3.3, we have

$$S^2_{\Lambda,\mathbb{J}} - S^2_{\Lambda,\mathbb{J}^c} = KK^*S_{\Lambda,\mathbb{J}} - S_{\Lambda,\mathbb{J}^c}KK^*.$$

Therefore

$$S^2_{\Lambda,\mathbb{J}} + S^2_{\Lambda,\mathbb{J}^c} = 2\Big(\frac{KK^*}{2} - S_{\Lambda,\mathbb{J}}\Big)^2 + \frac{(KK^*)^2}{2} \geq \frac{(KK^*)^2}{2}.$$

Thus

$$\begin{aligned}KK^*S_{\Lambda,\mathbb{J}}+S^2_{\Lambda,\mathbb{J}^c}+(KK^*S_{\Lambda,\mathbb{J}}+S^2_{\Lambda,\mathbb{J}^c})^* &= KK^*S_{\Lambda,\mathbb{J}}+S^2_{\Lambda,\mathbb{J}^c}+S_{\Lambda,\mathbb{J}}KK^*+S^2_{\Lambda,\mathbb{J}^c}\\ &= KK^*(S_{\Lambda,\mathbb{J}}+S_{\Lambda,\mathbb{J}^c})+S^2_{\Lambda,\mathbb{J}}+S^2_{\Lambda,\mathbb{J}^c}\\ &\geq \frac{3}{2}(KK^*)^2.\end{aligned}$$

Now, for each $f\in H$ we obtain

$$\begin{aligned}&\|\sum_{i\in\mathbb{J}}C^*\Lambda_i^*\Lambda_iC'f\|^2+\mathrm{Re}\Big(\sum_{i\in J^c}\langle\Lambda_iC'f,\Lambda_iC^*KK^*f\rangle\Big)\\ &=\frac{1}{2}\big(\langle KK^*S_{\Lambda,\mathbb{J}}f,f\rangle+\langle S^2_{\Lambda,\mathbb{J}^c}f,f\rangle+\langle f,KK^*S_{\Lambda,\mathbb{J}}f\rangle+\langle f,S^2_{\Lambda,\mathbb{J}^c}f\rangle\big)\\ &\geq\frac{3}{4}\|KK^*f\|^2.\end{aligned}$$

□

4. Perturbation of controlled K-g-frames

Stability of the wavelet and Gabor frames under perturbation is one of the important problems in frame theory. At first this problem was studied by Paley and Wienes for bases and then extended to frames. In recent years many authors extended the concept of perturbation to many kinds of frames such as g-frames, Banach frames, K and controlled frames [5, 11, 12, 19, 20]. But the most important results are obtained by Casazza and Christensen in [3]. Here we study the perturbation of CC'-controlled K-g-frames.

Theorem 4.1. *Let Λ be a CC'-controlled K-g-frame for H with bounds $A_{CC'}$ and $B_{CC'}$. Assume that $\Theta:=\{\Theta_i\in\mathcal{B}(H,H_i)\}_{i\in\mathbb{I}}$ is a sequence of operators such that for each $f\in H$ and $i\in\mathbb{I}$*

$$\begin{aligned}\|(C^*\Lambda_i^*\Lambda_iC'-C^*\Theta_i^*\Theta_iC')^{\frac12}f\| &\leq \lambda_1\|(C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|+\lambda_2\|(C^*\Theta_i^*\Theta_iC')^{\frac12}f\|\\ &\quad+c_i\|K^*f\|,\end{aligned}$$

where $\{c_i\}_{i\in\mathbb{I}}$ is a sequence of positive numbers such that $\eta:=\sum_{i\in\mathbb{I}}c_i^2<\infty$ and $0\leq\lambda_1,\lambda_2<1$. Then Θ is a CC'controlled K-g-frame for H with bounds:

$$\Big(\frac{(1-\lambda_1)\sqrt{A_{CC'}}-\eta}{1+\lambda_2}\Big)^2,\qquad \Big(\frac{(1+\lambda_1)\sqrt{B_{CC'}}+\eta\|K\|}{1-\lambda_2}\Big)^2.$$

Proof. For each $f\in H$ we have

$$\begin{aligned}\|(C^*\Theta_i^*\Theta_iC')^{\frac12}f\| &= \|(C^*\Theta_i^*\Theta_iC'-C^*\Lambda_i^*\Lambda_iC')^{\frac12}f+(C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|\\ &\leq\|(C^*\Theta_i^*\Theta_iC'-C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|+\|(C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|\\ &\leq\lambda_1\|(C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|+\lambda_2\|(C^*\Theta_i^*\Theta_iC')^{\frac12}f\|\\ &\quad+c_i\|K^*f\|+\|(C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|.\end{aligned}$$

