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DIRECT AND INVERSE THEOREMS IN VARIABLE EXPONENT SMIRNOV CLASSES

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Abstract. Let G be a simple connected bounded domain in the complex plane $\mathbb{C}$. Imposing some additional conditions on the variable exponent $p(\cdot)$, we prove direct and inverse theorems of approximation theory in the variable exponent Smirnov classes $E^{p(\cdot)}(G)$, when the boundary $\Gamma := \partial G$ is a Carleson curve or so called regular Jordan curve. A constructive characterization of Lipschitz subclass of $E^{p(\cdot)}(G)$ is also obtained.

1. Introduction

By F we denote the segment $\mathbb{T} : = [0, 2\pi]$ or any rectifiable Jordan curve $\Gamma \subset \mathbb{C}$.

We say that a Lebesgue measurable function $p(\cdot) : F \to [1, \infty)$ belongs to $\mathcal{P}_{\log}(F)$ if it satisfies the conditions:

$$1 < p_- := ess \inf_{z\in F} p(z) \leq ess \sup_{z\in F} p(z) =: p_+ < \infty$$

$$|p(z_1) - p(z_2)| \ln(|F| / |z_1 - z_2|) \leq c, \ \ \forall z_1, z_2 \in F,$$

where $|F|$ is the Lebesgue measure of F.

For a given $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$ we define the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ as the set of Lebesgue measurable functions f defined on Γ, such that $\int_\Gamma |f(z)|^{p(z)} |dz| < \infty$. It becomes a Banach space with respect to the norm:

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf \left\{ \lambda \geq 0 : \int_\Gamma |f(z)/\lambda|^{p(z)} |dz| \leq 1 \right\} < \infty.$$

In the case of $F := \mathbb{T}$ it reduces to the variable exponent Lebesgue space $L^{p(\cdot)}([0, 2\pi])$ and then we have

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf \left\{ \lambda \geq 0 : \int_0^{2\pi} |f(x)/\lambda|^{p(x)} dx \leq 1 \right\} < \infty.$$

Let G be a simple connected bounded domain in the complex plane $\mathbb{C}$ with the rectifiable Jordan boundary Γ and let $E^p(G)$, $1 \leq p < \infty$, be the classical Smirnov class of analytic functions in G. If $f \in E^p(G)$, then f is analytic in G and there exists a sequence (γ_n) of rectifiable Jordan curves γ_n, $n = 1, 2, ...$ in G,

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tending to the boundary in the sense that γ_n eventually surrounds each compact subdomain of G, such that

$$\int_{\gamma_n} |f(z)|^p |dz| \leq M < \infty$$

for some constant M nondepending of n.

Definition 1.1 *Let $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$. The variable exponent Smirnov class $E^{p(\cdot)}(G)$ of analytic functions in G is defined as*

$$E^{p(\cdot)}(G) := \left\{ f \in E^1(G) : f \in L^{p(\cdot)}(\Gamma) \right\}.$$

In the last thirty years variable exponent Lebesgue spaces $L^{p(\cdot)}$, defined on the real line and also on domains of the n dimensional Euclidean spaces were investigated intensively. There are excellent monographs relating to these investigations [5, 7], and also the monograph [27], where investigated the approximation problems in these spaces. Moreover, in the works [27]-[30], [32]-[34], [1]-[4], [21]-[23] and [10, 11, 16, 18] were proved different direct and inverse theorems and also were studied approximation properties of different summation methods in $L^{p(\cdot)}([0, 2\pi])$. Some of these results on approximation were later transferred to the complex plane. In particular, were obtained direct and inverse theorems in $E^{p(\cdot)}(G)$ and were considered constructive characterization problems in the Lipschitz subclasses of $E^{p(\cdot)}(G)$ [19, 1, 15, 13, 14]. Here, in all studies considered in the complex plane the boundary of domain assumed to be Dini-smooth curve.

Definition 1.2 *A smooth Jordan curve Γ is called Dini-smooth, if*

$$\int_0^\delta \frac{\omega(\theta, s)}{s} ds, \quad \delta > 0,$$

where $\theta(s)$ is the angle, between the tangent of Γ and the positive real axis expressed as a function of arc length s, with the modulus of continuity $\omega(\theta, s) := \sup\{|\theta(t_1) - \theta(t_2)| : |t_1 - t_2| \leq s\}$.

In this paper imposing some additional conditions on the variable exponent $p(\cdot)$, we study approximation problems in the variable exponent Smirnov classes $E^{p(\cdot)}(G)$, when Γ is a Carleson curve or so called regular Jordan curve.

