

PARA-COMPLEX STRUCTURES ON LINEAR COFRAME BUNDLE WITH SASAKIAN METRIC

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Abstract. By using a Riemannian metric on a differentiable manifold, the Sasakian metric is introduced on the linear coframe bundle of the Riemannian manifold. Geometric properties of Levi-Civita connection of Sasakian metric are investigated. Also, para-complex structures on the linear coframe bundle with Sasakian metric are constructed and some interesting properties of those structures are studied.

1. Introduction

Let M be an n -dimensional manifold of class C^∞ . The problem of extending differential-geometrical structures on M to its fiber bundles has been the subject of a number of papers. An account of these can be found in Yano and Ishihara [22] (see, also [2]). In [21], Sasaki by using a Riemannian metric g on a differentiable manifold M , constructed a Riemannian metric \tilde{g} on the tangent bundle $T(M)$ of M . Then, some geometers such as Kowalski, Aso, Musso and Tricerri studied interesting geometric properties of this metric, that is called Sasaki metric (see [1], [8], [11]). Some properties and applications for the Riemannian metrics of the cotangent, linear frame, linear coframe and tensor bundles are given in [3, 5, 9, 10, 16-18]. Also noteworthy are the papers devoted to the study of various differential geometric structures, including submanifolds of Sasakian manifolds. V.A.Khan and M.A.Khan studied pseudo-slant submanifolds of Sasakian manifolds (see [7]). The main results related to the projective curvature tensor in Sasakian manifolds are due to U.K.Gautam, A.Haseeb and R.Prasad (see [6]).

Let M_{2k} be a $2k$ -dimensional differentiable manifold endowed with an almost (para) complex structure φ and a pseudo-Riemannian metric g of signature (k, k) such that $g(\varphi X, Y) = g(X, \varphi Y)$ for arbitrary vector fields X and Y on M_{2k} , i.e. g is pure with respect to φ . The metric g is called Nordenian metric. Nordenian metrics are also referred to as anti-Hermitian metrics or B-metrics. They present extensive application in mathematics as well as in theoretical physics. Many authors considered almost (para) complex Nordenian structures on the tangent, cotangent and tensor bundles [4, 12-15].

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This paper is devoted to the investigation of para-Nordenian structures in the linear coframe bundle with Sasakian metric. In 2 we briefly describe the definitions and results that are needed later, after which the Sasakian metric Sg and the Levi-Civita connection ${}^S\nabla$ of this metric in the linear coframe bundle $F^*(M)$ over a Riemannian manifold (M, g) are studied in 3. The para-Nordenian structures on the linear coframe bundle $F^*(M)$ with Sasakian metric are introduced in 4. In 5 we study the almost para-holomorphic vector fields on the linear coframe bundle $F^*(M)$ with Sasakian metric. The integrabilities of almost para-Nordenian structures on the $F^*(M)$ with Sasakian metric are investigated in 6.

2. Preliminaries

In this section we shall summarize briefly the basic definitions and results which be used later. Let (M, g) be an n -dimensional Riemannian manifold and $F^*(M)$ its coframe bundle (see, [19,20]). The coframe bundle $F^*(M)$ over M consists of all pairs (x, u^*) , where x is a point of M and u^* is a basis (coframe) for the cotangent space T_x^*M . We denote by π the natural projection of $F^*(M)$ to M defined by $\pi(x, u^*) = x$. If $(U; x^1, x^2, \dots, x^n)$ is a system of local coordinates in M , then a coframe $u^* = (X^\alpha) = (X^1, X^2, \dots, X^n)$ for T_x^*M can be expressed uniquely in the form $X^\alpha = X_i^\alpha(dx^i)_x$ and hence

$$(\pi^{-1}(U); x^1, x^2, \dots, x^n, X_1^1, X_2^1, \dots, X_n^n)$$

is a system of local coordinates in $F^*(M)$ (see, [19]). Indices $i, j, k, \dots, \alpha, \beta, \gamma, \dots$ have range in $\{1, 2, \dots, n\}$, while indices A, B, C, \dots have range in

$$\{1, \dots, n, n+1, \dots, n+n^2\}.$$

We put $h_\alpha = \alpha \cdot n + h$. Summation over repeated indices is always implied.

We denote by $\mathfrak{S}_s^r(M)$ the set of all differentiable tensor fields of type (r, s) on M . Let $V = V^i\partial_i$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M$ of a vector and a covector (1-form) fields $V \in \mathfrak{S}_0^1(M)$ and $\omega \in \mathfrak{S}_1^0(M)$, respectively. Then the complete and horizontal lifts ${}^C V, {}^H V \in \mathfrak{S}_0^1(F^*(M))$ of V and the β -th vertical lifts $V_\beta \omega \in \mathfrak{S}_0^1(F^*(M))$ ($\beta = 1, 2, \dots, n$) of ω are given, respectively, by

$${}^C V = \begin{pmatrix} V^i \\ -X_j^\alpha \partial_i V^j \end{pmatrix}, \quad {}^H V = \begin{pmatrix} V^i \\ X_j^\alpha \Gamma_{ik}^j V^k \end{pmatrix}, \quad V_\beta \omega = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \omega_i \end{pmatrix} \quad (2.1)$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\} = \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial X_i^\alpha} \right\}$, (see [10] for more details).

