Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

NONLINEAR WAVE EQUATIONS WITH NONLINEAR TRANSMISSION ACOUSTIC CONDITION

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Abstract. In this paper we consider a mixed problem for nonlinear wave equations with nonlinear transmission acoustic condition. We prove a theorem on local existence and uniqueness of solutions for this problem by using the Faedo-Galerkin approximations combined with a contraction mapping theorem.

1. Introduction

Let Ω be a bounded domain in $R^n (n \geq 1)$ with smooth boundary Γ_1, $\Omega_2 \subset \Omega$ be a subdomain with smooth boundary Γ_2 and $\Omega_1 = \Omega \backslash (\Omega_2 \cup \Gamma_2)$ be a subdomain with boundary $\Gamma = \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 = \emptyset$. The nonlinear transmission acoustic problem considered here is

$$u_{tt} - \Delta u + |u_t|^{q_1 - 1} u_t = f(u) \quad \text{in } \Omega_1 \times (0, \infty), \tag{1.1}$$

$$\upsilon_{tt} - \Delta \upsilon + |\upsilon_t|^{q_2 - 1} \upsilon_t = g(\upsilon) \quad \text{in } \Omega_2 \times (0, \infty), \tag{1.2}$$

$$M\delta_{tt} + D\delta_t + K\delta = -u_t \ \text{ on } \Gamma_2 \times (0, \infty), \tag{1.3}$$

$$u = 0 \ \text{ on } \Gamma_1 \times (0, \infty), \tag{1.4}$$

$$u = \upsilon, \frac{\partial u}{\partial \nu} - \frac{\partial \upsilon}{\partial \nu} + \rho(u_t) = \delta_t \quad \text{on } \Gamma_2 \times (0, \infty), \tag{1.5}$$

$$u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in \Omega_1, \tag{1.6}$$

$$\upsilon(x, 0) = \upsilon_0(x), \ \upsilon_t(x, 0) = \upsilon_1(x), \ x \in \Omega_2, \tag{1.7}$$

$$\delta(x, 0) = \delta_0(x), \ \delta_t(x, 0) = \frac{\partial u_0}{\partial \nu} - \frac{\partial \upsilon_0}{\partial \nu} + \rho(u_1) \equiv \delta_1(x), \ x \in \Gamma_2, \tag{1.8}$$

where ν is the unit outward normal vector to Γ; $f, g, \rho : (-\infty, +\infty) \to (-\infty, +\infty)$, $M, D, K : \bar{\Gamma}_2 \to (-\infty, +\infty)$, $u_0, u_1 : \Omega_1 \to (-\infty, +\infty), \upsilon_0, \upsilon_1 : \Omega_2 \to (-\infty, +\infty)$, $\delta_0 : \Gamma_2 \to (-\infty, +\infty)$ are given functions; $q_i > 1, i = 1, 2$ are constants.

The problems like (1.1)-(1.8), called transmission acoustic problems, are related to the problem of two wave equations, which models the transverse acoustic vibrations of the membrane composed by two different materials Ω_1 and Ω_2.

Transmission problems were studied, for example, in [3, 4, 5, 13, 26, 28, 31]. The transmission problem to hyperbolic equations was investigated by Dautray and Lions [13] who proved the existence and regularity of solutions for the linear

2010 *Mathematics Subject Classification.* 35L05, 35L70.

Key words and phrases. Nonlinear wave equation, nonlinear transmission acoustic condition, local solution, contraction mapping theorem.

problem. Bae [4] studied the transmission problem, in which one component is clamped and the other is in a viscoelastic fluid producing a dissipation mechanism on the boundary, and established a decay result.

Aliev and Mammadhasanov [3] studied the initial boundary value problem on longitudinal impact on a composite linear viscoelastic bar and established a well-posedness result by the method of dynamic regularization of transmission and boundary conditions.

The acoustic boundary conditions were studied in [6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 29, 30, 32, 33, 34]. A mixed problem for wave equation with nonlinear acoustic boundary conditions was considered by Gao, Liang, Xiao [20], Graber [22, 23]. Graber and Said-Houari [24] studied the stability of a structural acoustic wave equation with semilinear porous acoustic boundary conditions and obtained several results in local existence, global existence, the decay rate and blow up results.

The problems like (1.1)-(1.8) with linear acoustic conditions were studied in [1, 2] in which some results in local existence, global existence, the exponential stability and blow up results were obtained.

In this paper we prove the theorem on local existence and uniqueness of solutions for the problem (1.1)-(1.8) by using the Faedo-Galerkin approximations, the compactness method and the fixed point theorem.

Our paper is organized as follows. In section 2 we introduce some notations, preliminaries and statement of well-posedness result for the problem (1.1)-(1.8), which is proved in section 3.

2. Preliminaries and main result

The inner product and norm in $L^2(\Omega_i)$, $i = 1,2$ and $L^2(\Gamma_2)$ are denoted respectively, by

$$(u,v)_i = \int_{\Omega_i} u(x)\, v(x) dx, \;\; \|u\|_i = \left(\int_{\Omega_i} (u(x))^2 dx \right)^{1/2}, \quad i = 1,2,$$

$$(\delta,\theta)_{\Gamma_2} = \int_{\Gamma_2} \delta(x)\, \theta(x) d\Gamma_2, \;\; \|\delta\|_{\Gamma_2} = \left(\int_{\Gamma_2} (\delta(x))^2 d\Gamma_2 \right)^{1/2}.$$

$H^1(\Omega_i)$, $i = 1,2$ are the usual real Sobolev spaces of first order. We define a closed subspace of the $H^1(\Omega_i)$ as

$$H^1_{\Gamma_1}(\Omega_1) = \left\{ u \in H^1(\Omega_1) : \gamma_0(u) = 0 \;\text{ a.e.on } \Gamma_1 \right\},$$

where $\gamma_0 : H^1(\Omega_1) \to H^{1/2}(\Gamma)$ is the trace map of order zero and $H^{1/2}(\Gamma)$ is the Sobolev space of order $\frac{1}{2}$ defined over Γ, as introduced by Lions and Magenes [27]. Observe that the norm in $H^1_{\Gamma_1}(\Omega_i)$:

$$\|u\|_{H^1_{\Gamma_1}(\Omega_1)} = \left(\sum_{i=1}^{n} \int_{\Omega_1} \left(\frac{\partial u}{\partial x_i} \right)^2 dx \right)^{1/2}$$

and the norm of the real Sobolev space $H^1(\Omega_i)$ are equivalent, because the Poincaré's inequality holds in $H^1_{\Gamma_1}(\Omega_i)$. Thus we consider $H^1_{\Gamma_1}(\Omega_i)$ with the above gradient norm.

The map $\gamma_1 : H(\Delta,\Omega_1)\cup H(\Delta,\Omega_2)\to H^{-1/2}(\Gamma_2)$ is the Neumann trace map on $H(\Delta,\Omega_1)\cup H(\Delta,\Omega_2)$ and

$$H(\Delta,\Omega_i)=\left\{u\in H^1(\Omega_i):\Delta u\in L^2(\Omega_i)\right\},\quad i=1,2$$

are equipped with the norms

$$\|u\|_{\Delta,\Omega_i}=\left(\|u\|^2_{H^1(\Omega_i)}+\|\Delta u\|_i^2\right)^{1/2},\quad i=1,2.$$

The well-posedness result is contained in the following theorem.

Theorem 2.1. *Assume that*

$$M,D,K\in C\left(\bar{\Gamma}_2\right),M>0,D>0,K>0\ \text{ for }\ \forall x\in\bar{\Gamma}_2; \tag{2.1}$$

$$f,g\in C^1(-\infty;+\infty),\ |f(s)|\le c_1|s|^p,$$
$$\left|f'(s)\right|\le c_2|s|^{p-1},|g(s)|\le c_3|s|^p,\ \left|g'(s)\right|\le c_4|s|^{p-1}\ \ (c_i>0,i=1,2,3,4); \tag{2.2}$$

$$p>1\ \text{if}\ \ n=1,2,\ 1<p\le\frac{n}{n-2}\ \ \text{if}\ \ n\ge 3; \tag{2.3}$$

$$\rho\in C^1(-\infty;+\infty),\ |\rho(s)|\le c_5|s|^{q_1}\ \ (c_5>0); \tag{2.4}$$

$$\rho(s)\text{ is monotone increasing function on }(-\infty;+\infty)\text{ with }\rho(0)=0. \tag{2.5}$$

Then for $\forall(u_0,\upsilon_0,\delta_0)\in H^1_{\Gamma_1}(\Omega_1)\times H^1(\Omega_2)\times L^2(\Gamma_2)$, $\forall(u_1,\upsilon_1,\delta_1)\in L^{2q_1}(\Omega_1)\times L^{2q_2}(\Omega_2)\times L^2(\Gamma_2)$ $\left(u_0|_{\Gamma_2}=\upsilon_0|_{\Gamma_2},\ u_1|_{\Gamma_2}=\upsilon_1|_{\Gamma_2}\right)$ *there exists* $T>0$ *such that the problem* (1.1)-(1.8) *has a unique solution* (u,υ,δ) *which satisfies*

$$u\in C\left([0,T];H^1_{\Gamma_1}(\Omega_1)\right),\ u_t\in C\left([0,T];L^2(\Omega_1)\right)\cap L^{q_1+1}(\Omega_1\times(0,T)),$$
$$\upsilon\in C\left([0,T];H^1(\Omega_2)\right),\ \upsilon_t\in C\left([0,T];L^2(\Omega_2)\right)\cap L^{q_2+1}(\Omega_2\times(0,T)),$$
$$\delta,\delta_t\in L^\infty\left(0,T;L^2(\Gamma_2)\right).$$

Moreover, if $T_{\max}>0$ *is the length of the maximal existence interval of the solution* (u,υ,δ), *then either* $T_{\max}=+\infty$, *or*

$$\lim_{t\to T-0}\left(\|u_t\|_1^2+\|\upsilon_t\|_2^2+\|\nabla u\|_1^2+\|\nabla\upsilon\|_2^2+\left\|\sqrt{M}\delta_t\right\|^2_{\Gamma_2}+\left\|\sqrt{K}\delta\right\|^2_{\Gamma_2}\right)=+\infty.$$

3. Proof of Theorem 2.1

We prove this theorem using the combination of the Faedo-Galerkin approximation, the compactness method and the fixed point theorem.