Definition ***1.3*** *A rectifiable Jordan curve Γ is called regular if there exists a constant $c > 0$ such that for every $r > 0$,*

$$\sup\{|\Gamma \cap D(z, r)| : z \in \Gamma\} \leq cr,$$

where $D(z, r)$ is the open disk with radius r and centered at z, and $|\Gamma \cap D(z, r)|$ is the length of the intersection $\Gamma \cap D(z, r)$.

We denote by S the set of all regular Jordan curves in the complex plane.

This class is very wide and includes smooth and piecewise smooth curves, quasismooth curves and also some other class of curves, in particular.

Let's give necessary definitions before giving the main results.

Let U be the unit disk, $G^{-} := Ext\Gamma$, $T := \partial U$, $U^{-} := ExtT$. We denote by φ the conformal mapping of G^{-} onto U^{-} normalized by the conditions $\varphi(\infty) = \infty$ and $\lim_{z\to\infty} \varphi(z)/z > 0$, with the inverse ψ. The mappings φ and ψ have continuous extensions to Γ and T, respectively, the derivatives φ' and ψ' have definite nontangential limit values a.e. on Γ and T, which are integrable with respect to the Lebesgue measure on Γ and T, respectively[9, pp. 419-438].

We define an important subclass of $\mathcal{P}_{\log}(\Gamma)$, that gives us an opportunity for investigation of approximation problems in $E^{p(\cdot)}(G)$, defined on the domains with regular boundary.

Definition 1.4 *We say that* $p(\cdot) \in \mathcal{P}_{\log}^{a}\left(\overline{G^{-}}\right)$, *if* $p(\cdot)$ *is a non-zero analytic in* G^{-} *and continuous on* $\overline{G^{-}}$ *function such that* $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$.

For $g \in L^{p(\cdot)}(T)$ we consider the mean value function

$$\sigma_h g(w) := \frac{1}{h}\int_0^h g\left(we^{it}\right) dt\ , \quad w \in T,\ 0 < h < \pi.$$

If $p(\cdot) \in \mathcal{P}_{\log}(T)$, then by boundedness [6] of maximal operator in $L^{p(\cdot)}(T)$, the operator σ_h is bounded from $L^{p(\cdot)}(T)$ to $L^{p(\cdot)}(T)$. Therefore, the following definition is correct.

Definition 1.5 [13, 28] *Let* $f \in L^{p(\cdot)}(T)$ *with* $p(\cdot) \in \mathcal{P}_{\log}(T)$ *and let*

$$\Delta_t^r f(w) := \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} f\left(we^{ist}\right)\ , \qquad r = 1, 2, \ldots .$$

We define the rth *modulus of smoothness* $\Omega_r(f,\delta)_{p(\cdot)}$ *as*

$$\Omega_r(f,\delta)_{p(\cdot),T} := \sup_{|h|\le\delta} \left\| \frac{1}{h}\int_0^h \Delta_t^r f(w)\, dt \right\|_{L^{p(\cdot)}(T)}, \qquad \delta > 0.$$

If $f,\ g \in L^{p(\cdot)}(T)$ with $p(\cdot) \in \mathcal{P}_{\log}(T)$, then the rth modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot),T}$ has the following properties :

i)$\Omega_r(f,\delta)_{p(\cdot),T}$ *is non-negative, continuous and non-decreasing function of* $\delta > 0$,

ii)$\Omega_r(f,\delta)_{p(\cdot),T}$ *is uniformly bounded function in* $L^{p(\cdot)}(T)$,

iii) $\lim_{\delta\to 0} \Omega_r(f,\delta)_{p(\cdot),T} = 0$,

iv)$\Omega_r(f+g,\delta)_{p(\cdot),T} \le \Omega_r(f,\delta)_{p(\cdot),T} + \Omega_r(g,\delta)_{p(\cdot),T}$.

Note that the properties *ii)* and *iii)* were established in [17, *Lemma 2 and 3*].

For a given $f \in E^{p(\cdot)}(G)$ we set

$$f_0(w) := f(\psi(w))\left(\psi'(w)\right)^{1/p_0(w)},\ w \in T,$$

where $p_0(w) := p(\psi(w))$ and define the Cauchy type integrals:

$$f_0^{+}(w) := \frac{1}{2\pi i}\int_T \frac{f_0(\tau)}{\tau - w} d\tau,\ w \in U \text{ and } f_0^{-}(w) := \frac{1}{2\pi i}\int_T \frac{f_0(\tau)}{\tau - w} d\tau,\ w \in U^{-}$$

which are analytic in U and U^{-}, respectively.