The vertical lift of a smooth function f on M is a function ${}^V f$ on $F^*(M)$ defined by ${}^V f = f \circ \pi$.

Let (U, x^i) be a coordinate system in M . In $U \in M$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \theta^{(i)} = dx^i, i = 1, 2, \dots, n.$$

Taking account of (2.1), we easily see that the components of ${}^H X_{(i)}$ and ${}^V \theta^{(i)}$ are respectively, given by

$$D_i = {}^H X_{(i)} = (A_i^H) = \begin{pmatrix} \delta_i^h \\ X_j^\alpha \Gamma_{ih}^j \end{pmatrix}, \quad (2.2)$$

$$D_{i_\alpha} = V_\alpha \theta^{(i)} = (A_{i_\alpha}^H) = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \delta_h^i \end{pmatrix} \quad (2.3)$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\}$. We call the set $\{{}^H X_{(i)}, V_\alpha \theta^{(i)}\}$ the frame adapted to the Levi-Civita connection ∇_g . On putting

$$D_i = {}^H X_{(i)}, \quad D_{i_\alpha} = V_\alpha \theta^{(i)},$$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_\alpha}\}$. From equations (2.2), (2.3), and (2.1) we see that ${}^H V$ and $V_\alpha \omega$ have respectively, components

$${}^H V = V^i D_i, \quad {}^H V = ({}^H V^I) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \quad (2.4)$$

$$V_\alpha \omega = \sum_i \omega_i \delta_\alpha^\beta D_{i_\alpha}, \quad V_\alpha \omega = (V_\alpha \omega^I) = \begin{pmatrix} 0 \\ \delta_\alpha^\beta \omega_i \end{pmatrix} \quad (2.5)$$

with respect to the adapted frame $\{D_I\}$, where V^i and ω_i being local components of $V \in \mathfrak{S}_0^1(M)$ and $\omega \in \mathfrak{S}_1^0(M)$, respectively.

For each $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T_x^*M by $g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j$ for all $\omega, \theta \in \mathfrak{S}_1^0(M)$.

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{aligned} [{}^{V_\beta} \omega, V_\gamma \theta] &= 0, \quad [{}^H X, V_\beta \omega] = V_\beta (\nabla_X \omega), \\ [{}^H X, {}^H Y] &= {}^H [X, Y] + \sum_{\sigma=1}^n V_\sigma (X^\sigma \circ R(X, Y)) \end{aligned} \quad (2.6)$$

for all $X, Y \in \mathfrak{S}_0^1(M)$ and $\omega, \theta \in \mathfrak{S}_1^0(M)$, where R denotes the curvature tensor field of the linear connection ∇ .

3. The Sasakian metric on the coframe bundle

We define a Riemannian metric ${}^S g$ on the coframe bundle $F^*(M)$ by the following three equations

$${}^S g({}^H X, {}^H Y) = V(g(X, Y)) = g(X, Y) \circ \pi, \quad (3.1)$$

$${}^S g({}^H X, V_\beta \omega) = 0, \quad (3.2)$$

$${}^S g(V_\beta \omega, V_\gamma \theta) = \delta^{\beta\gamma} V(g^{-1}(\omega, \theta)) = \delta^{\beta\gamma} (g^{-1}(\omega, \theta) \circ \pi) \quad (3.3)$$

for any $X, Y \in \mathfrak{S}_0^1(M)$ and $\omega, \theta \in \mathfrak{S}_1^0(M)$ (see, [5]). We call the metric ${}^S g$ the Sasakian metric on the coframe bundle $F^*(M)$ over the Riemannian manifold (M, g) . Since any tensor field of type $(0, 2)$ on the $F^*(M)$ is completely determined by its action on vector fields of type ${}^H X$ and $V_\beta \omega$ it follows that ${}^S g$ is completely determined by its eqs (3.1), (3.2) and (3.3). The metric ${}^S g$ is a Riemannian metric on $F^*(M)$ uniquely determined by the metric g .

From equations (3.1), (3.2) and (3.3) it follows that

$$\begin{aligned} {}^S g_{ij} &= {}^S g(D_i, D_j) = V(g(\partial_i, \partial_j)) = g_{ij}, \\ {}^S g_{i_\alpha j} &= {}^S g(D_{i_\alpha}, D_j) = 0, \\ {}^S g_{i_\alpha j_\beta} &= {}^S g(D_{i_\alpha}, D_{j_\beta}) = \delta^{\alpha\beta} V(g^{-1}(dx^i, dx^j)) = \delta^{\alpha\beta} g^{ij}, \end{aligned}$$

i.e. Sg has components in the form

$${}^Sg = \begin{pmatrix} g_{ij} & 0 \\ 0 & \delta^{\alpha\beta} g^{ij} \end{pmatrix} \quad (3.4)$$

with respect to the adapted frame $\{D_I\}$.