We consider the following problem:

$$U_{tt}-\Delta U+|U_t|^{q_1-1}U_t=F\ \text{ in }\ \Omega_1\times(0,T), \tag{3.1}$$

$$V_{tt}-\Delta V+|V_t|^{q_2-1}V_t=G\ \text{ in }\ \Omega_2\times(0,T), \tag{3.2}$$

$$M\delta_{tt}+D\delta_t+K\delta=-U_t\ \text{ on }\Gamma_2\times(0,T), \tag{3.3}$$

$$U=0\ \text{ on }\ \Gamma_1\times(0,T), \tag{3.4}$$

$$U=V,\ \frac{\partial U}{\partial\nu}-\frac{\partial V}{\partial\nu}+\rho(U_t)=\delta_t\ \text{ on }\ \Gamma_2\times(0,T), \tag{3.5}$$

$$U(x,0)=u_0(x),\ U_t(x,0)=u_1(x),\ x\in\Omega_1, \tag{3.6}$$

$$V(x,0)=\upsilon_0(x),\ V_t(x,0)=\upsilon_1(x),\ x\in\Omega_2, \tag{3.7}$$
$$\delta(x,0)=\delta_0(x),\ \delta_t(x,0)=\delta_1(x),\ x\in\Gamma_2; \tag{3.8}$$

here $T>0$; $F=F(x,t)$ and $G=G(x,t)$ are fixed functions on $\Omega_1\times[0,T)$ and $\Omega_2\times[0,T)$, respectively.

To prove Theorem 2.1 we need two lemmas (Lemma 3.1 and Lemma 3.2).

Lemma 3.1. *Suppose that* (2.1), (2.4)-(2.5) *hold and let*

$$F\in H^1\left(0,T;L^2(\Omega_1)\right),\ G\in H^1\left(0,T;L^2(\Omega_2)\right), \tag{3.9}$$
$$u_0\in H^1_{\Gamma_1}(\Omega_1)\cap H^2(\Omega_1),\ u_1\in H^1_{\Gamma_1}(\Omega_1)\cap L^{2q_1}(\Omega_1), \tag{3.10}$$
$$\upsilon_0\in H^2(\Omega_2),\ \upsilon_1\in H^1(\Omega_2)\cap L^{2q_2}(\Omega_2), \tag{3.11}$$
$$\delta_0,\delta_1\in L^2(\Gamma_2). \tag{3.12}$$

Then, there exists a unique solution (U,V,δ) *to the problem* (3.1)-(3.8) *such that*

$$U\in L^\infty\left(0,T;H^1_{\Gamma_1}(\Omega_1)\right),\ U_t\in L^\infty\left(0,T;H^1_{\Gamma_1}(\Omega_1)\right)\cap L^{q_1+1}(\Omega_1\times(0,T)),$$
$$U_{tt}\in L^\infty\left(0,T;L^2(\Omega_1)\right), \tag{3.13}$$
$$V\in L^\infty\left(0,T;H^1(\Omega_2)\right),V_t\in L^\infty\left(0,T;H^1(\Omega_2)\right)\cap L^{q_2+1}(\Omega_2\times(0,T)),$$
$$V_{tt}\in L^\infty\left(0,T;L^2(\Omega_2)\right), \tag{3.14}$$
$$\left(-\Delta U+|U_t|^{q_1-1}U_t\right)(t)\in L^2(\Omega_1),$$
$$\left(-\Delta V+|V_t|^{q_2-1}V_t\right)(t)\in L^2(\Omega_2)\ \ a.e.\ on\ \ (0,T), \tag{3.15}$$
$$\delta,\delta_t,\delta_{tt}\in L^\infty\left(0,T;L^2(\Gamma_2)\right). \tag{3.16}$$

Proof of Lemma 3.1. *Faedo-Galerkin approximation.* Let $\{(\Phi_j,\Psi_j,e_j)\}$ $(j\in N)$ be orthonormal basis in $W=\left\{(u,\upsilon,\delta)\in H^1_{\Gamma_1}(\Omega_1)\times H^1(\Omega_2)\times L^2(\Gamma_2),\right.$ $\left.u|_{\Gamma_2}=\upsilon|_{\Gamma_2}\right\}$. Since Γ_1 and Γ_2 are sufficiently smooth, we have that $\Phi_j\in H^1_{\Gamma_1}(\Omega_1)\cap L^\infty(\Omega_1)$ and $\Psi_j\in H^1(\Omega_2)\cap L^\infty(\Omega_2)$ for all $j\in N$. For each $m\in N$ we consider

$$U_m:\Omega_1\times[0,T_m]\to R,\ V_m:\Omega_2\times[0,T_m]\to R,\ \delta_m:\Gamma_2\times[0,T_m]\to R$$

defined by

$$U_m(x,t)=\sum_{j=1}^m\alpha_{jm}(t)\,\Phi_j(x),\ V_m(x,t)=\sum_{j=1}^m\beta_{jm}(t)\,\Psi_j(x),$$
$$\delta_m(x,t)=\sum_{j=1}^m\eta_{jm}(t)\,e_j(x),$$

which are solutions to the approximate problem :

$$(U_{mtt},\Phi_j)_1+(\nabla U_m,\nabla\Phi_j)_1-$$
$$-\left(\frac{\partial U_m}{\partial\nu},\gamma_0(\Phi_j)\right)_{\Gamma_2}+\left(|U_{mt}|^{q_1-1}U_{mt},\Phi_j\right)_1=(F,\Phi_j)_1, \tag{3.17}$$
$$(V_{mtt},\Psi_j)_2+(\nabla V_m,\nabla\Psi_j)_2+$$
$$+\left(\frac{\partial V_m}{\partial\nu},\gamma_0(\Psi_j)\right)_{\Gamma_2}+\left(|V_{mt}|^{q_2-1}V_{mt},\Psi_j\right)_2=(G,\Psi_j)_2, \tag{3.18}$$
$$(M\delta_{mtt}+D\delta_{mt}+K\delta_m,e_j)_{\Gamma_2}=-\left(\gamma_0(U_{mt}),e_j\right)_{\Gamma_2}, \tag{3.19}$$

$$U_m = V_m, \frac{\partial U_m}{\partial \nu} - \frac{\partial V_m}{\partial \nu} + \rho\left(U_{mt}\right) = \delta_{mt}, x \in \Gamma_2, \tag{3.20}$$

$$U_m\left(x,0\right) = U_{0m}\left(x\right) = \sum_{j=1}^{m} \left(u_0, \Phi_j\right)_1 \Phi_j, U_{m_t}\left(x,0\right) = U_{1m}\left(x\right) = \sum_{j=1}^{m} \left(u_1, \Phi_j\right)_1 \Phi_j,$$

$$V_m\left(x,0\right) = V_{0m}\left(x\right) = \sum_{j=1}^{m} \left(\upsilon_0, \Psi_j\right)_2 \Psi_j,\ V_{m_t}\left(x,0\right) = V_{1m}\left(x\right) = \sum_{j=1}^{m} \left(\upsilon_1, \Psi_j\right)_2 \Psi_j,$$

$$\delta_m\left(x,0\right) = \delta_{0m}\left(x\right) = \sum_{j=1}^{m} \left(\delta_0, e_j\right)_{\Gamma_2} e_j,$$

$$\delta_{m_t}\left(x,0\right) = \gamma_1\left(U_{0m} - V_{0m}\right) + \gamma_0\left(\rho\left(U_{1m}\right)\right) = \sum_{j=1}^{m} \left(\gamma_1\left(u_0 - \upsilon_0\right) + \gamma_0\left(\rho\left(u_1\right)\right), e_j\right)_{\Gamma_2} e_j.$$

The local existence of such solutions (U_m, V_m, δ_m), $m \in N$ of this problem on the interval $[0, T_m]$ is obvious. From (3.17)-(3.19) we have the approximate equations

$$\left(U_{mtt}, \Phi\right)_1 + \left(\nabla U_m, \nabla \Phi\right)_1 - \left(\frac{\partial U_m}{\partial \nu}, \gamma_0\left(\Phi\right)\right)_{\Gamma_2} + \left(\left|U_{mt}\right|^{q_1-1} U_{mt}, \Phi\right)_1 = \left(F, \Phi\right)_1, \tag{3.21}$$

$$\left(V_{mtt}, \Psi\right)_2 + \left(\nabla V_m, \nabla \Psi\right)_2 + \left(\frac{\partial V_m}{\partial \nu}, \gamma_0\left(\Psi\right)\right)_{\Gamma_2} + \left(\left|V_{mt}\right|^{q_2-1} V_{mt}, \Psi\right)_2 = \left(G, \Psi\right)_2, \tag{3.22}$$

$$\left(M\delta_{mtt} + D\delta_{m_t} + K\delta_m, e\right)_{\Gamma_2} = -\left(\gamma_0\left(U_{mt}\right), e\right)_{\Gamma_2} \tag{3.23}$$

for $\forall \Phi \in Span\left\{\Phi_1, \Phi_2, ..., \Phi_m, ...\right\}, \forall \Psi \in Span\left\{\Psi_1, \Psi_2, ..., \Psi_m, ...\right\}$, $\forall e \in Span\left\{e_1, e_2, ..., e_m, ...\right\}$.

