

## ON STRONG SOLVABILITY OF THE DIRICHLET PROBLEM FOR A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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**Abstract.** We study a strong solvability of the Dirichlet problem

$$\sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + g(x, u) = f(x), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0$$

for a class of semilinear elliptic equations with discontinuous coefficients satisfying the Cordes condition. For this problem we get the existence results in  $\dot{W}_2^2(\Omega)$  Sobolev space whenever the norm  $\|f\|_{L_2(\Omega)}$  is sufficiently small. We also have proved the strong solvability of the Dirichlet problem in space  $\dot{W}_p^2(\Omega)$  with  $1 < p < \infty$  for the equation considered above with continuous leading coefficients and a small  $\|f\|_{L_p(\Omega)}$  norm.

### 1. Introduction and the main results

Let  $E_n$  be  $n$ -dimensional Euclidean space of the points  $x = (x_1, x_2, \dots, x_n)$  and  $\Omega$  be a bounded domain in  $E_n$  with boundary  $\partial\Omega$  of the class  $C^2$ . We consider in  $\Omega$  the following Dirichlet problem:

$$\sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + g(x, u) = f(x), \quad x \in \Omega, \quad (1.1)$$

$$u|_{\partial\Omega} = 0. \quad (1.2)$$

It is assumed that the coefficients  $a_{ij}(x)$ ,  $i, j = 1, 2, \dots, n$  of the operator  $L = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$  are bounded measurable functions satisfying the conditions

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \leq \gamma^{-1}|\xi|^2, \quad \forall x \in \Omega, \quad \forall \xi \in E_n, \quad \gamma \in (0, 1), \quad (1.3)$$

$$\operatorname{ess\,sup}_{x \in \Omega} \frac{\sum_{i,j=1}^n a_{ij}^2(x)}{\left[ \sum_{i=1}^n a_{ii}(x) \right]^2} \leq \frac{1}{n-1} - \delta, \quad (1.4)$$

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where  $\delta \in (0, \frac{1}{n})$  is some number and  $g(x, u) : \Omega \times E_1 \rightarrow E_1$  is a Caratheodory function, measurable with respect to  $x \in \Omega$  and, for almost all  $x \in \Omega$  continuous with respect to  $u \in E_1$  and, in addition to this, satisfying the following growth condition:

$$|g(x, u)| \leq b_0 |u|^q, \quad b_0 > 0. \quad (1.5)$$

Condition (1.4) is called the Cordes condition. We understand it in the following sense: the domain  $\Omega$  can be covered by a finite number of domains  $\Omega_i$  so that in every  $\Omega_i$  there is a non-degenerate linear transformation of space, under which the equation coefficients satisfy the condition (1.4).

We denote by  $\dot{W}_p^2(\Omega)$  ( $p \geq 1$ ) the closure of the class of functions  $u \in C^\infty(\bar{\Omega}) \cap C(\bar{\Omega})$ ,  $u|_{\partial\Omega} = 0$  with respect to the norm

$$\|u\|_{\dot{W}_p^2(\Omega)} = \left[ \int_{\Omega} \left( |u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p \right) dx \right]^{1/p}.$$

Let  $u_i$  and  $u_{ij}$  denote the derivatives  $u_{x_i}$  and  $u_{x_i x_j}$  respectively  $i, j = 1, \dots, n$ . The conjugate number is denoted by  $p'$  i.e.,  $1 < p < \infty$ ,  $\frac{1}{p'} + \frac{1}{p} = 1$ .

The notation  $C_i(\dots)$  means that the positive constant  $C_i$  depends on the parameters indicated in parentheses.

A function  $u(x) \in \dot{W}_p^2(\Omega)$  is called a strong solution of problem (1.1)-(1.2) if it satisfies equation (1.1) a.e. in  $\Omega$ .

Let us recall some known results directly related to the subject. In the case of linear equations the questions of strong solvability of elliptic and parabolic equations with discontinuous leading coefficients satisfying the Cordes condition were considered in [3, 11, 15], where the problem of deriving the estimate

$$\int_{\Omega} (\Delta u)^2 dx \leq C_1(n, \gamma, \delta) \int_{\Omega} Lu \cdot \Delta u dx \quad (1.6)$$

is the essential point of the investigation. From the examples in [8, p. 48], it follows that if the coefficients of operator  $L$  are discontinuous and the Cordes condition is not fulfilled, then the equation  $Lu = f$  is unsolvable in  $\dot{W}_p^2(\Omega)$  for any  $p > 1$ . For the further results on strong solvability of linear elliptic equations with discontinuous coefficients in  $\dot{W}_p^2(\Omega)$  ( $p > 1$ ) we refer to the monographs [5, 8]. For such equations with VMO class coefficients the questions of  $\dot{W}_p^2(\Omega)$  strong solvability for  $p > 1$  were considered in [12]. Studies of nonlinear parabolic equation with discontinuous coefficients were considered in [3, 13, 14].