Let us consider local 1-forms $\tilde{\eta}^I$ in $\pi^{-1}(U)$ defined by

$$\tilde{\eta}^I = \bar{A}^I{}_J dx^J,$$

where

$$A^{-1} = (\bar{A}^I{}_J) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{j\beta} \\ \bar{A}^{\alpha}{}_j & \bar{A}^{\alpha}{}_{j\beta} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ -X_m^\alpha \Gamma_{ij}^m & \delta_\beta^\alpha \delta_i^j \end{pmatrix}. \quad (3.5)$$

The matrix (3.8) is the inverse of the matrix

$$A = (A_K{}^J) = \begin{pmatrix} A_k^j & A_{k\gamma}^j \\ A_k^{j\beta} & A_{k\gamma}^{j\beta} \end{pmatrix} = \begin{pmatrix} \delta_k^j & 0 \\ X_m^\beta \Gamma_{jk}^m & \delta_\gamma^\beta \delta_j^k \end{pmatrix} \quad (3.6)$$

of the transformation $D_K = A_K{}^J \partial_J$ (see (2.2) and (2.3)). It is easy to establish that the set $\{\tilde{\eta}^I\}$ is the coframe dual to the adapted frame $\{D_K\}$, i.e.

$$\tilde{\eta}^I(D_K) = \bar{A}^I{}_J A_K{}^J = \delta_K^I.$$

Since the adapted frame is non-holonomic, we put

$$[D_I, D_J] = \Omega_{IJ}{}^K D_K$$

from which we have

$$\Omega_{IJ}{}^K = (D_I A_J^L - D_J A_I^L) \bar{A}_L^K.$$

According to (2.2), (2.3), (3.5) and (3.6), the components of non-holonomic object $\Omega_{IJ}{}^K$ are given by

$$\begin{cases} \Omega_{ij\beta}{}^{k\gamma} = -\Omega_{j\beta i}{}^{k\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j, \\ \Omega_{ij}{}^{k\gamma} = X_m^\gamma R_{ijk}^m, \end{cases} \quad (3.7)$$

all the others being zero, where R_{ijk}^m local components of the curvature tensor field R of ∇_g .

Let ${}^S\nabla$ be the Levi-Civita connection determined by the Sasakian metric Sg on the coframe bundle $F^*(M)$. We put

$${}^S\nabla_{D_I} D_J = {}^S\Gamma_{IJ}^K D_K.$$

From the equation

$${}^S\nabla_X Y - {}^S\nabla_Y X = [X, Y], \forall X, Y \in \mathfrak{S}_0^1(F^*(M))$$

we have

$${}^S\Gamma_{IJ}^K - {}^S\Gamma_{JI}^K = \Omega_{IJ}{}^K. \quad (3.8)$$

The equation

$$({}^S\nabla_X {}^Sg)(Y, Z) = 0$$

has form

$$D_L {}^Sg_{IJ} - {}^S\Gamma_{LI}^K {}^Sg_{KJ} - {}^S\Gamma_{LJ}^K {}^Sg_{IK} = 0 \quad (3.9)$$

with respect to the adapted frame $\{D_K\}$. By using (3.8) and (3.9), we obtain:

$$\begin{aligned} {}^S\Gamma_{IJ}^K &= \frac{1}{2} {}^Sg^{KL} (D_I {}^Sg_{LJ} + D_J {}^Sg_{IL} - D_L {}^Sg_{IJ}) + \\ &+ \frac{1}{2} (\Omega_{IJ}{}^K + \Omega_{JI}{}^K + \Omega_{II}{}^K), \end{aligned} \quad (3.10)$$

where $\Omega^K_{IJ} = Sg^{KLS}g_{PJ}\Omega_{LI}^P$ and

$$(Sg)^{-1} = (Sg^{KJ}) = \begin{pmatrix} g^{kj} & 0 \\ 0 & \delta_{\gamma\beta}g_{kj} \end{pmatrix}. \quad (3.11)$$

Taking account (3.4), (3.7) and (3.11), we obtain from (3.10)

$$\begin{cases} S\Gamma_{ij}^k = \Gamma_{ij}^k, & S\Gamma_{i\alpha j\beta}^k = S\Gamma_{i\alpha j}^{k\gamma} = S\Gamma_{i\alpha j\beta}^{k\gamma} = 0, \\ S\Gamma_{ij\beta}^k = \frac{1}{2}X_m^\beta R_{.i.}^{kjm}, & S\Gamma_{i\alpha j}^k = \frac{1}{2}X_m^\alpha R_{.j.}^{kim}, \\ S\Gamma_{ij}^{k\gamma} = \frac{1}{2}X_m^\gamma R_{ijk}^m, & S\Gamma_{ij\beta}^{k\gamma} = -\delta_\gamma^\beta \Gamma_{ik}^j. \end{cases} \quad (3.12)$$

Let $\tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(F^*(M))$ and $\tilde{X} = \tilde{X}^I D_I, \tilde{Y} = \tilde{Y}^J D_J$. Then the covariant derivative ${}^S\nabla_{\tilde{Y}}\tilde{X}$ along \tilde{Y} has components in the form

$${}^S\nabla_{\tilde{Y}}\tilde{X}^I = \tilde{Y}^J D_J \tilde{X}^I + S\Gamma_{JK}^I \tilde{X}^K \tilde{Y}^J \quad (3.13)$$

with respect to the adapted frame $\{D_I\}$.