Estimate 1. Taking $\Phi = 2U_{mt}$ in (3.21), $\Psi = 2V_{mt}$ in (3.22), $e = 2\delta_{mt}$ in (3.23), we find

$$\tfrac{d}{dt}\left\|U_{m_t}\right\|_1^2 + \tfrac{d}{dt}\left\|\nabla U_m\right\|_1^2 - \left(\frac{\partial U_m}{\partial \nu}, \gamma_0\left(2U_{m_t}\right)\right)_{\Gamma_2} + 2\left(\left|U_{m_t}\right|^{q_1-1} U_{m_t}, U_{m_t}\right)_1 = 2\left(F, U_{m_t}\right)_1,$$

$$\tfrac{d}{dt}\left\|V_{m_t}\right\|_2^2 + \tfrac{d}{dt}\left\|\nabla V_m\right\|_2^2 + \left(\frac{\partial V_m}{\partial \nu}, \gamma_0\left(2V_{m_t}\right)\right)_{\Gamma_2} + 2\left(\left|V_{m_t}\right|^{q_2-1} V_{m_t}, V_{m_t}\right)_2 = 2\left(G, V_{m_t}\right)_2,$$

$$\tfrac{d}{dt}\left\|\sqrt{M}\delta_{m_t}\right\|_{\Gamma_2}^2 + 2\left\|\sqrt{D}\delta_{m_t}\right\|_{\Gamma_2}^2 + \tfrac{d}{dt}\left\|\sqrt{K}\delta_m\right\|_{\Gamma_2}^2 + \left(\gamma_0\left(U_{m_t}\right), 2\delta_{m_t}\right)_{\Gamma_2} = 0,$$

whence using (3.20), we obtain

$$\tfrac{d}{dt}\left(\left\|U_{m_t}\right\|_1^2 + \left\|\nabla U_m\right\|_1^2 + \left\|V_{m_t}\right\|_2^2 + \left\|\nabla V_m\right\|_2^2 + \left\|\sqrt{M}\delta_{m_t}\right\|_{\Gamma_2}^2 + \left\|\sqrt{K}\delta_m\right\|_{\Gamma_2}^2\right) + $$
$$+2\left(\rho\left(U_{m_t}\right), U_{m_t}\right)_{\Gamma_2} + 2\left\|\sqrt{D}\delta_{m_t}\right\|_{\Gamma_2}^2 + 2\left(\left|U_{m_t}\right|^{q_1+1}, 1\right)_1 + 2\left(\left|V_{m_t}\right|^{q_2+1}, 1\right)_2 =$$
$$= 2\left(F, U_{m_t}\right)_1 + 2\left(G, V_{m_t}\right)_2$$

or since by (2.5): $\rho(s)\,s \geq 0, \forall s \in (-\infty, +\infty)$, we have

$$\frac{d}{dt}\left(\|U_{m_t}\|_1^2 + \|\nabla U_m\|_1^2 + \|V_{m_t}\|_2^2 + \|\nabla V_m\|_2^2 + \left\|\sqrt{M}\delta_{m_t}\right\|_{\Gamma_2}^2 + \left\|\sqrt{K}\delta_m\right\|_{\Gamma_2}^2\right) +$$
$$+2\left\|\sqrt{D}\delta_{m_t}\right\|_{\Gamma_2}^2 + 2\left(|U_{m_t}|^{q_1+1}, 1\right)_1 + 2\left(|V_{m_t}|^{q_2+1}, 1\right)_2 =$$
$$= 2\left(F, U_{m_t}\right)_1 + 2\left(G, V_{m_t}\right)_2 .$$

Integrating this from 0 to $t(t \leq T_m)$ and using Young inequality, we get

$$\|U_{m_t}\|_1^2 + \|\nabla U_m\|_1^2 + \|V_{m_t}\|_2^2 + \|\nabla V_m\|_2^2 + \left\|\sqrt{M}\delta_{m_t}\right\|_{\Gamma_2}^2 + \left\|\sqrt{K}\delta_m\right\|_{\Gamma_2}^2 +$$
$$+2\int_0^t \left\|\sqrt{D}\delta_{m_t}\right\|_{\Gamma_2}^2 ds + 2\int_0^t \left[\left(|U_{m_t}|^{q_1+1}, 1\right)_1 + \left(|V_{m_t}|^{q_2+1}, 1\right)_2\right] ds \leq$$
$$\leq \|U_{1m}\|_1^2 + \|\nabla U_{0m}\|_1^2 + \|V_{1m}\|_2^2 + \|\nabla V_{0m}\|_2^2 +$$
$$+\left\|\sqrt{M}\left(\gamma_1\left(U_{0m} - V_{0m}\right) + \gamma_0\left(\rho\left(U_{1m}\right)\right)\right)\right\|_{\Gamma_2}^2 + \left\|\sqrt{K}\delta_{0m}\right\|_{\Gamma_2}^2 +$$
$$+\int_0^t \|F\|_1^2 ds + \int_0^t \|G\|_2^2 ds + \int_0^t \left(\|U_{m_t}\|_1^2 + \|V_{m_t}\|_2^2\right) ds,$$

whence by (2.1), (2.4), (3.9)-(3.12) and Gronwall's inequality, we deduce that

$$\|U_{m_t}\|_1^2 + \|\nabla U_m\|_1^2 + \|V_{m_t}\|_2^2 + \|\nabla V_m\|_2^2 + \|\delta_{m_t}\|_{\Gamma_2}^2 + \|\delta_m\|_{\Gamma_2}^2 +$$
$$+\int_0^t \|\delta_{m_t}\|_{\Gamma_2}^2 ds + \int_0^t \left[\left(|U_{m_t}|^{q_1+1}, 1\right)_1 + \left(|V_{m_t}|^{q_2+1}, 1\right)_2\right] ds \leq C_1,$$

where C_1 is a positive constant, which does not depend on m. This is Estimate 1.

Estimate 2. First of all, we estimate $\|U_{m_{tt}}(0)\|_1^2$, $\|V_{m_{tt}}(0)\|_2^2$ and $\|\delta_{m_{tt}}(0)\|_{\Gamma_2}^2$. Taking $\Phi = U_{m_{tt}}$ in (3.21), $\Psi = V_{m_{tt}}$ in (3.22), $e = \delta_{m_{tt}}$ in (3.23) and putting $t = 0$, we come to

$$\|U_{m_{tt}}(0)\|_1^2 = \left(\Delta U_{0m} - |U_{1m}|^{q_1-1} U_{1m} + F(x,0), U_{m_{tt}}(0)\right)_1 \leq$$
$$\leq \left\|\Delta U_{0m} - |U_{1m}|^{q_1-1} U_{1m} + F(x,0)\right\|_1 \|U_{m_{tt}}(0)\|_1 ,$$
$$\|V_{m_{tt}}(0)\|_2^2 = \left(\Delta V_{0m} - |V_{1m}|^{q_2-1} V_{1m} + G(x,0), V_{m_{tt}}(0)\right)_2 \leq$$
$$\leq \left\|\Delta V_{0m} - |V_{1m}|^{q_2-1} V_{1m} + G(x,0)\right\|_2 \|V_{m_{tt}}(0)\|_2 ,$$
$$\left\|\sqrt{M}\delta_{m_{tt}}(0)\right\|_{\Gamma_2}^2 =$$
$$= \left(-D\left(\gamma_1\left(U_{0m} - V_{0m}\right) + \gamma_0\left(\rho\left(U_{1m}\right)\right)\right) - K\delta_{0m} - \gamma_0\left(U_{1m}\right), \delta_{m_{tt}}(0)\right)_{\Gamma_2} \leq$$
$$\leq \left\|-D\left(\gamma_1\left(U_{0m} - V_{0m}\right) + \gamma_0\left(\rho\left(U_{1m}\right)\right)\right) - K\delta_{0m} - \gamma_0\left(U_{1m}\right)\right\|_{\Gamma_2} \|\delta_{m_{tt}}(0)\|_{\Gamma_2} ,$$

hence by (2.1), (2.4), (3.9)-(3.12) we have

$$\|U_{m_{tt}}(0)\|_1^2 + \|V_{m_{tt}}(0)\|_2^2 + \|\delta_{m_{tt}}(0)\|_{\Gamma_2}^2 \leq C_2, \tag{3.24}$$

where C_2 is a positive constant, which does not depend on m.

Differentiating (3.21), (3.22), (3.23) and taking $\Phi = 2U_{m_{tt}}$, $\Psi = 2V_{m_{tt}}$, $e = 2\delta_{m_{tt}}$, after standard calculations we obtain

$$\frac{d}{dt}\left(\|U_{m_{tt}}\|_1^2 + \|\nabla U_{m_t}\|_1^2 + \|V_{m_{tt}}\|_2^2 + \|\nabla V_{m_t}\|_2^2 + \left\|\sqrt{M}\delta_{m_{tt}}\right\|_{\Gamma_2}^2 + \left\|\sqrt{K}\delta_{m_t}\right\|_{\Gamma_2}^2\right) -$$

$$-\left(\frac{\partial U_{m_t}}{\partial \nu}, \gamma_0(2U_{m_{tt}})\right)_{\Gamma_2} + \left(\frac{\partial V_{m_t}}{\partial \nu}, \gamma_0\left(2V_{m_{tt}}\right)\right)_{\Gamma_2} + 2\left\|\sqrt{D}\delta_{m_{tt}}\right\|_{\Gamma_2}^2 +$$

$$+\left(\gamma_0\left(U_{m_{tt}}\right), 2\delta_{m_{tt}}\right)_{\Gamma_2} + 2q_1 \int\limits_{\Omega_1} \left|U_{m_t}\right|^{q_1-1} U_{m_{tt}}^2 dx + 2q_2 \int\limits_{\Omega_2} \left|V_{m_t}\right|^{q_2-1} V_{m_{tt}}^2 dx =$$

$$= 2\left(F_t, U_{m_{tt}}\right)_1 + 2\left(G_t, V_{m_{tt}}\right)_2,$$

whence using (3.20) and Young inequality, we get

$$\frac{d}{dt}\left(\|U_{m_{tt}}\|_1^2 + \|\nabla U_{m_t}\|_1^2 + \|V_{m_{tt}}\|_2^2 + \|\nabla V_{m_t}\|_2^2 + \left\|\sqrt{M}\delta_{m_{tt}}\right\|_{\Gamma_2}^2 + \left\|\sqrt{K}\delta_{m_t}\right\|_{\Gamma_2}^2\right) +$$

$$+2\left\|\sqrt{D}\delta_{m_{tt}}\right\|_{\Gamma_2}^2 + \left(\rho'\left(U_{m_t}\right)U_{m_{tt}}, \gamma_0\left(2U_{m_{tt}}\right)\right)_{\Gamma_2} +$$

$$+\frac{8q_1}{(q_1+1)^2}\int\limits_{\Omega_1}\left(\frac{\partial}{\partial t}\left(\left|U_{m_t}\right|^{\frac{q_1-1}{2}} U_{m_t}\right)\right)^2 dx +$$