If $f \in E^{p(\cdot)}(G)$, then we define its modulus of smoothness as

$$\Omega_r(f,\delta)_{p(\cdot),G} := \Omega_r\left(f_0^+,\delta\right)_{p(\cdot),T} \tag{1.1}$$

Let $\mathcal{F}_n$ be the class of algebraic polynomials with degree not exceeding n and let

$$E_n(f)_{p(\cdot),G} := \inf\left\{\|f-p_n\|_{L^{p(\cdot)}(\Gamma)} : p_n \in \mathcal{F}_n\right\}$$

be the best polynomial approximation number of f in $\mathcal{F}_n$.

Main results of this work can be formulated as following:

Theorem 1.1 *Let $\Gamma \in S$, $p(\cdot) \in \mathcal{P}_{\log}^a\left(\overline{G^-}\right)$ and let $p_0 \in \mathcal{P}_{\log}^a\left(\overline{U^-}\right)$. If $f \in E^{p(\cdot)}(G)$, then there exists a positive constant $c(p,r)$ such that for every $n \in \mathbb{N}$ the inequality*

$$E_n(f)_{p(\cdot),G} \leq c(p,r)\Omega_r(f,1/n)_{p(\cdot),G}$$

holds.

Note that in the case of $r=1$ *Theorem 1.1* was proved in [19].

If for a given $\alpha > 0$ and $r := [\alpha]+1 > 0$ we define the variable exponent Lipschitz subclass $Lip(\alpha,p(\cdot),G)$ as

$$Lip(\alpha,p(\cdot),G) := \left\{f \in E^{p(\cdot)}(G) : \Omega_r(f,\delta)_{p(\cdot),G} = \mathcal{O}(\delta^\alpha),\ \delta > 0\right\},$$

then from *Theorem 1.1* we have

Corollary 1.1 *Let $\Gamma \in S$ and let $p(\cdot) \in \mathcal{P}_{\log}^a\left(\overline{G^-}\right)$, $p_0 \in \mathcal{P}_{\log}^a\left(\overline{U^-}\right)$. If $f \in Lip(\alpha,p(\cdot),G)$, then there exists a positive constant $c(p,r)$ such that for every number $n \in \mathbb{N}$ the inequality*

$$E_n(f)_{p(\cdot),G} \leq c(p,r)n^{-\alpha}$$

holds.

Theorem 1.2 *Let $\Gamma \in S$, $p(\cdot) \in \mathcal{P}_{\log}^a\left(\overline{G^-}\right)$ and let $p_0 \in \mathcal{P}_{\log}^a\left(\overline{U^-}\right)$. If $f \in E^{p(\cdot)}(G)$, then there exists a positive constant $c(p,r)$ such that for every number $n \in \mathbb{N}$ the inequality*

$$\Omega_r(f,1/n)_{p(\cdot),G} \leq \frac{c(p,r)}{n^r}\sum_{\upsilon=0}^{n}(\upsilon+1)^{r-1}E_\upsilon(f)_{p(\cdot),G}$$

holds.

After simply computations, *Theorem 1.2* implies

Corollary 1.2 *Let $\Gamma \in S$ and let $p(\cdot) \in \mathcal{P}_{\log}^a\left(\overline{G^-}\right)$, $p_0(\cdot) \in \mathcal{P}_{\log}^a\left(\overline{U^-}\right)$. If $E_n(f)_{p(\cdot),G} = \mathcal{O}(n^{-\alpha})$, then $f \in Lip(\alpha,p(\cdot),G)$.*

Combining *Corollaries 1.1 and 1.2* we obtain the following constructive characterization of $Lip(\alpha,p(\cdot),G)$:

Theorem 1.3 *Let* $\Gamma \in S$ *and let* $p(\cdot) \in \mathcal{P}^{a}_{\log}\left(\overline{G^{-}}\right)$, $p_0 \in \mathcal{P}^{a}_{\log}\left(\overline{U^{-}}\right)$. *Then the following equivalence holds:*

$$f \in Lip(\alpha, p(\cdot), G) \Leftrightarrow E_n(f)_{p(\cdot),G} \leq c(p)n^{-\alpha}.$$

Definition 1.6 *Let* $0 < \beta \leq 1$. *If* $|\psi(w_1) - \psi(w_2)| \leq c|w_1 - w_2|^{\beta}$ *for* $\forall w_1, w_2 \in T$ *with a constant* c, *then we say that* $\Gamma \in D^{\beta}$.