Using (2.4)-(2.6), (3.1)-(3.3), (3.12) and (3.13), we have

Theorem 3.1. *Let M be a Riemannian manifold with metric g and ${}^S\nabla$ be the Levi-Civita connection of the coframe bundle $F^*(M)$ equipped with Sasakian metric Sg . Then ${}^S\nabla$ satisfies*

$$\begin{aligned} i) & {}^S\nabla_{V_{\alpha\omega}} V_\beta \theta = 0, \\ ii) & {}^S\nabla_{V_{\alpha\omega}} H Y = \frac{1}{2} H \left(R(\tilde{X}^\alpha, \tilde{\omega}) Y \right), \\ iii) & {}^S\nabla_{H X} V_\beta \theta = V_\beta (\nabla_X \theta) + \frac{1}{2} H \left(R(\tilde{X}^\beta, \tilde{\theta}) X \right), \\ iv) & {}^S\nabla_{H X} H Y = H (\nabla_X Y) + \frac{1}{2} \sum_{\sigma=1}^n V_\sigma (X^\sigma \circ R(X, Y)) \end{aligned}$$

for all $X, Y \in \mathfrak{S}_0^1(M)$ and $\omega, \theta \in \mathfrak{S}_1^0(M)$, where $\tilde{\omega} = g^{-1} \circ \omega \in \mathfrak{S}_0^1(M)$, $\tilde{\theta} = g^{-1} \circ \theta \in \mathfrak{S}_0^1(M)$, $\tilde{X}^\alpha = g^{-1} \circ X^\alpha \in \mathfrak{S}_0^1(M)$.

We note that the analogue of Theorem 3.1 in the case of cotangent bundle is proved in [18].

4. Para-Nordenian structures on $(F^*(M), {}^Sg)$

An almost para-complex manifold is an almost product manifold (M_n, φ) , $\varphi^2 = I$, such that the two eigenbundles $T^+(M_n)$ and $T^-(M_n)$ associated to the two eigenvalues $+1$ and -1 of φ , respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even.

A tensor field $t \in \mathfrak{S}_q^0(M_{2n})$ is said to be a pure with respect to the para-complex structure φ , if

$$t(\varphi X_1, X_2, \dots, X_q) = t(X_1, \varphi X_2, \dots, X_q) = t(X_1, X_2, \dots, \varphi X_q)$$

for any $X_1, X_2, \dots, X_q \in \mathfrak{S}_0^1(M_{2n})$.

We define the following operator ϕ_φ associated with φ and apply to the pure tensor field t :

$$(\phi_\varphi t)(Y, X_1, X_2, \dots, X_q) = (\varphi Y)(t(X_1, X_2, \dots, X_q))$$

$$-Y(t(\varphi X_1, X_2, \dots, X_q)) + t((L_{X_1}\varphi)Y, X_2, \dots, X_q) \\ + \dots + t(X_1, X_2, \dots, (L_{X_q}\varphi)Y).$$

We note that $\phi_\varphi t \in \mathfrak{S}_q^0(M_{2n})$. If $\phi_\varphi t = 0$ then t is said to be an almost para-holomorphic (see [5, 15, 16, 18]).

Definition 4.1. In a manifold with almost para-complex structure φ , a vector field X is called an almost para-holomorphic vector field if $L_X\varphi = 0$.

A Riemannian manifold (M_{2n}, g) with an almost para-complex structure φ is said to be almost para-Nordenian if the Riemannian metric g is pure with respect to φ . It is well known that the almost para-Nordenian manifold is para-Kahler ($\nabla_g\varphi = 0$) if and only if g is para-holomorphic ($\phi_\varphi g = 0$) (see [18]).

Let $(F^*(M), Sg)$ be the linear coframe bundle with the Sasakian metric Sg . Define a tensor field F_α of type $(1, 1)$ on $F^*(M)$ for each $\alpha = 1, 2, \dots, n$, by

$$F_\alpha({}^H X) = V_\alpha \tilde{X}, \quad F_\alpha({}^{V_\beta} \omega) = \delta_\alpha^\beta {}^H \tilde{\omega} \quad (4.1)$$

for any $X \in \mathfrak{S}_0^1(M)$ and $\omega \in \mathfrak{S}_1^0(M)$, where $\tilde{X} = g \circ X \in \mathfrak{S}_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \mathfrak{S}_0^1(M)$ and the horizontal lifts are considered with respect to the Levi-Civita connection of g . Each F_α satisfies the condition

$$F_\alpha^2 = I.$$

Indeed, by virtue of (4.1), we have

$$F_\alpha^2({}^H X) = F_\alpha(F_\alpha({}^H X)) = F_\alpha(V_\alpha \tilde{X}) = \delta_\alpha^\alpha {}^H \tilde{X} = {}^H X, \\ F_\alpha^2({}^{V_\beta} \omega) = F_\alpha(F_\alpha({}^{V_\beta} \omega)) = F_\alpha(\delta_\alpha^\beta {}^H \tilde{\omega}) = \delta_\alpha^\beta {}^{V_\alpha} \tilde{\omega} = {}^{V_\beta} \omega$$

for any $X \in \mathfrak{S}_0^1(M)$ and $\omega \in \mathfrak{S}_1^0(M)$, which implies $F_\alpha^2 = I$ for each $\alpha = 1, 2, \dots, n$.