$$+\frac{8q_2}{(q_2+1)^2}\int\limits_{\Omega_2}\left(\frac{\partial}{\partial t}\left(\left|V_{m_t}\right|^{\frac{q_2-1}{2}} V_{m_t}\right)\right)^2 dx \le \|F_t\|_1^2 + \|G_t\|_2^2 + \|U_{m_{tt}}\|_1^2 + \|V_{m_{tt}}\|_2^2;$$

integrating this over $(0,t)$ and using the fact that by (2.5): $\rho'(s) \ge 0, \forall s \in (-\infty, +\infty)$, we have

$$\|U_{m_{tt}}\|_1^2 + \|\nabla U_{m_t}\|_1^2 + \|V_{m_{tt}}\|_2^2 + \|\nabla V_{m_t}\|_2^2 + \|\sqrt{M}\delta_{m_{tt}}\|_{\Gamma_2}^2 + \|\sqrt{K}\delta_{m_t}\|_{\Gamma_2}^2 +$$

$$+2\int\limits_0^t \|\sqrt{D}\delta_{m_{tt}}\|_{\Gamma_2}^2 ds + \frac{8q_1}{(q_1+1)^2}\int\limits_0^t\int\limits_{\Omega_1}\left(\frac{\partial}{\partial t}\left(\left|U_{m_t}\right|^{\frac{q_1-1}{2}} U_{m_t}\right)\right)^2 dxds +$$

$$+\frac{8q_2}{(q_2+1)^2}\int\limits_0^t\int\limits_{\Omega_2}\left(\frac{\partial}{\partial t}\left(\left|V_{m_t}\right|^{\frac{q_2-1}{2}} V_{m_t}\right)\right)^2 dxds \le$$

$$\le \|U_{m_{tt}}(0)\|_1^2 + \|\nabla U_{1_m}\|_1^2 + \|V_{m_{tt}}(0)\|_2^2 + \|\nabla V_{1_m}\|_2^2 + \|\sqrt{M}\delta_{m_{tt}}(0)\|_{\Gamma_2}^2 +$$

$$+\|\sqrt{K}(\gamma_1(U_{0_m} - V_{0_m}) + \gamma_0(\rho(U_{1_m})))\|_{\Gamma_2}^2 +$$

$$+\int\limits_0^t \|F_t\|_1^2 ds + \int\limits_0^t \|G_t\|_2^2 ds + \int\limits_0^t \left(\|U_{m_{tt}}\|_1^2 + \|V_{m_{tt}}\|_2^2\right) ds.$$

Using (3.9)-(3.12), (3.24) and Gronwall's inequality we can obtain

$$\|U_{m_{tt}}\|_1^2 + \|\nabla U_{m_t}\|_1^2 + \|V_{m_{tt}}\|_2^2 + \|\nabla V_{m_t}\|_2^2 + \|\sqrt{M}\delta_{m_{tt}}\|_{\Gamma_2}^2 + \|\sqrt{K}\delta_{m_t}\|_{\Gamma_2}^2 +$$

$$+2\int\limits_0^t \|\sqrt{D}\delta_{m_{tt}}\|_{\Gamma_2}^2 ds + \frac{8q_1}{(q_1+1)^2}\int\limits_0^t\int\limits_{\Omega_1}\left(\frac{\partial}{\partial t}\left(\left|U_{m_t}\right|^{\frac{q_1-1}{2}} U_{m_t}\right)\right)^2 dxds +$$

$$+\frac{8q_2}{(q_2+1)^2}\int_0^t\int_{\Omega_2}\left(\frac{\partial}{\partial t}\left(|V_{m_t}|^{\frac{q_2-1}{2}}V_{m_t}\right)\right)^2 dxds\le C_3,$$

where C_3 is a positive constant, which does not depend on m. This is Estimate 2.

Passage to the limit. Using the estimates 1,2 and compactness argument, we can see that there exist a subsequence of $\{U_m\}$, a subsequence of $\{V_m\}$ and a subsequence of $\{\delta_m\}$ which will be denoted by the same notations, and the functions U , V and δ, such that

$$U_m\to U \text{ weakly star in } L^\infty(0,T;H^1_{\Gamma_1}(\Omega_1)),$$
$$V_m\to V \text{ weakly star in } L^\infty(0,T;H^1(\Omega_2)); \tag{3.25}$$
$$U_{m_t}\to U_t \text{ weakly star in } L^\infty(0,T;H^1_{\Gamma_1}(\Omega_1)),$$
$$V_{m_t}\to V_t \text{ weakly star in } L^\infty(0,T;H^1(\Omega_2)); \tag{3.26}$$
$$U_{m_{tt}}\to U_{tt} \text{ weakly star in } L^\infty(0,T;L^2(\Omega_1)),$$
$$V_{m_{tt}}\to V_{tt} \text{ weakly star in } L^\infty(0,T;L^2(\Omega_2)); \tag{3.27}$$
$$\delta_m\to \delta \text{ weakly star in } L^\infty(0,T;L^2(\Gamma_2)),$$
$$\delta_{m_t}\to \delta_t \text{ weakly star in } L^\infty(0,T;L^2(\Gamma_2)); \tag{3.28}$$
$$\delta_{m_{tt}}\to \delta_{tt} \text{ weakly star in } L^\infty(0,T;L^2(\Gamma_2)). \tag{3.29}$$

From [27, Theorem 3.1] we obtain that

$$U_m\to U \text{ strongly in } C([0,T];H^1_{\Gamma_1}(\Omega_1)),$$
$$U_{m_t}\to U_t \text{ strongly in } C([0,T];L^2(\Omega_1)),$$
$$V_m\to V \text{ strongly in } C([0,T];H^1(\Omega_2)),$$
$$V_{m_t}\to V_t \text{ strongly in } C([0,T];L^2(\Omega_2)),$$

and consequently:

$$U_m\to U,\ U_{m_t}\to U_t \text{ a.e. in } \Omega_1\times(0,T)\ ,V_m\to V,\ V_{m_t}\to V_t \text{ a.e. in } \Omega_2\times(0,T);$$

therefore

$$|U_{m_t}|^{q_1-1}U_{m_t}\to |U_t|^{q_1-1}U_t,\ \text{ a.e. in } \Omega_1\times(0,T),$$
$$|V_{m_t}|^{q_2-1}V_{m_t}\to |V_t|^{q_2-1}V_t,\ \text{ a.e. in } \Omega_2\times(0,T). \tag{3.30}$$

Since by Estimate 1:

$$U_{m_t}\in L^{q_1+1}(\Omega_1\times(0,T)),\ \ V_{m_t}\in L^{q_2+1}(\Omega_2\times(0,T)),$$

we obtain that

$$|U_{m_t}|^{q_1-1}U_{m_t}\in (L^{q_1+1}(\Omega_1\times(0,T)))'=L^{\frac{q_1+1}{q_1}}(\Omega_1\times(0,T)),$$
$$|V_{m_t}|^{q_2-1}V_{m_t}\in (L^{q_2+1}(\Omega_2\times(0,T)))'=L^{\frac{q_2+1}{q_2}}(\Omega_2\times(0,T)).$$

Therefore using estimate 1 we can see that there exist a subsequence of $\{U_m\}$ and a subsequence of $\{V_m\}$, which we still denote by the same notations, and the functions ψ,χ such that as $m\to\infty$

$$|U_{m_t}|^{q_1-1}U_{m_t}\to\psi \text{ weakly star in } L^{\frac{q_1+1}{q_1}}(\Omega_1\times(0,T)),$$
$$|V_{m_t}|^{q_2-1}V_{m_t}\to\chi \text{ weakly star in } L^{\frac{q_2+1}{q_2}}(\Omega_2\times(0,T)),$$

whence using (3.30) and [26, Lemma 1.3] we get

$$\psi = |U_t|^{q_1-1}U_t,\ \chi = |V_t|^{q_2-1}V_t$$

or

$$|U_{m_t}|^{q_1-1}U_{m_t} \to |U_t|^{q_1-1}U_t \text{ weakly star in } L^{\frac{q_1+1}{q_1}}(\Omega_1\times(0,T)), \tag{3.31}$$

$$|V_{m_t}|^{q_2-1}V_{m_t} \to |V_t|^{q_2-1}V_t \text{ weakly star in } L^{\frac{q_2+1}{q_2}}(\Omega_2\times(0,T)). \tag{3.32}$$

Taking into account the convergences in (3.25)-(3.29), (3.31), (3.32) we can pass to the limit in (3.21)-(3.23) and (3.20) as $m\to\infty$:

$$(U_{tt},\Phi)_1 + (\nabla U,\nabla\Phi)_1 - \\ -\left(\frac{\partial U}{\partial\nu},\gamma_0(\Phi)\right)_{\Gamma_2} + \left(|U_t|^{q_1-1}U_t,\Phi\right)_1 = (F,\Phi)_1, \tag{3.33}$$

$$(V_{tt},\Psi)_2 + (\nabla V,\nabla\Psi)_2 + \\ +\left(\frac{\partial V}{\partial\nu},\gamma_0(\Psi)\right)_{\Gamma_2} + \left(|V_t|^{q_2-1}V_t,\Psi\right)_2 = (G,\Psi)_2, \tag{3.34}$$

$$(M\delta_{tt}+D\delta_t+K\delta,e)_{\Gamma_2} = -(\gamma_0(U_t),e)_{\Gamma_2} \tag{3.35}$$

for all $(\Phi,\Psi,e)\in W$ a.e. in $(0,T)$ and

$$U=V,\ \delta_t = \frac{\partial U}{\partial\nu} - \frac{\partial V}{\partial\nu} + \rho(U_t),\ \ x\in\Gamma_2.$$

From (3.33), (3.34) we obtain

$$\int_{\Omega_1} U_{tt}\Phi dx - (\Delta U,\Phi)_{D'(\Omega_1)\times D(\Omega_1)} + \int_{\Omega_1}|U_t|^{q_1-1}U_t\Phi dx = \int_{\Omega_1}F\Phi dx,$$

$$\int_{\Omega_2} V_{tt}\Psi dx - (\Delta V,\Psi)_{D'(\Omega_2)\times D(\Omega_2)} + \int_{\Omega_2}|V_t|^{q_2-1}V_t\Psi dx = \int_{\Omega_2}G\Psi dx$$

for all $\Phi\in D(\Omega_1)$, $\Psi\in D(\Omega_2)$ a. e. in $(0,T)$. Therefore $\left(-\Delta U+|U_t|^{q_1-1}U_t\right)(t)\in L^2(\Omega_1)$, $\left(-\Delta V+|V_t|^{q_2-1}V_t\right)(t)\in L^2(\Omega_2)$ a.e. on $(0,T)$ and

$$U_{tt}-\Delta U+|U_t|^{q_1-1}U_t = F \text{ a.e. in } \Omega_1\times(0,T),$$

$$V_{tt}-\Delta V+|V_t|^{q_2-1}V_t = G \text{ a.e. in } \Omega_2\times(0,T).$$

From (3.35) we can see that (U,V,δ) satisfies the boundary condition (3.3).