Class D^{β} is sufficiently wide; every Dini-smooth curve belongs to D^1. If G is convex, then $\psi \in D^1$(*see*, [26, p.78]). If Γ is smooth, then $\Gamma \in D^{\beta}$ for every $\beta \in (0,1)$. Every rectifiable quasiconformal curve belongs toD^{β} for some $\beta \in (0,1)$. There are(*see*, [8, remarks]) also some other class of curves belonging to D^{β}.

As was proved in [19] if $L \in D^{\beta}$, then the condition $p(\cdot) \in \mathcal{P}^{a}_{\log}\left(\overline{G^{-}}\right)$ implies that $p_0(\cdot) \in \mathcal{P}^{a}_{\log}\left(\overline{U^{-}}\right)$. Hence in the case of $\Gamma \in S \cap D^{\beta}$ the condition $p_0 \in \mathcal{P}^{a}{}_{\log}\left(\overline{U^{-}}\right)$ can be omitted in the formulations of *Theorems 1.1-1.3,* and then we have in particular the following theorems:

Theorem 1.4 *Let* $\Gamma \in S \cap D^{\beta}$, $0 < \beta \leq 1$. *If* $f \in E^{p(\cdot)}(G)$, $p(\cdot) \in \mathcal{P}^{a}_{\log}\left(\overline{G^{-}}\right)$, *then there exists a positive constant* $c(p,r)$ *such that for every* $n \in \mathbb{N}$ *the inequality*

$$E_n(f)_{p(\cdot),G} \leq c(p,r)\Omega_r(f, 1/n)_{p(\cdot),G}$$

holds.

Theorem 1.5 *Let* $\Gamma \in S \cap D^{\beta}$, $0 < \beta \leq 1$. *If* $f \in E^{p(\cdot)}(G)$, *with* $p(\cdot) \in \mathcal{P}^{a}_{\log}\left(\overline{G^{-}}\right)$, *then there exists a positive constant* $c(p,r)$ *such that for every* $n \in \mathbb{N}$ *the inequality*

$$\Omega_r(f, 1/n)_{p(\cdot),G} \leq \frac{c(p,r)}{n^r} \sum_{v=0}^{n} (v+1)^{r-1} E_v(f)_{p(\cdot),G}$$

holds.

Theorem 1.6 *Let* $\Gamma \in S \cap D^{\beta}$, $0 < \beta \leq 1$, *and let* $p(\cdot) \in \mathcal{P}^{a}_{\log}\left(\overline{G^{-}}\right)$, *then the following equivalence holds:*

$$f \in Lip(\alpha, p(\cdot), G), \Leftrightarrow E_n(f)_{p(\cdot),G} \leq c(p)n^{-\alpha}.$$

2. Some Definitions, Notations, and Auxiliary Results

Let $f \in L^1(\Gamma)$. Then the functions f^{+} and f^{-} defined by

$$f^{+}(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G, \;\; f^{-}(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G^{-}$$

are analytic in G and G^{-}, and $f^{-}(\infty) = 0$. By [24], if $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$, then Cauchy's singular integral

$$S_{\Gamma}(f)(z_0) := (P.V.)\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta := \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int_{\Gamma \cap \{\zeta : |\zeta - z_0| > \varepsilon\}} \frac{f(\zeta)}{\zeta - z_0} d\zeta$$

exists *a.e.* on Γ and the operator S_Γ is a bounded operator from $L^{p(\cdot)}(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$.

Hence, by Privalov's theorem [9, pp. 431] the formulas

$$f^{+}(z)=S_L(f)(z)+f(z)/2 \quad \text{and} \quad f^{-}(z)=S_L(f)(z)-f(z)/2 \tag{2.1}$$

and

$$f(z)=f^{+}(z)-f^{-}(z) \tag{2.2}$$

hold *a.e.* on Γ.

The following two lemmas were proved in [19].

Lemma 2.1 *If* $p(\cdot)\in\mathcal{P}_{\log}(\Gamma)$, $\Gamma\in S$, *then* $f^{+}\in E^{p(\cdot)}(G)$ *and* $f^{-}\in E^{p(\cdot)}(G^{-})$ *for each* $f\in L^{p(\cdot)}(\Gamma)$.

Lemma 2.2 *If* $p(\cdot)\in\mathcal{P}_{\log}(\Gamma)$ *then* $f\in L^{p(\cdot)}(\Gamma)\Longleftrightarrow f_0\in L^{p_0(\cdot)}(\Gamma)$.