Note that, in the study of semilinear equations (1.1) with small nonlinearity ( $0 < q < 1$ ), the condition in the norm of a function of type (2.1) on the right-hand side is not required (see, [4]).

Different from semilinear equations with regular coefficients, the case of discontinuous coefficients was not considered before on the subject of strong solvability. The aim of this paper is to prove a strong solvability of Dirichlet problem for a class of semilinear elliptic equations (1.1) in Sobolev's space  $\dot{W}_2^2(\Omega)$  when the leading coefficients are discontinuous and satisfy the Cordes condition and the function  $f(x)$  has a sufficiently small  $L_2(\Omega)$  (Cf.[10]) norm. We also consider the results on strong solvability of Dirichlet problem in space  $\dot{W}_p^2(\Omega)$  with  $1 < p < \infty$

for a class of elliptic equations with continuous leading coefficients and function  $f(x)$  having a sufficiently small  $L_p(\Omega)$  norm.

## 2. Equations with Discontinuous Coefficients

**Theorem 2.1.** *Let  $n > 4$ ,  $1 \leq q < \frac{n}{n-4}$  and conditions (1.3)-(1.5) be satisfied,  $\partial\Omega \in C^2$ . Then there exists a sufficiently small positive constant  $C_2 = C_2(n, \gamma, \delta, q, b_0)$  such that problem (1.1)- (1.2) has at least one solution from  $\dot{W}_2^2(\Omega)$  for any  $f(x) \in L_2(\Omega)$  satisfying*

$$\|f\|_{L_2(\Omega)} \leq C_2(\text{mes}_n \Omega)^{\frac{-n+(n-4)q}{2n(q-1)}}. \quad (2.1)$$

**Proof.** We apply Schauder's theorem on continuous mappings of a convex and the compact set into itself in a Banach space (see, e.g. [5, p. 257]).

Take  $L_{2q}(\Omega)$  as a Banach space. In this space we define the set of functions

$$V_2 = \left\{ u \in \dot{W}_2^2(\Omega) : \|u\|_{W_2^2(\Omega)} \leq K \right\},$$

where the number  $K$  will be chosen later. For every  $u(x) \in L_{2q}(\Omega)$ ,  $f(x) \in L_2(\Omega)$  we denote by  $v(x) \in \dot{W}_2^2(\Omega)$  the solution of the problem:

$$Lv + g(x, u) = f(x), \quad x \in \Omega, \quad (2.2)$$

$$v|_{\partial\Omega} = 0. \quad (2.3)$$

For any  $u(x) \in V_2$ ,  $f(x) \in L_2(\Omega)$  problem (2.2)-(2.3) is uniquely solvable in the space  $\dot{W}_2^2(\Omega)$ , since under our assumptions we come to the Dirichlet problem for the equation

$$Lv = F(x), \quad x \in \Omega, \quad (2.4)$$

where  $F = f(x) - g(x, u) \in L_2(\Omega)$  (see [7]).

Indeed,

$$\|F\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)} \leq \|f\|_{L_2(\Omega)} + b_0 \|u\|_{L_{2q}(\Omega)}^q.$$

$\|u\|_{L_{2q}(\Omega)}^q$  is finite since the space  $\dot{W}_2^2(\Omega)$  is continuously embedded into  $L_{2q}(\Omega)$  for  $1 \leq q < \frac{n}{n-4}$ , therefore  $F \in L_2(\Omega)$  [5, see p. 154].

Denote by  $A$  the operator which transforms  $u$  to  $v$  in the problem (2.2), (2.3):

$$Au = v.$$

Let us show that the operator  $A$  is continuous in  $L_{2q}(\Omega)$ .