The initial conditions (3.6)-(3.8) can be proved in a standard way and this completes the proof of the existence of solutions.

Uniqueness. The uniqueness of solution to the problem (3.1)-(3.8) is obtained by energy method as follows.

Let (U_1,V_1,δ_1) and (U_2,V_2,δ_2) be two solutions of the problem (3.1)-(3.8). We have that $U_1-U_2=\widetilde{U}$, $V_1-V_2=\widetilde{V}$, $\delta_1-\delta_2=\widetilde{\delta}$ satisfy

$$\left(\widetilde{U}_{tt},\Phi\right)_1 + \left(\nabla\widetilde{U},\nabla\Phi\right)_1 - \\ -\left(\frac{\partial\widetilde{U}}{\partial\nu},\gamma_0(\Phi)\right)_{\Gamma_2} + \int_{\Omega_1}\left(|U_{1t}|^{q_1-1}U_{1t} - |U_{2t}|^{q_1-1}U_{2t}\right)\Phi dx = 0, \tag{3.36}$$

$$\left(\widetilde{V}_{tt}, \Psi\right)_2 + \left(\nabla \widetilde{V}, \nabla \Psi\right)_2 +$$

$$+\left(\frac{\partial \widetilde{V}}{\partial \nu}, \gamma_0(\Psi)\right)_{\Gamma_2} + \int_{\Omega_2} \left(|V_{1t}|^{q_2-1} V_{1t} - |V_{2t}|^{q_2-1} V_{2t}\right) \Psi dx = 0, \tag{3.37}$$

$$\left(M\widetilde{\delta}_{tt} + D\widetilde{\delta}_t + K\widetilde{\delta}, e\right)_{\Gamma_2} = -\left(\gamma_0\left(\widetilde{U}_t\right), e\right)_{\Gamma_2} \tag{3.38}$$

for all $(\Phi, \Psi, e) \in W$ a. e. in $(0, T)$;

$$\widetilde{U} = 0 \text{ a.e. on } \Gamma_1 \times (0, T), \tag{3.39}$$

$$\widetilde{U} = \widetilde{V},\ \frac{\partial \widetilde{U}}{\partial \nu} - \frac{\partial \widetilde{V}}{\partial \nu} + \rho(U_{1t}) - \rho(U_{2t}) = \widetilde{\delta}_t \text{ a.e. on } \Gamma_2 \times (0, T), \tag{3.40}$$

$$\widetilde{U}(x, 0) = 0,\ \widetilde{U}_t(x, 0) = 0,\ x \in \Omega_1, \tag{3.41}$$

$$\widetilde{V}(x, 0) = 0,\ \widetilde{V}_t(x, 0) = 0,\ x \in \Omega_2, \tag{3.42}$$

$$\widetilde{\delta}(x, 0) = 0,\ \widetilde{\delta}_t(x, 0) = 0,\ x \in \Gamma_2, \tag{3.43}$$

Taking $\Phi = 2\widetilde{U}_t$ in (3.36), $\Psi = 2\widetilde{V}_t$ in (3.37), $e = 2\widetilde{\delta}_t$ in (3.38) and using (3.39), (3.40), we get

$$\frac{d}{dt}\left(\|\widetilde{U}_t\|_1^2 + \|\nabla \widetilde{U}\|_1^2 + \|\widetilde{V}_t\|_2^2 + \|\nabla \widetilde{V}\|_2^2 + \|\sqrt{M}\widetilde{\delta}_t\|_{\Gamma_2}^2 + \|\sqrt{K}\widetilde{\delta}\|_{\Gamma_2}^2\right) +$$

$$+2(\rho(U_{1t}) - \rho(U_{2t}), U_{1t} - U_{2t})_{\Gamma_2} + 2\|\sqrt{D}\widetilde{\delta}_t\|_{\Gamma_2}^2 +$$

$$+2(|U_{1t}|^{q_1-1}U_{1t} - |U_{2t}|^{q_1-1}U_{2t}, U_{1t} - U_{2t})_1 +$$

$$+2(|V_{1t}|^{q_2-1}V_{1t} - |V_{2t}|^{q_2-1}V_{2t}, V_{1t} - V_{2t})_2 = 0.$$

Since

$$(|U_{1t}|^{q_1-1}U_{1t} - |U_{2t}|^{q_1-1}U_{2t}, U_{1t} - U_{2t})_1 \geq 0,$$

$$(|V_{1t}|^{q_2-1}V_{1t} - |V_{2t}|^{q_2-1}V_{2t}, V_{1t} - V_{2t})_2 \geq 0$$

and by (2.5) :

$$(\rho(U_{1t}) - \rho(U_{2t}), U_{1t} - U_{2t})_{\Gamma_2} \geq 0,$$

then using (3.41)-(3.43) in last equality, we obtain

$$\|\widetilde{U}_t\|_1^2 + \|\nabla \widetilde{U}\|_1^2 + \|\widetilde{V}_t\|_2^2 + \|\nabla \widetilde{V}\|_2^2 +$$

$$+\|\sqrt{M}\widetilde{\delta}_t\|_{\Gamma_2}^2 + \|\sqrt{K}\widetilde{\delta}\|_{\Gamma_2}^2 + 2\int_0^t \|\sqrt{D}\widetilde{\delta}_t\|_{\Gamma_2}^2 ds \leq 0.$$

This inequality yields $\widetilde{U} = 0$, $\widetilde{V} = 0$, $\widetilde{\delta} = 0$. Lemma 3.1 is proved.

For

$$u \in C([0, T]; H^1_{\Gamma_1}(\Omega_1)) \cap C^1([0, T]; L^2(\Omega_1)),$$

$$v \in C([0, T]; H^1(\Omega_2)) \cap C^1([0, T]; L^2(\Omega_2))$$

$(u|_{\Gamma_2} = v|_{\Gamma_2})$ given we consider the following problem:

$$\begin{cases} U_{tt} - \Delta U + |U_t|^{q_1-1}U_t = f(u) \text{ in } \Omega_1 \times (0,T), \\ V_{tt} - \Delta V + |V_t|^{q_2-1}V_t = g(v) \text{ in } \Omega_2 \times (0,T), \\ M\delta_{tt} + D\delta_t + K\delta = -U_t \text{ on } \Gamma_2 \times (0,T), \\ U = 0 \text{ on } \Gamma_1 \times (0,T), \\ U = V,\ \frac{\partial U}{\partial \nu} - \frac{\partial V}{\partial \nu} + \rho(U_t) = \delta_t \text{ on } \Gamma_2, \\ U(x,0) = u_0(x),\ U_t(x,0) = u_1(x),\ x \in \Omega_1, \\ V(x,0) = v_0(x),\ V_t(x,0) = v_1(x),\ x \in \Omega_2, \\ \delta(x,0) = \delta_0(x),\ \delta_t(x,0) = \delta_1(x),\ x \in \Gamma_2. \end{cases} \tag{3.44}$$

Lemma 3.2. *Suppose that* (2.1)-(2.5) *hold and let* $u_0 \in H^1_{\Gamma_1}(\Omega_1)$, $u_1 \in L^{2q_1}(\Omega_1)$, $v_0 \in H^1(\Omega_2)$, $v_1 \in L^{2q_2}(\Omega_2)$, $\delta_0 \in L^2(\Gamma_2)$, $\delta_1 \in L^2(\Gamma_2)$. *Then, there exist* $T > 0$ *and a unique solution* (U, V, δ) *to the problem* (3.44) *such that*

$$U \in C([0,T]; H^1_{\Gamma_1}(\Omega_1)),\ U_t \in C([0,T]; L^2(\Omega_1)) \cap L^{q_1+1}(\Omega_1 \times (0,T)),$$

$$V \in C([0,T]; H^1(\Omega_2)),\ V_t \in C([0,T]; L^2(\Omega_2)) \cap L^{q_2+1}(\Omega_2 \times (0,T)),$$

$$\delta, \delta_t \in L^\infty(0,T; L^2(\Gamma_2)).$$

Proof of Lemma 3.2. By the same methods as in [21], we approximate $u \in C([0,T]; H^1_{\Gamma_1}(\Omega_1)) \cap C^1([0,T]; L^2(\Omega_1))$, $v \in C([0,T]; H^1(\Omega_2)) \cap C^1([0,T]; L^2(\Omega_2))$ by sequences $\{u_\mu\}_{\mu\in N}$ in $C([0,T]; C_0^\infty(\Omega_1))$, $\{v_\mu\}_{\mu\in N}$ in $C([0,T]; C^\infty(\Omega_2))$ by standard convolution argument as in [11]. Next we approximate the initial data $u_0 \in H^1_{\Gamma_1}(\Omega_1)$ by a sequence $\{u^0_\mu\}_{\mu\in N}$ in $H^1_{\Gamma_1}(\Omega_1) \cap H^2(\Omega_1)$, the initial data $v_0 \in H^1(\Omega_2)$ by a sequence $\{v^0_\mu\}_{\mu\in N}$ in $H^2(\Omega_2)$, the initial data $u_1 \in L^{2q_1}(\Omega_1)$ by a sequence $\{u^1_\mu\}_{\mu\in N}$ in $C^\infty(\Omega_1)$, the initial data $v_1 \in L^{2q_2}(\Omega_2)$ by a sequence $\{v^1_\mu\}_{\mu\in N}$ in $C^\infty(\Omega_2)$, $\delta_0 \in L^2(\Gamma_2)$ by a sequence $\{\delta^0_\mu\}_{\mu\in N}$ in $C^\infty(\Gamma_2)$ and $\delta_1 \in L^2(\Gamma_2)$ by a sequence $\{\delta^1_\mu\}_{\mu\in N}$ in $C^\infty(\Gamma_2)$. Then we consider the set of following problems