Let k be a nonnegative integer. It is obvious that if $p(\cdot)\in\mathcal{P}^{a}_{\log}\left(\overline{G^{-}}\right)$, then $\varphi^{k}(z)\left(\varphi'(z)\right)^{1/p(z)}$ is analytic in $\mathbb{C}\diagdown\overline{\mathbb{G}}$ and have a pole of order k at the point ∞. So there exists a polynomial $F_{k,p(\cdot)}(z)$ of degree k and an analytic function $E_{k,p(\cdot)}(z)$ in G^{-} such that $E_{k,p(\cdot)}(\infty)=0$ and $\varphi^{k}(z)\left(\varphi'(z)\right)^{1/p(z)}=F_{k,p(\cdot)}(z)+E_{k,p(\cdot)}(z)$ for every $z\in G^{-}$. The polynomials $F_{k,p(\cdot)}(z)$ $(k=1,2,3...)$ we call the $p(\cdot)-$ *Faber polynomials* for $\overline{G}$. As in the classical case [31] by means of Cauchy's integral formula it is easily seen that

$$F_{k,p(\cdot)}(z)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\varphi^{k}(\varsigma)\left(\varphi'(\varsigma)\right)^{1/p(\varsigma)}}{\varsigma-z}d\varsigma,\ z\in G$$

where $\Gamma_R:=\{z\in G^{-}:|\varphi(z)|=R>1\}$. Moreover, the polynomials $F_{k,p(\cdot)}(z)$ $(k=1,2,3...)$ can be defined as the coefficients of the series expansion [19]

$$\frac{\left(\psi'(w)\right)^{1-1/p_0(w)}}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{F_{k,p(\cdot)}(z)}{w^{k+1}},\ z\in G,w\in U^{-}.$$

Let $f\in E^{p(\cdot)}(G)$. Since $f\in E^{1}(G)$, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma-z}d\varsigma \\ &= \frac{1}{2\pi i}\int_{T}\frac{f(\psi(w))\,\psi'(w)}{\psi(w)-z}dw \\ &= \frac{1}{2\pi i}\int_{T}f(\psi(w))\,\psi'(w)^{1/p_0(w)}\,\frac{\psi'(w)^{1-1/p_0(w)}}{\psi(w)-z}dw \end{aligned}$$

for every $z\in G$. Considering this formula and the series representation stated above, we can associate with f the formal series

$$f(z)\sim\sum_{k=0}^{\infty}a_k(f)\,F_{k,p(\cdot)}(z),$$

where

$$a_k(f) := \frac{1}{2\pi i}\int_T \frac{f(\psi(w))\,\psi'(w)^{1/p_0(w)}}{w^{k+1}}dw, \quad k = 0,1,2,....$$

This series we call the $p(\cdot)-$*Faber series expansion* of f with the $p(\cdot)-$ *Faber coefficients* $a_k(f)$, $k = 0,1,2,..$.

Let $\mathcal{F}$ be the set of all algebraic polynomials (with no restrictions on the degree), and let $\mathcal{F}(U)$ be the set of traces of members of $\mathcal{F}$ on U. Defining the operator $T_{p(\cdot)}$ on $\mathcal{F}(U)$ as

$$T_{p(\cdot)}(P)(z) := \frac{1}{2\pi i}\int_T \frac{P(w)(\psi'(w))^{1-p_0(w)}}{\psi(w)-z}dw \quad , \quad z \in G,$$

we have that $T_{p(\cdot)}\left(\sum_{k=0}^{n} a_k w^k\right) = \sum_{k=0}^{n} a_k F_{k,p(\cdot)}(z)$. On the other hand, considering

$$\begin{aligned} T_{p(\cdot)}(P)(z') &= \frac{1}{2\pi i}\int_\Gamma \frac{P(\varphi(z))\,\varphi'(z)^{1/p(z)}}{\zeta - z'}dz \\ &= \left[(P\circ\varphi)(\varphi')^{1/p(\cdot)}\right]^+(z') \end{aligned}$$

for $z' \in G$ and taking the limit $z' \to z \in \Gamma$ over all non-tangential paths inside Γ, by (2.1) we have

$$T_{p(\cdot)}(P)(z) = S_\Gamma\left[(P\circ\varphi)(\varphi')^{1/p(\cdot)}\right](z) + \frac{1}{2}\left[(P\circ\varphi)(\varphi')^{1/p(\cdot)}\right](z) \qquad (2.3)$$

a. e. on Γ.