The following theorem holds.

Theorem 4.1. *The triple $(F^*(M), Sg, F_\alpha)$, for each $\alpha = 1, 2, \dots, n$, is an almost para-Nordenian manifold.*

Proof. If we put

$$A(\tilde{X}, \tilde{Y}) = Sg(F_\alpha \tilde{X}, \tilde{Y}) - Sg(\tilde{X}, F_\alpha \tilde{Y})$$

for any $\tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(F^*(M))$. Then from (3.1)-(3.3) and (4.1), we have

$$A({}^H X, {}^H Y) = Sg(F_\alpha({}^H X), {}^H Y) - Sg({}^H X, F_\alpha({}^H Y)) \\ = Sg(V_\alpha \tilde{X}, {}^H Y) - Sg({}^H X, V_\alpha \tilde{Y}) = 0, \\ A({}^H X, {}^{V_\beta} \omega) = Sg(F_\alpha({}^H X), {}^{V_\beta} \omega) - Sg({}^H X, F_\alpha({}^{V_\beta} \omega)) \\ = Sg(V_\alpha \tilde{X}, {}^{V_\beta} \omega) - Sg({}^H X, \delta_\alpha^\beta {}^H \tilde{\omega}) \\ \delta^{\alpha\beta} g^{-1}(gX, \omega) - g(X, g^{-1}\omega) \delta_\alpha^\beta = 0, \\ A({}^{V_\beta} \omega, {}^H Y) = -A({}^H Y, {}^{V_\beta} \omega) = 0, \\ A({}^{V_\beta} \omega, {}^{V_\gamma} \theta) = Sg(F_\alpha({}^{V_\beta} \omega), {}^{V_\gamma} \theta) - Sg({}^{V_\beta} \omega, F_\alpha({}^{V_\gamma} \theta)) \\ = Sg(\delta_\alpha^\beta {}^H \tilde{\omega}, {}^{V_\gamma} \theta) - Sg({}^{V_\beta} \omega, \delta_\alpha^\gamma {}^H \tilde{\theta}) \\ = \delta_\alpha^\beta Sg({}^H \tilde{\omega}, {}^{V_\gamma} \theta) - \delta_\alpha^\gamma Sg({}^{V_\beta} \omega, {}^H \tilde{\theta}) = 0,$$

i.e. Sg is pure with respect to F_α , for each $\alpha = 1, 2, \dots, n$. Thus Theorem 4.1 is proved.

Let us consider the covariant derivative of F_α , for each $\alpha = 1, 2, \dots, n$, with respect to the Levi-Civita connection ${}^S\nabla$ of metric Sg . Taking into account (i) – (iv) of Theorem 3.1, we obtain

$$\begin{aligned}
 ({}^S\nabla_{H_X} F_\alpha)({}^H Y) &= {}^H\nabla_{H_X} F_\alpha({}^H Y) - F_\alpha({}^S\nabla_{H_X} {}^H Y) \\
 &= {}^S\nabla_{H_X} V_\alpha \tilde{Y} - F_\alpha({}^H(\nabla_X Y) + \frac{1}{2} \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y))) \\
 &= V_\alpha(\nabla_X \tilde{Y}) + \frac{1}{2}(R(\tilde{X}^\alpha, \tilde{Y})X) - F_\alpha({}^H(\nabla_X Y) + \frac{1}{2} \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y))) \\
 &= \frac{1}{2} {}^H(R(\tilde{X}^\alpha, Y)X) - \frac{1}{2} \delta_\alpha^\beta \sum_{\beta=1}^n {}^H(g^{-1} X^\beta \circ R(X, Y)) \tag{4.2}
 \end{aligned}$$

$$= \frac{1}{2} {}^H(g^{-1} X^\alpha (R(\quad, Y)X - R(X, Y))),$$

$$({}^S\nabla_{V_{\beta\omega}} F_\alpha)({}^H Y) = {}^H\nabla_{V_{\beta\omega}} F_\alpha({}^H Y) - F_\alpha({}^S\nabla_{V_{\beta\omega}} {}^H Y) \tag{4.3}$$

$$\begin{aligned}
 &= {}^S\nabla_{V_{\beta\omega}} V_\alpha \tilde{Y} - \frac{1}{2} F_\alpha({}^H(R(\tilde{X}^\beta, \tilde{\omega})Y) - \frac{1}{2} F_\alpha({}^H(X^\beta g^{-1} \circ R(\quad, \tilde{\omega})Y) \\
 &= -\frac{1}{2} V_\alpha(X^\beta \circ R(\quad, \tilde{\omega})Y),
 \end{aligned}$$