$$\begin{cases} U_{\mu_{tt}} - \Delta U_\mu + |U_{\mu_t}|^{q_1-1}U_{\mu_t} = f(u_\mu) \text{ in } \Omega_1 \times (0,T), \\ V_{\mu_{tt}} - \Delta V_\mu + |V_{\mu_t}|^{q_2-1}V_{\mu_t} = g(v_\mu) \text{ in } \Omega_2 \times (0,T), \\ M\delta_{\mu_{tt}} + D\delta_{\mu_t} + K\delta_\mu = -U_{\mu_t} \text{ on } \Gamma_2 \times (0,T), \\ U_\mu = 0 \text{ on } \Gamma_1 \times (0,T), \\ U_\mu = V_\mu,\ \frac{\partial U_\mu}{\partial \nu} - \frac{\partial V_\mu}{\partial \nu} + \rho(U_{\mu_t}) = \delta_{\mu_t} \text{ on } \Gamma_2, \\ U_\mu(x,0) = u^0_\mu(x),\ U_{\mu_t}(x,0) = u^1_\mu(x),\ x \in \Omega_1, \\ V_\mu(x,0) = v^0_\mu(x),\ V_{\mu_t}(x,0) = v^1_\mu(x),\ x \in \Omega_2, \\ \delta_\mu(x,0) = \delta^0_\mu(x),\ \delta_{\mu_t}(x,0) = \delta^1_\mu(x),\ x \in \Gamma_2. \end{cases} \tag{3.45}$$

By (2.2)-(2.3) we obtain $f(u_\mu) \in H^1(0,T; L^2(\Omega_1))$, $g(v_\mu) \in H^1(0,T; L^2(\Omega_2))$. Consequently, Lemma 3.1 guarantees the existence of a sequence of unique solutions $(U_\mu, V_\mu, \delta_\mu)$ to the problem (3.45) satisfying (3.13)-(3.16). Our goal now is to show that the sequence of solutions $(U_\mu, V_\mu, \delta_\mu)$ converges to the solution (U, V, δ) of (3.44); it is suffices to show that $(U_\mu, V_\mu, \delta_\mu)$ is a Cauchy sequence in the space

$$Z_T = \big\{(U,V,\delta) : U \in C([0,T]; H^1_{\Gamma_1}(\Omega_1)) \cap C^1([0,T]; L^2(\Omega_1)),$$

$$V \in C([0,T]; H^1(\Omega_2)) \cap C^1([0,T]; L^2(\Omega_2)), \delta \in L^\infty(0,T; L^2(\Gamma_2)),$$

$$\delta_t \in L^\infty(0,T; L^2(\Gamma_2))\big\}$$

endowed with the norm

$$\|(U,V,\delta)\|_{Z_T}^2 =$$

$$= \max_{0\le t\le T}(\|U_t\|_1^2 + \|\nabla U\|_1^2 + \|V_t\|_2^2 + \|\nabla V\|_2^2 + \|\delta_t\|_{\Gamma_2}^2 + \|\delta\|_{\Gamma_2}^2).$$

We set $\tilde{u} = u_\mu - u_\tau$, $\tilde{\upsilon} = \upsilon_\mu - \upsilon_\tau$, $\tilde{U} = U_\mu - U_\tau$, $\tilde{V} = V_\mu - V_\tau$, $\tilde{\delta} = \delta_\mu - \delta_\tau$. It is easy to see that $(\tilde{U}, \tilde{V}, \tilde{\delta})$ satisfies

$$\begin{cases} \widetilde{U}_{tt} - \Delta\widetilde{U} + |U_{\mu t}|^{q_1-1}U_{\mu t} - |U_{\tau t}|^{q_1-1}U_{\tau t} = f(u_\mu) - f(u_\tau) \text{ in } \Omega_1\times(0,T), \\ \widetilde{V}_{tt} - \Delta\widetilde{V} + |V_{\mu t}|^{q_2-1}V_{\mu t} - |V_{\tau t}|^{q_2-1}V_{\tau t} = g(\upsilon_\mu) - g(\upsilon_\tau) \text{ in } \Omega_2\times(0,T), \\ M\widetilde{\delta}_{tt} + D\widetilde{\delta}_t + K\widetilde{\delta} = -\widetilde{U}_t \text{ on } \Gamma_2\times(0,T), \\ \widetilde{U} = 0 \text{ on } \Gamma_1\times(0,T), \\ \widetilde{U} = \widetilde{V},\ \frac{\partial\widetilde{U}}{\partial\nu} - \frac{\partial\widetilde{V}}{\partial\nu} + \rho(U_{\mu t}) - \rho(U_{\tau t}) = \widetilde{\delta}_t \text{ on } \Gamma_2, \\ \widetilde{U}(x,0) = \widetilde{U}_0(x) = u_\mu^0(x) - u_\tau^0(x), \\ \widetilde{U}_t(x,0) = \widetilde{U}_1(x) = u_\mu^1(x) - u_\tau^1(x),\ x\in\Omega_1, \\ \widetilde{V}(x,0) = \widetilde{V}_0(x) = \upsilon_\mu^0(x) - \upsilon_\tau^0(x), \\ \widetilde{V}_t(x,0) = \widetilde{V}_1(x) = \upsilon_\mu^1(x) - \upsilon_\tau^1(x),\ x\in\Omega_2, \\ \widetilde{\delta}(x,0) = \widetilde{\delta}_0(x) = \delta_\mu^0(x) - \delta_\tau^0(x), \\ \widetilde{\delta}_t(x,0) = \widetilde{\delta}_1(x) = \delta_\mu^1(x) - \delta_\tau^1(x),\ x\in\Gamma_2. \end{cases} \tag{3.46}$$

Multiplying the first equation of (3.46) by $2\widetilde{U}_t$, the second equation by $2\widetilde{V}_t$ and the third equation by $2\widetilde{\delta}_t$, integrating over $\Omega_1\times(0,T)$, $\Omega_2\times(0,T)$ and $\Gamma_2\times(0,T)$, respectively, then using the fifth condition in (3.46) and the fact that

$$\left(|U_{\mu t}|^{q_1-1}U_{\mu t} - |U_{\tau t}|^{q_1-1}U_{\tau t}, U_{\mu t} - U_{\tau t}\right)_1 \ge 0,$$

$$\left(|V_{\mu t}|^{q_2-1}V_{\mu t} - |V_{\tau t}|^{q_2-1}V_{\tau t}, V_{\mu t} - V_{\tau t}\right)_2 \ge 0,$$

we get

$$\frac{d}{dt}\left(\|\widetilde{U}_t\|_1^2 + \|\nabla\widetilde{U}\|_1^2 + \|\widetilde{V}_t\|_2^2 + \|\nabla\widetilde{V}\|_2^2 + \|\sqrt{M}\widetilde{\delta}_t\|_{\Gamma_2}^2 + \|\sqrt{K}\widetilde{\delta}\|_{\Gamma_2}^2\right) +$$

$$+2(\rho(U_{\mu t}) - \rho(U_{\tau t}), U_{\mu t} - U_{\tau t})_{\Gamma_2} + 2\|\sqrt{D}\widetilde{\delta}_t\|_{\Gamma_2}^2 \le$$

$$\le 2(f(u_\mu) - f(u_\tau), \widetilde{U}_t)_1 + 2(g(\upsilon_\mu) - g(\upsilon_\tau), \widetilde{V}_t)_2. \tag{3.47}$$

Using Hölder's inequality (with $\frac{1}{n} + \frac{n-2}{2n} + \frac{1}{2} = 1$) and (2.2) we estimate the right hand side of (3.47) as follows:

$$\int_{\Omega_1}(f(u_\mu) - f(u_\tau))\,\widetilde{U}_t dx \le \int_{\Omega_1} f'(\theta u_\mu + (1-\theta)u_\tau)|\tilde{u}||\widetilde{U}_t|dx \le$$

$$\le c_2\int_{\Omega_1}\sup\left(|u_\mu|^{p-1}, |u_\tau|^{p-1}\right)|\tilde{u}||\widetilde{U}_t|dx \le$$

$$\le c_2\left(\|u_\mu\|_{L^{(p-1)n}(\Omega_1)}^{p-1} + \|u_\tau\|_{L^{(p-1)n}(\Omega_1)}^{p-1}\right)\times$$

$$\times\|\tilde{u}\|_{L^{2n/(n-2)}(\Omega_1)}\|\widetilde{U}_t\|_1\ \ (0<\theta<1); \tag{3.48}$$

by a similar way:

$$\int\limits_{\Omega_2} (g(\upsilon_\mu) - g(\upsilon_\tau))\, \widetilde{V}_t dx \le$$

$$\le c_4 \left(\|\upsilon_\mu\|^{p-1}_{L^{(p-1)n}(\Omega_2)} + \|\upsilon_\tau\|^{p-1}_{L^{(p-1)n}(\Omega_2)} \right) \|\tilde{\upsilon}\|_{L^{2n/(n-2)}(\Omega_2)} \|\widetilde{V}_t\|_2. \tag{3.49}$$