Theorem 2.1 *Let* $\Gamma \in S$, $p(\cdot) \in \mathcal{P}_{\log}(\Gamma)$ *and* $p_0 \in \mathcal{P}_{\log}(T)$. *Then the operator* $T_{p(\cdot)} : \mathcal{F}(U) \subset E^{p_0(\cdot)}(U) \to E^{p(\cdot)}(G)$ *is linear and bounded.*

Proof The linearity of $T_{p(\cdot)}$ is clear. On the other hand, by (2.3) and by boundedness of the operator S_Γ, we have that

$$\begin{aligned} &\left\|T_{p(\cdot)}(P)\right\|_{L^{p(\cdot)}(\Gamma)} \\ \leq\ &\left\|S_\Gamma\left[(P\circ\varphi)(\varphi')^{1/p(\cdot)}\right]\right\|_{L^{p(\cdot)}(\Gamma)} + \left\|\frac{1}{2}(P\circ\varphi)(\varphi')^{1/p(\cdot)}\right\|_{L^{p(\cdot)}(\Gamma)} \\ &\leq c(p)\left\|(P\circ\varphi)(\varphi')^{1/p(\cdot)}\right\|_{L^{p(\cdot)}(\Gamma)} = c(p)\left\|P\right\|_{L^{p_0(\cdot)}(T)}. \qquad \blacksquare \end{aligned}$$

Since the set $\mathcal{F}$ is dense in $E^{p_0(\cdot)}(U)$, $p_0 \in \mathcal{P}_{\log}(T)$, the operator $T_{p(\cdot)}$ can be extended to the whole of $E^{p_0(\cdot)}(U)$ as a bounded linear operator and hence we have the representation

$$T_{p(\cdot)}(g)(z) = \frac{1}{2\pi i}\int_T \frac{g(w)(\psi'(w))^{1-p_0(w)}}{\psi(w)-z}dw, \quad z \in G, \qquad (2.4)$$

for every $g \in E^{p_0(\cdot)}(U)$.

Theorem 2.2 *Let* $\Gamma \in S$, $p(\cdot) \in \mathcal{P}^a_{\log}\left(\overline{G^-}\right)$ *and* $p_0 \in \mathcal{P}_{\log}(T)$. *Then the operator* $T_{p(\cdot)} : E^{p_0(\cdot)}(U) \to E^{p(\cdot)}(G)$ *is one-to-one and onto. Moreover* $T_{p(\cdot)}\left(f_0^+\right) = f$ *for* $f \in E^{p(\cdot)}(G)$.

Proof Let $g \in E^{p_0(\cdot)}(U)$ and let $g(w) = \sum_{k=0}^{\infty} a_k w^k$ be its Taylor series expansion in U. Since $p_0 \in \mathcal{P}_{\log}(T)$, denoting $g_r(w) := g(rw)$, $0 < r < 1$, we have [25]: $\|g_r - g\|_{L^{p_0(\cdot)}(T)} \to 0$, $r \to 1^-$ and then boundedness of $T_{p(\cdot)}$ implies that

$$\left\|T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right\|_{L^{p(\cdot)}(\Gamma)} \le c\,\|g_r - g\|_{L^{p_0(\cdot)}(T)} \to 0, \quad r \to 1^-. \tag{2.5}$$

Since the series $\sum_{k=0}^{\infty} a_k w^k$ converges uniformly on $|w| = r < 1$, the series $\sum_{k=0}^{\infty} a_k r^k w^k$ converges uniformly on T. Therefore, by (2.4)we have

$$\begin{aligned} T_{p(\cdot)}(g_r)(z) &= \frac{1}{2\pi i}\int_T \frac{g_r(w)(\psi'(w))^{1-p_0(w)}}{\psi(w) - z}dw \\ &= \frac{1}{2\pi i}\int_T \sum_{k=0}^{\infty} a_k r^k w^k \frac{(\psi'(w))^{1-p_0(w)}}{\psi(w) - z}dw \\ &= \sum_{k=0}^{\infty} a_k r^k \frac{1}{2\pi i}\int_T w^k \frac{(\psi'(w))^{1-p_0(w)}}{\psi(w) - z}dw \\ &= \sum_{k=0}^{\infty} a_k r^k F_{k,p(\cdot)}, \quad z \in G. \end{aligned}$$