$$({}^S\nabla_{H_X} F_\alpha)({}^{V_\gamma} \theta) = {}^H\nabla_{H_X} F_\alpha({}^{V_\gamma} \theta) - F_\alpha({}^S\nabla_{H_X} {}^{V_\gamma} \theta)$$

$$= {}^S\nabla_{H_X} \delta_\alpha^\gamma {}^H \tilde{\theta} - F_\alpha({}^{V_\gamma}(\nabla_X \theta) + \frac{1}{2} {}^H(R(\tilde{X}^\gamma, \tilde{\theta})X))$$

$$= \delta_\alpha^\gamma {}^H(\nabla_X \tilde{\theta}) + \delta_\alpha^\gamma \frac{1}{2} \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, \tilde{\theta})) - \delta_\alpha^\gamma {}^H(g^{-1} \circ \nabla_X \theta)$$

$$-\frac{1}{2} V_\alpha(g \circ R(\tilde{X}^\gamma, \tilde{\theta})X) = \delta_\alpha^\gamma \frac{1}{2} \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, \tilde{\theta})) \tag{4.4}$$

$$-\frac{1}{2} V_\alpha(g \circ R(\tilde{X}^\gamma, \tilde{\theta})X),$$

$$({}^S\nabla_{V_{\beta\omega}} F_\alpha)({}^{V_\gamma} \theta) = {}^H\nabla_{V_{\beta\omega}} F_\alpha({}^{V_\gamma} \theta) - F_\alpha({}^S\nabla_{V_{\beta\omega}} {}^{V_\gamma} \theta)$$

$$= {}^S\nabla_{V_{\beta\omega}} (\delta_\alpha^\gamma {}^H \tilde{\theta}) = \delta_\alpha^\gamma {}^S\nabla_{V_{\beta\omega}} {}^H \tilde{\theta} = \delta_\alpha^\gamma \frac{1}{2} {}^H(R(\tilde{X}^\beta, \tilde{\omega})\tilde{\theta}) \tag{4.5}$$

$$= \delta_\alpha^\gamma \frac{1}{2} {}^H(X^\beta g^{-1} \circ R(\quad, \tilde{\omega})\tilde{\theta}).$$

From (4.2)-(4.5), we get

Theorem 4.2. *The linear coframe bundle $F^*(M)$ of a Riemannian manifold (M, g) is para-Kahlerian (para-holomorphic Nordenian) with respect to the metric Sg and almost para-complex structure F_α , for each $\alpha = 1, 2, \dots, n$, if and only if the Riemannian manifold (M, g) is flat.*

5. Para-holomorphic vector-fields on $(F^*(M), {}^Sg)$

Let (M, g) be a Riemannian manifold, and let $F^*(M)$ be its linear coframe bundle with Sasakian metric Sg and with the almost para-Nordenian structures F_α , $\alpha = 1, 2, \dots, n$. A vector field $\tilde{X} \in \mathfrak{S}_0^1(F^*(M))$ with respect to which the almost para-Nordenian structure F_α has a vanishing Lie derivative ($L_{\tilde{X}}F_\alpha = 0$) is said to be almost para-holomorphic (see, [18]). It is well known that [5]

$$[{}^C X, {}^H Y] = {}^H [X, Y] + \sum_{\alpha=1}^n V_\alpha (X^\alpha \circ (L_X \nabla) Y), \quad (5.1)$$

$$[{}^C V, {}^{V_\gamma} \omega] = {}^{V_\gamma} (L_V \omega) \quad (5.2)$$

for any $X, Y \in \mathfrak{S}_0^1(M)$ and $\omega \in \mathfrak{S}_1^0(M)$, where

$$(L_X \nabla) Y = \nabla_Y \nabla X + R(X, Y)$$

and

$$(L_X \nabla)(Y, Z) = L_X(\nabla_Y X) - \nabla_Y(L_X Z) - \nabla_{[X, Y]} Z.$$

A vector field $X \in \mathfrak{S}_0^1(M)$ is called a Killing vector field (or infinitesimal isometry) if $L_X g = 0$ and X called an infinitesimal affine transformation if $L_V \nabla g = 0$. A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $L_V \nabla g = 0$ as a consequence of $L_X g = 0$. Now we consider the Lie derivative of F_α ($\alpha = 1, 2, \dots, n$) with respect to the complete lift ${}^C X$. Taking account of (4.1), (5.1) and (5.2), we find