By the Sobolev embedding $H^1_{\Gamma_1}(\Omega_1) \mapsto L^{2n/(n-2)}(\Omega_1)$, Phriedrich's inequality and the condition that $\tilde{u}|_{\Gamma_2} = \tilde{\upsilon}|_{\Gamma_2}$ we can obtain

$$\|\tilde{u}\|_{L^{2n/(n-2)}(\Omega_1)} + \|\tilde{\upsilon}\|_{L^{2n/(n-2)}(\Omega_2)} \le C_4(\|\nabla\tilde{u}\|_1 + \|\nabla\tilde{\upsilon}\|_2);$$

and in a similar way, by (2.3) we have

$$\|u_\mu\|^{p-1}_{L^{(p-1)n}(\Omega_1)} + \|u_\tau\|^{p-1}_{L^{(p-1)n}(\Omega_1)} + \|\upsilon_\mu\|^{p-1}_{L^{(p-1)n}(\Omega_2)} + \|\upsilon_\tau\|^{p-1}_{L^{(p-1)n}(\Omega_2)} \le$$

$$\le C_5(\|\nabla u_\mu\|^{p-1}_1 + \|\nabla u_\tau\|^{p-1}_1 + \|\nabla \upsilon_\mu\|^{p-1}_2 + \|\nabla \upsilon_\tau\|^{p-1}_2),$$

where C_4, C_5 are positive constants depending only on Ω_1, Ω_2 and p. By these inequalities from (3.48), (3.49) we have

$$\left(f(u_\mu) - f(u_\tau), \widetilde{U}_t \right)_1 + \left(g(\upsilon_\mu) - g(\upsilon_\tau), \widetilde{V}_t \right)_2 \le$$

$$\max\{c_2, c_4\} \left(\|u_\mu\|^{p-1}_{L^{(p-1)n}(\Omega_1)} + \|u_\tau\|^{p-1}_{L^{(p-1)n}(\Omega_1)} + \|\upsilon_\mu\|^{p-1}_{L^{(p-1)n}(\Omega_2)} + \|\upsilon_\tau\|^{p-1}_{L^{(p-1)n}(\Omega_2)} \right) \times$$

$$\times \left(\|\tilde{u}\|_{L^{2n/(n-2)}(\Omega_1)} \|\widetilde{U}_t\|_1 + \|\tilde{\upsilon}\|_{L^{2n/(n-2)}(\Omega_2)} \|\widetilde{V}_t\|_2 \right) \le$$

$$\le \max\{c_2, c_4\} C_4 C_5 (\|\nabla u_\mu\|^{p-1}_1 + \|\nabla u_\tau\|^{p-1}_1 + \|\nabla \upsilon_\mu\|^{p-1}_2 + \|\nabla \upsilon_\tau\|^{p-1}_2)\times$$

$$\times(\|\nabla\tilde{u}\|_1 + \|\nabla\tilde{\upsilon}\|_2)(\|\widetilde{U}_t\|_1 + \|\widetilde{V}_t\|_2). \tag{3.50}$$

Using (3.50) and (2.5) in (3.47) we obtain

$$\|\widetilde{U}_t\|^2_1 + \|\nabla\widetilde{U}\|^2_1 + \|\widetilde{V}_t\|^2_2 + \|\nabla\widetilde{V}\|^2_2 + \|\sqrt{M}\widetilde{\delta}_t\|^2_{\Gamma_2} +$$

$$+ \|\sqrt{K}\widetilde{\delta}\|^2_{\Gamma_2} + 2\int\limits_0^t \|\sqrt{D}\widetilde{\delta}_t\|^2_{\Gamma_2} ds \le$$

$$\le \|\widetilde{U}_1\|^2_1 + \|\nabla\widetilde{U}_0\|^2_1 + \|\widetilde{V}_1\|^2_2 + \|\nabla\widetilde{V}_0\|^2_2 + \|\sqrt{M}\widetilde{\delta}_1\|^2_{\Gamma_2} + \|\sqrt{K}\widetilde{\delta}_0\|^2_{\Gamma_2} +$$

$$+2\max\{c_2, c_4\} C_4 C_5 \int\limits_0^t \left(\|\nabla u_\mu\|^{p-1}_1 + \|\nabla u_\tau\|^{p-1}_1 + \|\nabla \upsilon_\mu\|^{p-1}_2 + \|\nabla \upsilon_\tau\|^{p-1}_2 \right) \times$$

$$\times(\|\nabla\tilde{u}\|_1 + \|\nabla\tilde{\upsilon}\|_2)(\|\widetilde{U}_t\|_1 + \|\widetilde{V}_t\|_2) ds.$$

By (2.1) the last inequality gives

$$\|\widetilde{U}_t\|^2_1 + \|\nabla\widetilde{U}\|^2_1 + \|\widetilde{V}_t\|^2_2 + \|\nabla\widetilde{V}\|^2_2 + \|\widetilde{\delta}_t\|^2_{\Gamma_2} +$$

$$+ \|\widetilde{\delta}\|^2_{\Gamma_2} + \int\limits_0^t \|\widetilde{\delta}_t\|^2_{\Gamma_2} ds \le \frac{1}{C_6} \left(\|\widetilde{U}_1\|^2_1 + \|\nabla\widetilde{U}_0\|^2_1 + \|\widetilde{V}_1\|^2_2 + \|\nabla\widetilde{V}_0\|^2_2 + \right.$$

$$\left. + \max_{x\in\overline{\Gamma}_2} M(x) \|\widetilde{\delta}_1\|^2_{\Gamma_2} + \max_{x\in\overline{\Gamma}_2} K(x) \|\widetilde{\delta}_0\|^2_{\Gamma_2} \right) +$$

$$+\frac{K_1}{C_6}\int\limits_0^t(\|\nabla\tilde{u}\|_1+\|\nabla\tilde{\upsilon}\|_2)(\|\widetilde{U}_t\|_1+\|\widetilde{V}_t\|_2)ds,$$

where $C_6=\min\{1,m_0,2d_0,k_0\}$, $\min\limits_{x\in\overline{\Gamma}_2}M(x)=m_0>0$, $\min\limits_{x\in\overline{\Gamma}_2}D(x)=d_0>0$, $\min\limits_{x\in\overline{\Gamma}_2}K(x)=k_0>0$ and K_1 is a positive constant depending only on Ω_1, Ω_2, p, T. The Gronwall's lemma guarantees that:

$$\|(\widetilde{U},\widetilde{V},\widetilde{\delta})\|^2_{Z_T}\le K_2\left(\|\widetilde{U}_1\|_1^2+\|\nabla\widetilde{U}_0\|_1^2+\|\widetilde{V}_1\|_2^2+\|\nabla\widetilde{V}_0\|_2^2+\right.$$

$$\left.+\max_{x\in\overline{\Gamma}_2}M(x)\|\widetilde{\delta}_1\|^2_{\Gamma_2}+\max_{x\in\overline{\Gamma}_2}K(x)\|\widetilde{\delta}_0\|^2_{\Gamma_2}\right)+K_3T\|(\widetilde{u},\widetilde{\upsilon},\widetilde{\delta})\|^2_{Z_T},$$

where K_2,K_3 are positive constants depending only on Ω_1, Ω_2, p, T.

Since $\{u^0_\mu\}$ is a converging sequence in $H^1_{\Gamma_1}(\Omega_1)$, $\{\upsilon^0_\mu\}$ is a converging sequence in $H^1(\Omega_2)$, $\{u^1_\mu\}$ is a converging sequence in $L^2(\Omega_1)$, $\{\upsilon^1_\mu\}$ is a converging sequence in $L^2(\Omega_2)$, $\{\delta^0_\mu\}$ is a converging sequence in $L^2(\Gamma_2)$ and $\{\delta^1_\mu\}$ is a converging sequence in $L^2(\Gamma_2)$, we conclude that (U_μ,V_μ,δ_μ) is a Cauchy sequence in Z_T. Thus (U_μ,V_μ,δ_μ) converges to the limit $(U,V,\delta)\in Z_T$. By the same procedure used by Georgiev and Todorova in [21] we can prove that this limit is a solution of the problem (3.44). Lemma 3.2 is proved.

Proof of Theorem 2.1. For $T>0$, we define the convex closed subset of Z_T:

$$X_T=\{(U,V,\delta)\in Z_T: U|_{t=0}=u_0, V|_{t=0}=\upsilon_0, U_t|_{t=0}=u_1, V_t|_{t=0}=\upsilon_1$$

$$\delta|_{t=0}=\delta_0,\delta_t|_{t=0}=\delta_1\}.$$

For $R>0$ let us denote

$$B_R(X_T)=\{(U,V,\delta)\in X_T:\|(U,V,\delta)\|_{Z_T}\le R\}.$$

Then, Lemma 3.2 implies that for any $(u,\upsilon,\delta)\in X_T$, we define $(U,V,\delta)=\Phi(u,\upsilon,\delta)$ as a unique solution of problem (3.44) corresponding to (u,υ,δ).

Φ is a contractive map satisfying $\Phi(B_R(X_T))\subset B_R(X_T)$. Indeed, let $(u,\upsilon,\delta)\in B_R(X_T)$ and $(U,V,\delta)=\Phi(u,\upsilon,\delta)$. Then for all $t\in(0,T)$ (as in the proof of Lemma 3.2) we have

$$C_6\left(\|U_t\|_1^2+\|\nabla U\|_1^2+\|V_t\|_2^2+\|\nabla V\|_2^2+\|\delta_t\|^2_{\Gamma_2}+\right.$$

$$\left.+\|\delta\|^2_{\Gamma_2}+\int\limits_0^t\|\delta_t\|^2_{\Gamma_2}ds\right)+2\int\limits_0^t(\rho(U_t),U_t)_{\Gamma_2}ds+2\int\limits_0^t\int\limits_{\Omega_1}|U_t|^{q_1+1}dxds+$$

$$+2\int\limits_0^t\int\limits_{\Omega_2}|V_t|^{q_2+1}dxds\le\|U_1\|_1^2+\|\nabla U_0\|_1^2+\|V_1\|_2^2+\|\nabla V_0\|_2^2+$$

$$+\max_{x\in\overline{\Gamma}_2}M(x)\|\delta_1\|^2_{\Gamma_2}+\max_{x\in\overline{\Gamma}_2}K(x)\|\delta_0\|^2_{\Gamma_2}+2\int\limits_0^t\int\limits_{\Omega_1}f(u)U_tdxds+2\int\limits_0^t\int\limits_{\Omega_2}g(\upsilon)V_tdxds.$$