Let  $u_n \rightarrow u_0$  in  $L_{2q}(\Omega)$  as  $n \rightarrow \infty$ , where  $u_n, u_0 \in L_{2q}(\Omega)$  and  $v_n = Au_n$ ;  $v_0 = Au_0$ . Then

$$\begin{aligned} Lv_n &= -g(x, u_n) + f, \\ Lv_0 &= -g(x, u_0) + f. \end{aligned} \quad (2.5)$$

We show that  $v_n \rightarrow v_0$  in the norm of the space  $L_{2q}(\Omega)$ . We have

$$L(v_n - v_0) = -(g(x, u_n) - g(x, u_0)). \quad (2.6)$$

Having multiplied both parts of equality (2.5) by  $\Delta(v_n - v_0)$ , we obtain:

$$L(v_n - v_0)\Delta(v_n - v_0) = -(g(x, u_n) - g(x, u_0))\Delta(v_n - v_0).$$

Hence, using estimate (1.6) and Holder inequality, we obtain

$$\|\Delta(v_n - v_0)\|_{L_2(\Omega)} \leq C_3(n, \gamma, \delta) \|g_n - g\|_{L_2(\Omega)}, \quad (2.7)$$

where  $g_n = g(x, u_n)$ ,  $g = g(x, u_0)$ .

Let us show that

$$\|g_n - g\|_{L_2(\Omega)} \rightarrow 0 \quad \text{for } n \rightarrow \infty. \quad (2.8)$$

From  $u_m \rightarrow u$  in  $L_{2q}(\Omega)$  it follows that  $u_{m_k} - u \rightarrow 0$  almost everywhere in  $\Omega$  for some subsequence  $\{u_{m_k}\}$ . Since  $u_{m_k} \rightarrow u$  is convergent in  $L_{2q}(\Omega)$ , the sequence  $\{|u_{m_k}|^{2q}\}$  is equi integrable. Therefore the Caratheodory condition and the growth condition (1.5) imply the equicontinuity of the sequence  $\{|g_{m_k} - g|^2\}$  and the convergence of  $g_{m_k} - g \rightarrow 0$  almost everywhere in  $\Omega$  as  $m_k \rightarrow \infty$ . Then, applying Vitali's theorem, we obtain

$$\|g_{m_k} - g\|_{L_2(\Omega)} \rightarrow 0 \quad \text{for } m_k \rightarrow \infty. \quad (2.9)$$

The same reasoning also holds for the sequence  $\{g_m - g\}$ . Indeed, let  $\lim_{m_s \rightarrow \infty} \|g_{m_s} - g\|_{L_2(\Omega)} = \delta, \delta \neq 0$  for some subsequence  $\{g_{m_s} - g\}$ . Repeating the above arguments for this subsequence, we find that it has a subsequence for which (2.9) is fulfilled.

The obtained contradiction proves the convergence of  $\|g_m - g\|_{L_2(\Omega)} \rightarrow 0$  as  $m \rightarrow \infty$ .

Now taking (2.8) into account, from (2.7) we have

$$\|\Delta(v_n - v_0)\|_{L_2(\Omega)} \rightarrow 0, \quad (n \rightarrow \infty).$$

Applying the estimate

$$\|v_n - v_0\|_{W_2^2(\Omega)} \leq C_3 \|\Delta(v_n - v_0)\|_{L_2(\Omega)},$$

we obtain

$$\|v_n - v_0\|_{W_2^2(\Omega)} \rightarrow 0 \quad (n \rightarrow \infty).$$

By virtue of the embedding theorem  $W_2^2(\Omega) \hookrightarrow L_{2q}(\Omega)$ , we have

$$\|v_n - v_0\|_{L_{2q}(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The continuity of the operator  $A$  has been proved.

Let us prove that the set  $V_2$  is convex and compact in  $L_{2q}(\Omega)$  and that the operator  $A$  transforms it to itself.

For  $u_1, u_2 \in V_2$  and  $v = tu_1 + (1-t)u_2, t \in [0, 1]$ , we have

$$\begin{aligned} \|v\|_{W_2^2(\Omega)} &= \|tu_1 + (1-t)u_2\|_{W_2^2(\Omega)} \leq t\|u_1\|_{W_2^2(\Omega)} + \\ &+ (1-t)\|u_2\|_{W_2^2(\Omega)} \leq tK + (1-t)K = K, \end{aligned}$$

which means the convexity of  $V_2$ .

The set  $V_2 \subset L_{2q}(\Omega)$  is compact by virtue of the theorem on compact embedding  $W_2^2 \hookrightarrow L_{2q}(\Omega)$  whenever

$$1 \leq q < \frac{n}{n-4}. \quad (2.10)$$

Let us show that for a certain choice of  $K$  the operator  $A$  transforms  $V_2$  to itself.