$$\begin{aligned} (L_{{}^C X} F_\alpha)({}^{V_\gamma} \theta) &= L_{{}^C X}(F_\alpha({}^{V_\gamma} \theta) - F_\alpha(L_X {}^{V_\gamma} \theta)) \\ &= \delta_\alpha^\gamma L_{{}^C X} {}^H \tilde{\theta} - \delta_\alpha^\gamma (g^{-1} \circ (L_X \theta)) = \delta_\alpha^\gamma \left[{}^H [X, \tilde{\theta}] + \sum_{\beta=1}^n V_\beta (X^\beta \circ L_X \nabla \tilde{\theta}) \right. \\ &\quad \left. - {}^H (g^{-1} \circ (L_X \theta)) \right] = \delta_\alpha^\gamma \left[{}^H (L_X (g^{-1} \circ \theta) - g^{-1} \circ (L_X \theta)) \right. \\ &\quad \left. + \sum_{\beta=1}^n V_\beta (X^\beta \circ L_X \nabla \tilde{\theta}) \right], \end{aligned} \quad (5.3)$$

$$\begin{aligned} (L_{{}^C X} F_\alpha)({}^H Y) &= L_{{}^C X}(F_\alpha({}^H Y) - F_\alpha(L_X {}^H Y)) \\ &= L_{{}^C X} V_\alpha \tilde{Y} - F_\alpha({}^H [X, Y] + \sum_{\beta=1}^n V_\beta (X^\beta \circ L_X \nabla Y)) \\ &= L_{{}^C X} V_\alpha \tilde{Y} - V_\alpha (g \circ [X, Y]) + \sum_{\beta=1}^n \delta_\alpha^\beta V_\beta (g^{-1} X^\beta \circ L_X \nabla Y) \\ &= V_\alpha (L_X (g \circ Y) - g \circ L_X Y) + \sum_{\beta=1}^n \delta_\alpha^\beta V_\beta (g^{-1} X^\beta \circ L_X \nabla Y). \end{aligned} \quad (5.4)$$

Let now X be a Killing vector field ($L_X g = 0$). Then by virtue of $L_X \nabla = 0$, from (5.3) and (5.4), we have $L_{{}^C X} F_\alpha = 0$, $\alpha = 1, 2, \dots, n$, i.e. ${}^C X$ is para-holomorphic with respect to each F_α . Hence, we have

Theorem 5.1. *Let (M, g) be a Riemannian manifold and let $(F^*(M), {}^Sg, F_\alpha)$ be para-Nordenian manifold for each $\alpha = 1, 2, \dots, n$. Then the complete lift ${}^C X$ of vector field $X \in \mathfrak{X}_0^1(M)$ to $F^*(M)$ is almost para-holomorphic vector field with respect to the each almost para-Nordenian structure $(F_\alpha, {}^Sg)$, if X is a Killing vector field on a Riemannian manifold (M, g) .*

6. Integrability of para-Nordenian structures on $(F^*(M), {}^Sg)$

In this section, we study the integrability of almost para-Nordenian structure F_α for each $\alpha = 1, 2, \dots, n$, on $(F^*(M), {}^Sg)$. We assume that ∇ is a Levi-Civita connection of a Riemannian metric g . Denoting by N_{F_α} the Nijenhuis tensor of F_α , we have

$$N_{F_\alpha}(\tilde{X}, \tilde{Y}) = [F_\alpha \tilde{X}, F_\alpha \tilde{Y}] - F_\alpha[F_\alpha \tilde{X}, \tilde{Y}] - F_\alpha[\tilde{X}, F_\alpha \tilde{Y}] + [\tilde{X}, \tilde{Y}] \quad (6.1)$$

for all $\tilde{X}, \tilde{Y} \in \mathfrak{X}_0^1(F^*(M))$. Then taking account of (2.6) and (6.1), we obtain