By Hölder's inequality, the condition (2.2) and the facts that $\rho(s)s \geq 0$ ($\forall s \in (-\infty, +\infty)$) and $(u, \upsilon, \delta) \in B_R(X_T)$ we get

$$C_6\left(\|U_t\|_1^2 + \|\nabla U\|_1^2 + \|V_t\|_2^2 + \|\nabla V\|_2^2 + \|\delta_t\|_{\Gamma_2}^2 + \right.$$

$$\left. + \|\delta\|_{\Gamma_2}^2 + \int_0^t \|\delta_t\|_{\Gamma_2}^2 ds\right) + 2\int_0^t\int_{\Omega_1} |U_t|^{q_1+1} dxds + 2\int_0^t\int_{\Omega_2} |V_t|^{q_2+1} dxds \leq$$

$$\leq \|U_1\|_1^2 + \|\nabla U_0\|_1^2 + \|V_1\|_2^2 + \|\nabla V_0\|_2^2 +$$

$$+ \max_{x\in\overline{\Gamma}_2} M(x)\|\delta_1\|_{\Gamma_2}^2 + \max_{x\in\overline{\Gamma}_2} K(x)\|\delta_0\|_{\Gamma_2}^2 + 2CR^p \int_0^t (\|U_t\|_1 + \|V_t\|_2) ds,$$

where C is a positive constant depending only on Ω_1, Ω_2, p, T. This leads to

$$\|(U, V, \delta)\|_{Z_T}^2 \leq C_6^{-1}\left(\|U_1\|_1^2 + \|\nabla U_0\|_1^2 + \|V_1\|_2^2 + \|\nabla V_0\|_2^2 + \right.$$

$$\left. + \max_{x\in\overline{\Gamma}_2} M(x)\|\delta_1\|_{\Gamma_2}^2 + \max_{x\in\overline{\Gamma}_2} K(x)\|\delta_0\|_{\Gamma_2}^2\right) + K_4 R^p T \|(U, V, \delta)\|_{Z_T}, \tag{3.51}$$

where K_4 – a positive constant which is independent on R. Using Young's inequality in the last term on the right-hand side of (3.51), we get

$$\|(U, V, \delta)\|_{Z_T}^2 \leq C_6^{-1}\left(\|U_1\|_1^2 + \|\nabla U_0\|_1^2 + \|V_1\|_2^2 + \|\nabla V_0\|_2^2 + \right.$$

$$\left. + \max_{x\in\overline{\Gamma}_2} M(x)\|\delta_1\|_{\Gamma_2}^2 + \max_{x\in\overline{\Gamma}_2} K(x)\|\delta_0\|_{\Gamma_2}^2\right) + TR^p\left(\frac{TR^p}{2} K_4^2 + \frac{1}{2TR^p}\|(U, V, \delta)\|_{Z_T}^2\right).$$

Hence, we obtain

$$\frac{1}{2}\|(U, V, \delta)\|_{Z_T}^2 \leq C_6^{-1}\left(\|U_1\|_1^2 + \|\nabla U_0\|_1^2 + \|V_1\|_2^2 + \|\nabla V_0\|_2^2 + \right.$$

$$\left. + \max_{x\in\overline{\Gamma}_2} M(x)\|\delta_1\|_{\Gamma_2}^2 + \max_{x\in\overline{\Gamma}_2} K(x)\|\delta_0\|_{\Gamma_2}^2\right) + \frac{1}{2} R^{2p} T^2 K_4^2. \tag{3.52}$$

By choosing R large enough so that

$$C_6^{-1}\left(\|U_1\|_1^2 + \|\nabla U_0\|_1^2 + \|V_1\|_2^2 + \|\nabla V_0\|_2^2 + \right.$$

$$\left. + \max_{x\in\overline{\Gamma}_2} M(x)\|\delta_1\|_{\Gamma_2}^2 + \max_{x\in\overline{\Gamma}_2} K(x)\|\delta_0\|_{\Gamma_2}^2\right) \leq \frac{1}{4} R^2,$$

then T sufficiently small so that $R^{2p} T^2 K_4^2 \leq \frac{1}{2} R^2$, from (3.52) we obtain: $(U, V, \delta) \in B_R(X_T)$.

Next, we verify that Φ is a contraction mapping. To this end, we set $\widetilde{u} = u - \bar{u}$, $\widetilde{\upsilon} = \upsilon - \bar{\upsilon}$, $\widetilde{U} = U - \overline{U}$, $\widetilde{V} = V - \overline{V}$ and $\widetilde{\delta} = \delta - \overline{\delta}$, where $(U, V, \delta) = \Phi(u, \upsilon, \delta)$

and $(\overline{U},\overline{V},\overline{\delta})=\Phi(\bar{u},\bar{v},\bar{\delta})$. It is straightforward to verify that $(\widetilde{U},\widetilde{V},\widetilde{\delta})$ satisfies

$$\begin{cases} \widetilde{U}_{tt}-\Delta\widetilde{U}+|U_t|^{q_1-1}U_t-|\overline{U}_t|^{q_1-1}\overline{U}_t=f(u)-f(\bar{u}) \text{ in } \Omega_1\times(0,T), \\ \widetilde{V}_{tt}-\Delta\widetilde{V}+|V_t|^{q_2-1}V_t-|\overline{V}_t|^{q_2-1}\overline{V}_t=g(v)-g(\overline{v}) \text{ in } \Omega_2\times(0,T), \\ M\widetilde{\delta}_{tt}+D\widetilde{\delta}_t+K\widetilde{\delta}=-\widetilde{U}_t \text{ on } \Gamma_2\times(0,T), \\ \widetilde{U}=0 \text{ on } \Gamma_1\times(0,T), \\ \widetilde{U}=\widetilde{V},\ \frac{\partial\widetilde{U}}{\partial\nu}-\frac{\partial\widetilde{V}}{\partial\nu}+\rho(U_t)-\rho(\overline{U}_t)=\widetilde{\delta}_t \text{ on } \Gamma_2, \\ \widetilde{U}(x,0)=0,\ \widetilde{U}_t(x,0)=0,\ x\in\Omega_1, \\ \widetilde{V}(x,0)=0,\ \widetilde{V}_t(x,0)=0,\ x\in\Omega_2, \\ \widetilde{\delta}(x,0)=0,\ \widetilde{\delta}_t(x,0)=0,\ x\in\Gamma_2. \end{cases} \tag{3.53}$$

Multiplying the first equation by $2\widetilde{U}_t$, the second equation by $2\widetilde{V}_t$ and the third equation by $2\widetilde{\delta}_t$ in (3.53) and integrating over $(0,t)\times\Omega_1$, $(0,t)\times\Omega_2$, and $(0,t)\times\Gamma_2$, respectively, we obtain

$$\begin{gathered} \frac{d}{dt}\left(\|\widetilde{U}_t\|_1^2+\|\nabla\widetilde{U}\|_1^2+\|\widetilde{V}_t\|_2^2+\|\nabla\widetilde{V}\|_2^2+\|\sqrt{M}\widetilde{\delta}_t\|_{\Gamma_2}^2+\|\sqrt{K}\widetilde{\delta}\|_{\Gamma_2}^2\right)+ \\ +2(\rho(U_t)-\rho(\overline{U}_t),U_t-\overline{U}_t)_{\Gamma_2}+2\|\sqrt{D}\widetilde{\delta}_t\|_{\Gamma_2}^2+ \\ +2(|U_t|^{q_1-1}U_t-|\overline{U}_t|^{q_1-1}\overline{U}_t,U_t-\overline{U}_t)_1+2(|V_t|^{q_2-1}V_t-|\overline{V}_t|^{q_2-1}V_t,V_t-\overline{V}_t)_2= \\ =2(f(u)-f(\bar{u}),\widetilde{U}_t)_1+2(g(v)-g(\bar{v}),\widetilde{V}_t)_2. \end{gathered} \tag{3.54}$$

Since

$$\begin{gathered} (\rho(U_t)-\rho(\overline{U}_t),U_t-\overline{U}_t)_{\Gamma_2}\geq 0, \\ (|U_t|^{q_1-1}U_t-|\overline{U}_t|^{q_1-1}\overline{U}_t,U_t-\overline{U}_t)_1\geq 0, \\ (|V_t|^{q_2-1}V_t-|\overline{V}_t|^{q_2-1}\overline{V}_t,V_t-\overline{V}_t)_2\geq 0, \\ \min_{x\in\bar{\Gamma}_2}M(x)=m_0>0,\ \min_{x\in\bar{\Gamma}_2}D(x)=d_0>0,\ \min_{x\in\bar{\Gamma}_2}K(x)=k_0>0 \end{gathered}$$

and by (3.50):

$$\begin{gathered} \left(f(u)-f(\bar{u}),\widetilde{U}_t\right)_1+\left(g(v)-g(\bar{v}),\widetilde{V}_t\right)_2\leq \\ \leq\max\{c_2,c_4\}C_4C_5(\|\nabla u\|_1^{p-1}+\|\nabla\bar{u}\|_1^{p-1}+\|\nabla v\|_2^{p-1}+\|\nabla\bar{v}\|_2^{p-1})\times \\ \times(\|\nabla\tilde{u}\|_1+\|\nabla\tilde{v}\|_2)(\|\widetilde{U}_t\|_1+\|\widetilde{V}_t\|_2), \end{gathered}$$

then from (3.54) we obtain

$$\begin{gathered} \|\widetilde{U}_t\|_1^2+\|\nabla\widetilde{U}\|_1^2+\|\widetilde{V}_t\|_2^2+\|\nabla\widetilde{V}\|_2^2+\|\widetilde{\delta}_t\|_{\Gamma_2}^2+\|\widetilde{\delta}\|_{\Gamma_2}^2+\int_0^t\|\widetilde{\delta}_t\|_{\Gamma_2}^2ds\leq \\ \leq\frac{2\max\{c_2,c_4\}C_4C_5}{C_6}\int_0^t\left(\|\nabla u\|_1^{p-1}+\|\nabla\bar{u}\|_1^{p-1}+\|\nabla v\|_2^{p-1}+\|\nabla\bar{v}\|_2^{p-1}\right)\times \\ \times(\|\nabla\tilde{u}\|_1+\|\nabla\tilde{v}\|_1)(\|\widetilde{U}_t\|_1+\|\widetilde{V}_t\|_2)ds. \end{gathered}$$

Thus we have

$$\|(\widetilde{U},\widetilde{V},\widetilde{\delta})\|_{Z_T}\leq K_5TR^{p-1}\|(\tilde{u},\tilde{v},\tilde{\delta})\|_{Z_T}, \tag{3.55}$$

where K_5 is a positive constant depending only on Ω_1, Ω_2 and p.

By choosing T small enough in order to have $K_5TR^{p-1}<1$, the estimate (3.55) shows that Φ is a contraction. By the contraction mapping theorem we obtain the existence of a unique solution (U, V, δ) satisfying $(U, V, \delta) = \Phi(U, V, \delta)$. Theorem 2.1 is proved.

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Received: November 12, 2020; Accepted: April 30, 2021