For the solution of the Dirichlet problem for equation (2.4) we have

$$\|v\|_{W_2^2(\Omega)} \leq C_4(n, \gamma, \delta) \|F\|_{L_2(\Omega)} \leq C_4 \left[ \|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Omega)} \right]. \quad (2.11)$$

Furthermore,

$$\begin{aligned} \|g\|_{L_2(\Omega)} &\leq \left( \int_{\Omega} b_0^2 |u|^{2q} dx \right)^{1/2} \leq b_0 (\text{mes}_n \Omega)^{\frac{1}{2} - \frac{n-4}{n}q} \left( \int_{\Omega} u^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{2n}q} \leq \\ &\leq b_0 (\text{mes}_n \Omega)^{\frac{1}{2}(1 - \frac{n-4}{n}q)} C_5(n, q) \|u\|_{W_2^2(\Omega)}^q. \end{aligned} \quad (2.12)$$

Here we have used the above-mentioned embedding theorem for (2.10).

Using the estimate (2.12) in (2.11), we obtain

$$\begin{aligned} \|v\|_{W_2^2(\Omega)} &\leq C_6 \left[ b_0 (\text{mes}_n \Omega)^{\frac{1}{2}(1 - \frac{n-4}{n}q)} \|u\|_{W_2^2(\Omega)}^q + \|f\|_{L_2(\Omega)} \right] \leq \\ &\leq C_6 \left[ K^q b_0 (\text{mes}_n \Omega)^{\frac{1}{2}(1 - \frac{n-4}{n}q)} + \|f\|_{L_2(\Omega)} \right], \end{aligned}$$

where  $C_6 = C_6(n, \delta, \gamma, q)$ .

Let us assume that  $K$  satisfies the following estimate

$$\left[ K^q b_0 (\text{mes}_n \Omega)^{\frac{1}{2}(1 - \frac{n-4}{n}q)} + \|f\|_{L_2(\Omega)} \right] \leq K. \quad (2.13)$$

For such a number  $K$  to exist it is sufficient that

$$\|f\|_{L_2(\Omega)} \leq C_7 (\text{mes}_n \Omega)^{-\left(\frac{n-(n-4)q}{2n(q-1)}\right)},$$

where  $C_7 = C_7(n, \delta, \gamma, q)$ .

Indeed, if we introduce the notation

$$a = b_0 (\text{mes}_n \Omega)^{\frac{1}{2}(1 - \frac{n-4}{n}q)}, \quad b = \|f\|_{L_2(\Omega)},$$

then inequality (2.13) reads

$$aK^q + b \leq K, \quad aK^q - K + b \leq 0, \quad K > 0. \quad (2.14)$$

The function  $\Psi(K) = aK^q - K$ ,  $K \geq 0$ , takes a minimal value for  $K_0 = \left(\frac{1}{qa}\right)^{\frac{1}{q-1}}$ . Indeed,  $\Psi'(K) = aqK^{q-1} - 1$ , then for  $K_0^{q-1} = \frac{1}{qa}$  we have  $\Psi'(K_0) = 0$ ,  $\Psi''(K_0) > 0$ . Therefore, for  $b \leq \Psi(K_0)$  inequality (2.13) is solvable with respect to  $K$ . Theorem 2.1 is proved.

In the case  $1 \leq n \leq 4$  the following statement is true.

**Theorem 2.2.** *Let conditions (1.3)-(1.5) be fulfilled and  $1 \leq n < 4$  ( $n = 4$ ),  $1 < q < \infty$ ,  $\partial\Omega \in C^2$ . Then there exists a sufficiently small positive constant  $C_8 = C_8(n, \gamma, \delta, q, b_0)$  ( $C_9 = C_9(n, \gamma, \delta, q, b_0)$ ) such that problem (1.1), (1.2) has at least one solution from  $\dot{W}_2^2(\Omega)$  for any  $f(x) \in L_2(\Omega)$  satisfying the condition*

$$\|f\|_{L_2(\Omega)} \leq C_8 (\text{mes}_n \Omega)^{-\frac{n+q(n-4)}{2n(q-1)}} \left( \|f\|_{L_2(\Omega)} \leq C_9 (\text{mes}_n \Omega)^{-\frac{1}{2(q-1)}} \right).$$

In proving this theorem, as a Banach space we take the space  $C(\overline{\Omega})(L_{2q}(\Omega))$  and argue as above applying the theorem of compact embedding  $W_2^2(\Omega) \hookrightarrow C(\overline{\Omega})(W_2^2(\Omega) \hookrightarrow L_{2q}(\Omega))$ .