Hence for Faber coefficients of $T_{p(\cdot)}(g_r)$ we have $a_k\left(T_{p(\cdot)}(g_r)\right) = a_k(g)\,r^k$, $k = 0, 1, 2...$ and hence

$$a_k\left(T_{p(\cdot)}(g_r)\right) \to a_k(g), \quad r \to 1^-.$$

On the other hand, using Hölder's inequality and (2.5) we obtain that

$$\begin{aligned} &\left|a_k\left(T_{p(\cdot)}(g_r)\right) - a_k\left(T_{p(\cdot)}(g)\right)\right| \\ = &\left|\frac{1}{2\pi i}\int_T \frac{\left[T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right](\psi(w))(\psi'(w))^{1/p_0(w)}}{w^{k+1}}dw\right| \\ \le &\frac{1}{2\pi}\int_T \left|\left[T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right](\psi(w))\right| \left|(\psi'(w))^{1/p_0(w)}\right| |dw| \\ = &\frac{1}{2\pi}\int_\Gamma \left|\left[T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right](z)\right| \left|(\varphi'(z))^{1-1/p(z)}\right| |dz| \\ = &\frac{1}{2\pi}\int_\Gamma \left|\left[T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right](z)\right| \left|(\varphi'(z))^{1/q(z)}\right| |dz| \\ \le &\frac{c}{2\pi}\left\|\left(T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right)\right\|_{L^{p(\cdot)}(\Gamma)} \left\|(\varphi')^{1/q(\cdot)}\right\|_{L^{q(\cdot)}(\Gamma)} \\ \le &\frac{c_1}{2\pi}\left\|\left(T_{p(\cdot)}(g_r) - T_{p(\cdot)}(g)\right)\right\|_{L^{p(\cdot)}(\Gamma)} \le c_2\,\|g_r - g\|_{L^{p_0(\cdot)}(T)} \to 0, \quad r \to 1^-, \end{aligned}$$

and hence $a_k\left(T_{p(\cdot)}(g_r)\right) \to a_k\left(T_{p(\cdot)}(g)\right)$, $r \to 1^-$, which yields that

$$a_k\left(T_{p(\cdot)}(g)\right) = a_k(g), \quad k = 0, 1, 2, \ldots .$$

Hence, if $T_{p(\cdot)}(g) = 0$, then $a_k(g) = a_k\left(T_{p(\cdot)}(g)\right) = 0$ for $k = 0, 1, 2, \ldots$, and thus $g = 0$. This proves that the operator $T_{p(\cdot)}$ is one-to-one.

Let $f \in E^{p(\cdot)}(G)$. Then $f_0(w) = f(\psi(w))(\psi'(w))^{1/p_0(w)} \in L^{p_0(\cdot)}(T)$ and by *Lemma 2.1* we have that $f_0^+ \in E^{p_0(\cdot)}(U)$ and $f_0^- \in E^{p_0(\cdot)}(U^-)$. Using the definition of $p(\cdot) - Faber\ coefficients$ of f, (2.2) and generalized Cauchy's formula we obtain that

$$\begin{aligned} a_k(f) &= \frac{1}{2\pi i}\int_T \frac{f(\psi(w))\,\psi'(w)^{1/p_0(w)}}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_T \frac{f_0(w)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_T \frac{f_0^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int_T \frac{f_0^-(w)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_T \frac{f_0^+(w)}{w^{k+1}}dw \,. \end{aligned}$$

This means that the $p(\cdot) - Faber\ coefficients$ of f are the Taylor coefficients of f_0^+ at the origin, that is

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f)\, w^k, \quad w \in U.$$

By the first part of the proof we have

$$T_{p(\cdot)}\left(f_0^+\right)(z) = \sum_{k=0}^{\infty} a_k(f)\, F_{k,p(\cdot)}(z)$$

and hence $T_{p(\cdot)}\left(f_0^+\right) = f$, since there is no two different functions in $E^{p(\cdot)}(G)$ that have the same $p(\cdot) - Faber\ series$ representation. This means that the operator $T_{p(\cdot)}$ is onto. ∎

Lemma 2.3 *Let $\Gamma \in S$, $p(\cdot) \in \mathcal{P}^a_{\log}\left(\overline{G^-}\right)$ and let $p_0(\cdot) \in \mathcal{P}_{\log}(T)$. If $f \in E^{p(\cdot)}(G)$, then*

$$E_n\left(f_0^+\right)_{U,p_0(\cdot)} \le c_3(p)\, E_n(f)_{G,p(\cdot)} \le c_4(p)\, E_n\left(f_0^+\right)_{U,p_0(\cdot)}$$

with some positive constants $c_3(p)$ and $c_4(p)$, independent of n.