$$\begin{aligned} N_{F_\alpha}({}^H X, {}^H Y) &= [F_\alpha {}^H X, F_\alpha {}^H Y] - F_\alpha[F_\alpha {}^H X, {}^H Y] \\ &\quad - F_\alpha[{}^H X, F_\alpha {}^H Y] + [{}^H X, {}^H Y] = [{}^{V_\alpha} \tilde{X}, {}^{V_\alpha} \tilde{Y}] - F_\alpha[{}^{V_\alpha} \tilde{X}, {}^H Y] \\ &\quad - F_\alpha[{}^H X, {}^{V_\alpha} \tilde{Y}] + {}^H[X, Y] + \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)) \\ &= F_\alpha {}^{V_\alpha}(\nabla_Y \tilde{X}) - F_\alpha {}^{V_\alpha}(\nabla_X \tilde{Y}) + {}^H[X, Y] + \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)) \\ &= \delta_\alpha^{\alpha H}(\nabla_Y X - \nabla_X Y) + {}^H[X, Y] + \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)) \\ &= \sum_{\beta=1}^n V_\beta(X^\beta \circ R(X, Y)), \\ N_{F_\alpha}({}^H X, {}^{V_\beta} \omega) &= [F_\alpha {}^H X, F_\alpha {}^{V_\beta} \omega] - F_\alpha[F_\alpha {}^H X, {}^{V_\beta} \omega] \\ &\quad - F_\alpha[{}^H X, F_\alpha {}^{V_\beta} \omega] + [{}^H X, {}^{V_\beta} \omega] = [{}^{V_\alpha} \tilde{X}, \delta_\alpha^{\beta H} \tilde{\omega}] - F_\alpha[{}^{V_\alpha} \tilde{X}, {}^{V_\beta} \omega] \\ &\quad - F_\alpha[{}^H X, \delta_\alpha^{\beta H} \tilde{\omega}] + {}^{V_\beta}(\nabla_X \omega) = \delta_\alpha^\beta [{}^{V_\alpha} \tilde{X}, {}^H \tilde{\omega}] + {}^{V_\beta}(\nabla_X \omega) \\ &\quad - \delta_\alpha^\beta F_\alpha({}^H[X, \tilde{\omega}] + \sum_{\gamma=1}^n V_\gamma(X^\gamma \circ R(X, \tilde{\omega}))) = -\delta_\alpha^\beta {}^{V_\alpha}(\nabla_{\tilde{\omega}} \tilde{X}) \\ &\quad + {}^{V_\beta}(\nabla_X \omega) - \delta_\alpha^\beta {}^{V_\alpha}(g \circ [X, \tilde{\omega}]) - \delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^{\gamma H}(g^{-1} X^\gamma \circ R(X, \tilde{\omega})) \\ &= -{}^{V_\beta}(g \circ \nabla_{\tilde{\omega}} X) + {}^{V_\beta}(g \circ \nabla_X \tilde{\omega}) - {}^{V_\beta}(g \circ [X, \tilde{\omega}]) \\ &\quad - \delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^{\gamma H}(g^{-1} X^\gamma \circ R(X, \tilde{\omega})) = {}^{V_\beta}(g \circ [X, \tilde{\omega}]) - {}^{V_\beta}(g \circ [X, \tilde{\omega}]) \\ &= -\delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^{\gamma H}(g^{-1} X^\gamma \circ R(X, \tilde{\omega})), \\ N_{F_\alpha}({}^{V_\beta} \omega, {}^{V_\gamma} \theta) &= [F_\alpha {}^{V_\beta} \omega, F_\alpha {}^{V_\gamma} \theta] - F_\alpha[F_\alpha {}^{V_\beta} \omega, {}^{V_\gamma} \theta] \end{aligned}$$

$$\begin{aligned}
& -F_\alpha[V^\beta\omega, F_\alpha V^\gamma\theta] + [V^\beta\omega, V^\gamma\theta] = [\delta_\alpha^\beta H\tilde{\omega}, \delta_\alpha^\gamma H\tilde{\theta}] \\
& -F_\alpha[\delta_\alpha^\beta H\tilde{\omega}, V^\gamma\theta] - F_\alpha[V^\beta\omega, \delta_\alpha^\gamma H\tilde{\theta}] = \delta_\alpha^\beta \delta_\alpha^\gamma [H\tilde{\omega}, H\tilde{\theta}] \\
& -\delta_\alpha^\beta F_\alpha[H\tilde{\omega}, V^\gamma\theta] - \delta_\alpha^\gamma F_\alpha[V^\beta\omega, H\tilde{\theta}] = \delta_\alpha^\beta \delta_\alpha^\gamma H[\tilde{\omega}, \tilde{\theta}] \\
& + \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) - \delta_\alpha^\beta F_\alpha V_\gamma(\nabla_{\tilde{\omega}}\theta) \\
& + \delta_\alpha^\gamma F_\alpha V_\beta(\nabla_{\tilde{\theta}}\omega) = \delta_\alpha^\beta \delta_\alpha^\gamma H[\tilde{\omega}, \tilde{\theta}] + \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) \\
& - \delta_\alpha^\beta \delta_\alpha^\gamma H(g^{-1} \circ \nabla_{\tilde{\omega}}\theta) + \delta_\alpha^\gamma \delta_\alpha^\beta H(g^{-1} \circ \nabla_{\tilde{\theta}}\omega) = \delta_\alpha^\beta \delta_\alpha^\gamma H[\tilde{\omega}, \tilde{\theta}] \\
& + \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) - \delta_\alpha^\beta \delta_\alpha^\gamma H(\nabla_{\tilde{\omega}}\tilde{\theta}) \\
& + \delta_\alpha^\gamma \delta_\alpha^\beta H(\nabla_{\tilde{\theta}}\tilde{\omega}) = \delta_\alpha^\beta \delta_\alpha^\gamma \sum_{\sigma=1}^n V_\sigma(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta}))
\end{aligned}$$

for all $X, Y \in \mathfrak{S}_0^1(M)$, $\omega, \theta \in \mathfrak{S}_1^0(M)$ and $1 \leq \alpha \leq n$.

From the above equations we conclude that $N_{F_\alpha} = 0$ if and only if $R(X, Y) = 0$ for all $X, Y \in \mathfrak{S}_0^1(M)$. Therefore, we have

Theorem 6.1. *Let (M, g) be a Riemannian manifold and $F^*(M)$ be its linear coframe bundle equipped with Sasakian metric Sg . Then the almost para-Nordenian manifold $(F^*(M), {}^Sg, F_\alpha)$, for each $\alpha = 1, 2, \dots, n$, is para-Nordenian if and only if $R = 0$.*

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