### 3. Equations with Continuous Coefficients

In this section, we will consider the Dirichlet problem for semilinear elliptic equations (1.1) with continuous leading coefficients. In other words, we will consider, in  $\Omega$ , the Dirichlet problem

$$\sum_{i,j=1}^n a_{ij}(x)u_{ij} + g(x, u) = f(x), \quad x \in \Omega, \quad (3.1)$$

$$u|_{\partial\Omega} = 0, \quad (3.2)$$

when the coefficients  $a_{ij}(x)$ ,  $i, j = 1, 2, \dots, n$  of the operator  $L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$  are functions satisfying conditions (1.3) such that

$$a_{ij}(x) \in C(\bar{\Omega}), i, j = 1, 2, \dots, n. \quad (3.3)$$

Note that in the preceding section our consideration was confined to the case  $1 \leq q < \frac{n}{n-4}$ ,  $n > 4$ . This was caused by the application of existing a priori estimate (1.6) to an equation with discontinuous coefficients.

In the case of equations with continuous coefficients we will apply a priori estimate in  $W_p^2(\Omega)$  ( see, e.g. [5, Lemma 9.17]).

The following statement is true.

**Theorem 3.1.** *Let conditions (1.3), (1.5) and (3.3) be fulfilled,  $1 < q < \infty$ ,  $p > \frac{n}{2q}$ ,  $p \geq 1$ ,  $\partial\Omega \in C^2$ . Then there exists a sufficiently small positive constant  $C_{10} = C_{10}(n, \gamma, \delta, q, b_0)$  such that problem (3.1), (3.2) has at least one solution from  $\dot{W}_p^2(\Omega)$  for any  $f(x) \in L_p(\Omega)$  satisfying*

$$\|f\|_{L_p(\Omega)} \leq C_{10}(\text{mes}_n \Omega)^{-\frac{n+(n-2p)q}{pn(q-1)}}. \quad (3.4)$$

**Proof.** In this case, as a Banach space take:  $L_{pq}(\Omega)$  if  $\frac{n}{2q'} < p < \frac{n}{2}$ ,  $p \geq 1$ ;  $C^\alpha(\Omega)$  ( $0 < \alpha < 2 - \frac{n}{p}$ ) if  $\frac{n}{2} < p \leq n$ ;  $C^{1,\alpha}(\Omega)$  ( $0 < \alpha < 1 - \frac{n}{p}$ ) if  $p > n$ .

Assume  $V_p = \left\{ u \in \dot{W}_p^2(\Omega) : \|u\|_{W_p^2(\Omega)} \leq K \right\}$ .

Denote by  $v(x) \in \dot{W}_p^2(\Omega)$  the solution of the problem

$$Lv + g(x, u) = f(x), \quad x \in \Omega, \quad (3.5)$$

$$v|_{\partial\Omega} = 0, \quad (3.6)$$

where  $u(x)$  is an arbitrary function from  $L_{pq}$  if  $\frac{n}{2q'} < p < \frac{n}{2}$ ,  $p \geq 1$ ;  $C^\alpha(\Omega)$  ( $0 < \alpha < 2 - \frac{n}{p}$ ) if  $\frac{n}{2} < p \leq n$ ;  $C^{1,\alpha}(\Omega)$  ( $0 < \alpha < 1 - \frac{n}{p}$ ) if  $p > n$ .

For any  $u(x) \in V_p$ ,  $f \in L_p(\Omega)$  problem (3.5)-(3.6) is solvable in the spaces  $\dot{W}_p^2(\Omega)$  since under these assumptions we deal with the solvability of the Dirichlet problem for equation (2.4), where  $F = f(x) - g(x, u) \in L_p(\Omega)$  (for the solvability of the Dirichlet problem [5, Theorem 9.15]).

By virtue of the embedding  $W_p^2(\Omega) \hookrightarrow L_{pq}(\Omega)$  we have

$$\|F\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)} + \|g\|_{L_p(\Omega)} \leq \|f\|_{L_p(\Omega)} + b_0 \|u\|_{L_{pq}(\Omega)}^q,$$

therefore  $F \in L_p(\Omega)$ .