Proof By *Theorem 2.2*, the operator $T_{p(\cdot)} : E^{p_0(\cdot)}(U) \to E^{p(\cdot)}(G)$ is bounded, one-to-one and onto. The same properties hold for the inverse operator $T^{-1}_{p(\cdot)}$: $E^{p(\cdot)}(G) \to E^{p_0(\cdot)}(U)$. Hence if $f \in E^{p(\cdot)}(G)$, then $T^{-1}_{p(\cdot)}(f) = f_0^+ \in E^{p_0(\cdot)}(U)$. If $P_n^* \in \mathcal{P}_n$ is the polynomial of the best approximation to f in $E^{p(\cdot)}(G)$, then $T^{-1}_{p(\cdot)}(P_n^*) \in E^{p_0(\cdot)}(U)$ and by boundedness of $T^{-1}_{p(\cdot)}$ we get

$$\begin{aligned} E_n\left(f_0^+\right)_{U,p_0(\cdot)} &\le \left\| f_0^+ - T^{-1}_{p(\cdot)}(P_n^*)\right\|_{L^{p_0(\cdot)}(T)} = \left\| T^{-1}_{p(\cdot)}(f) - T^{-1}_{p(\cdot)}(P_n^*)\right\|_{L^{p_0(\cdot)}(T)} \\ &\le \left\| T^{-1}_{p(\cdot)}\right\| \left\| f - P_n^*\right\|_{L^{p(\cdot)}(\Gamma)} = c_3(p) E_n(f)_{G,p(\cdot)} \end{aligned}$$

with some constant $c_3(p) > 0$. Hence the first inequality is proved. Similarly, can be proved the second inequality. ∎

The following *Theorems 2.3 and 2.4* are the disk versions of direct and inverse theorems proved in [17, *Theorem 1 and Theorem 2*] .

Theorem 2.3 *Let* $p(\cdot) \in \mathcal{P}_{\log}(T)$ *and let* $g \in E^{p(\cdot)}(U)$. *Then there exists a positive constant* $c(p,r) > 0$ *such that for every number* $n \in \mathbb{N}$ *the inequality* $E_n(g)_{U,p(\cdot)} \leq c(p,r)\,\Omega_r(g,1/n)_{p(\cdot),T}$ *holds.*

Theorem 2.4 *If* $g \in E^{p(\cdot)}(U)$, $p(\cdot) \in \mathcal{P}_{\log}(T)$, *then there exists a positive constant* $c(p,r) > 0$ *such that for every number* $n \in \mathbb{N}$ *the inequality*

$$\Omega_r(g,1/n)_{p(\cdot),T} \leq \frac{c(p,r)}{n^r} \sum_{\upsilon=0}^{n} (\upsilon+1)^{r-1} E_\upsilon(f)_{U,p(\cdot)} \quad n = 1,2,,...$$

holds.

3. Proofs of Main Results

Proof of Theorem 1.1 Let $f \in E^{p(\cdot)}(G)$. Then $f_0^+ = T_p^{-1}(f) \in E^{p_0(\cdot)}(U)$ and applying the second inequality of *Lemma 2.3* and *Theorem 2.3,* respectively, we have that

$$E_n(f)_{p(\cdot),G} \leq c_4(p) E_n\left(f_0^+\right)_{,p_0(\cdot),U} \leq c(p,r)\Omega_r\left(f_0^+, 1/n\right)_{p_0(\cdot),U},$$

which by (1.1) implies that $E_n(f)_{p(\cdot),G} \leq c(p)\Omega_r(f,1/n)_{p(\cdot),G}$. ■

Proof of Theorem 1.2 Let $f \in E^{p(\cdot)}(G)$. Then $f_0^+ \in E^{p_0(\cdot)}(U)$ and by applying *Theorem 2.4* in the case of $g := f_0^+$ and the first inequality of *Lemma 2.3*, we have

$$\begin{aligned} \Omega_r(f,1/n)_{p(\cdot),G} &= \Omega_r\left(f_0^+, 1/n\right)_{p(\cdot),T} \leq \frac{c(p,r)}{n^r} \sum_{\upsilon=0}^{n} (\upsilon+1)^{r-1} E_\upsilon\left(f_0^+\right)_{p(\cdot),D} \\ &\leq \frac{c(p,r)}{n^r} \sum_{\upsilon=0}^{n} (\upsilon+1)^{r-1} E_\upsilon(f)_{p(\cdot),G}. \qquad ■ \end{aligned}$$

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