Hence

$$(1-\lambda_2)\|(C^*\Theta_i^*\Theta_iC')^{\frac12}f\|\leq(1+\lambda_1)\|(C^*\Lambda_i^*\Lambda_iC')^{\frac12}f\|+c_i\|K^*f\|.$$

Since Λ is a CC'-controlled K-g-frame, so

$$\begin{aligned}\|T^*_{CC'}f\|^2 &= \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\|^2\\ &= \sum_{i\in\mathbb{I}}\langle\Lambda_i C'f, \Lambda_i Cf\rangle\\ &\leq B_{CC'}\|f\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\|(C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\| &\leq \frac{(1+\lambda_1)\|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\| + c_i\|K^*f\|}{1-\lambda_2}\\ &\leq \Big(\frac{(1+\lambda_1)\sqrt{B_{CC'}}+\eta\|K\|}{1-\lambda_2}\Big)\|f\|.\end{aligned}$$

Now, for the lower bound we get

$$\begin{aligned}\|(C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\| &= \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f - (C^*\Lambda_i^*\Lambda_i C' - C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\|\\ &\geq \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\| - \|(C^*\Lambda_i^*\Lambda_i C' - C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\|\\ &\geq \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\| - \lambda_1\|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\|\\ &\quad - \lambda_2\|(C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\| - c_i\|K^*f\|.\end{aligned}$$

Therefore

$$\begin{aligned}&(1+\lambda_2)\|(C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\|\\ &\qquad\geq (1-\lambda_1)\|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\| - c_i\|K^*f\|,\end{aligned}$$

or

$$\|(C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\| \geq \frac{(1-\lambda_1)\|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\| - c_i\|K^*f\|}{1+\lambda_2}.$$

Since,

$$\|T^*_{CC'}f\|^2 = \|(C^*\Lambda_i^*\Lambda_i C')^{\frac{1}{2}}f\|^2 \geq A_{CC'}\|K^*f\|^2,$$

thus

$$\|(C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\| \geq \Big(\frac{(1-\lambda_1)\sqrt{A_{CC'}}-\eta}{1+\lambda_2}\Big)\|K^*f\|.$$

This completes the proof. □

Proposition 4.1. *Let Λ be a CC'-controlled K-g-frame for H with bounds $A_{CC'}$ and $B_{CC'}$. Assume that $\Theta := \{\Theta_i \in \mathcal{B}(H, H_i)\}_{i\in\mathbb{I}}$ is a sequence of operators such that for each $f \in H$ and $i \in \mathbb{I}$*

$$\|(C^*\Lambda_i^*\Lambda_i C' - C^*\Theta_i^*\Theta_i C')^{\frac{1}{2}}f\| \leq c_i\|K^*f\|,$$

where $\{c_i\}_{i\in\mathbb{I}}$ is a sequence of positive numbers such that $\eta := \sum_{i\in\mathbb{I}} c_i^2 < \infty$. Then Θ is a CC'-controlled K-g-frame for H with bounds:

$$(\sqrt{A_{CC'}}-\eta)^2\ , \quad (\eta\|K\| + \sqrt{B_{CC'}})^2.$$

Proof. For each $f \in H$ we have

$$\begin{aligned}\|(C^*\Theta_i^*\Theta_iC')^{\frac{1}{2}}f\| &= \|(C^*\Theta_i^*\Theta_iC' - C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}f + (C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}f\| \\ &\leq \|(C^*\Theta_i^*\Theta_iC' - C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}f\| + \|(C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}f\| \\ &\leq (\eta\|K\| + \sqrt{B_{CC'}})\|f\|.\end{aligned}$$

Therefore Θ is a CC'-controlled g-Bessel sequence for H. On the other hand

$$\begin{aligned}\|(C^*\Theta_i^*\Theta_iC')^{\frac{1}{2}}f\| &\geq \|(C^*\Lambda_i^*\Lambda_iC')^{\frac{1}{2}}f\| - \|(C^*\Lambda_i^*\Lambda_iC' - C^*\Theta_i^*\Theta_iC')^{\frac{1}{2}}f\| \\ &\geq (\sqrt{A_{CC'}} - \eta)\|K^*f\|,\end{aligned}$$

and this completes the proof. □

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