Denote by  $A$  the operator transforming  $u$  to  $v$  and show that the operator  $A$  is continuous. For that, let  $u_n \rightarrow u_0$  in  $L_{pq}(\Omega)$  as  $n \rightarrow \infty$ ,  $u_n, u_0 \in L_{pq}(\Omega)$  and  $v_n, v_0$  be solutions of (2.4) in  $W_p^2(\Omega)$ . Then  $v_n - v_0$  satisfies (2.5). By virtue of the priori estimate

$$\|v\|_{W_p^2(\Omega)} \leq C_{11}(n, \gamma, p) \|Lv\|_{L_p(\Omega)}, \quad v \in W_p^2(\Omega),$$

it follows

$$\|v_n - v_0\|_{W_p^2(\Omega)} \leq C_{11} \|g_n - g\|_{L_p(\Omega)}.$$

Repeating the reasoning of the proof of Theorem 2.1 we get

$$\|v_n - v_0\|_{W_p^2(\Omega)} \rightarrow 0 (n \rightarrow \infty).$$

This implies

$$\|v_n - v_0\|_{L_{pq}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The compactness of  $V_p$  follows from the compact embedding  $W_p^2(\Omega)$  into  $L_{pq}(\Omega)$ , while the convexity is obvious.

Let us show that for a certain choice of  $K$  the operator  $A$  transforms  $V_p$  to itself. For the solution of the Dirichlet problem (3.1) we have the estimate

$$\begin{aligned} \|v\|_{W_p^2(\Omega)} &\leq C_{12}(\gamma, n) \|Lv\|_{L_p(\Omega)} \leq C_{12} \left[ \|g\|_{L_p(\Omega)} + \|f\|_{L_p(\Omega)} \right] \leq \\ &\leq C_{13} \left[ (mes_n \Omega)^{\frac{1}{p} \left(1 - \frac{n-2p}{n} q\right)} b_0 \|u\|_{W_p^2(\Omega)}^q + \|f\|_{L_p(\Omega)} \right] \leq \\ &\leq C_{13} \left[ (mes_n \Omega)^{\frac{1}{p} \left(1 - \frac{n-2p}{n} q\right)} b_0 K^q + \|f\|_{L_p(\Omega)} \right], \end{aligned}$$

where  $C_{13} = C_{13}(n, \gamma, \delta, q, b_0)$ .

From (3.4) it follows that, the inequality

$$C_{13} \left[ K^q b_0 (mes_n \Omega)^{\frac{1}{2} \left(1 - \frac{n-2p}{n} q\right)} + \|f\|_{L_p(\Omega)} \right] \leq K$$

is solvable with respect to  $K > 0$ , i.e., the operator  $A$  transforms  $V_p$  to itself.

Theorem 3.1 has been proved.

**Remark 3.1.** *The assertion of Theorem 3.1 remains valid in the case of nonlinear equation*

$$\sum_{i,j=1}^n a_{ij}(x)(x, u, u_x) u_{x_i x_j} + g(x, u) = f(x)$$

with the Cordes type condition

$$esssup_{x \in \Omega} \frac{\sum_{i,j=1}^n a_{ij}^2(x, \xi, \eta)}{\left( \sum_{i=1}^n a_{ii}^2(x, \xi, \eta) \right)^2} \leq \frac{1}{n-1} - \delta; \quad \xi \in E_1, \quad \eta \in E_n.$$

As to the existence of positive solutions of problem (3.1), (3.2) we make the following

**Remark 3.2.** *Let all the conditions of Theorem 3.1 be fulfilled for the operator  $L$ , domain  $\Omega$  and positive functions  $g(x, u)$  and  $f(x)$ . Let further  $p \geq n$ . Then*

there exists a sufficiently small positive constant  $C_{14} = C_{14}(\gamma, n, p, q, b_0)$  such that problem (3.1), (3.2) has at least one positive solution from  $\dot{W}_p^2(\Omega)$  for any  $f(x) \in L_p(\Omega)$  satisfying the condition

$$\|f\|_{L_p(\Omega)} \leq C_{14}(\text{mes}_n \Omega)^{-\frac{n+(n-2p)q}{pn(q-1)}}.$$

The proof of this statement is based on that of Theorem 3.1. In this case the class  $V_p$  is defined in the form

$$V_p = \left\{ u \in \dot{W}_p^2(\Omega) : \|v\|_{W_p^2(\Omega)} \leq K, \quad u(x) > 0, \quad x \in \Omega \right\}.$$

The class  $V_p$  is preserved by the operator  $A$  if we use the reasoning of Theorem 3.1 and the maximum principle [2] for solutions of elliptic equations in the space  $W_p^2(\Omega)$  for  $p \leq n$ .

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