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CALDERÓN-ZYGMUND OPERATORS WITH KERNELS OF DINI'S TYPE AND THEIR MULTILINEAR COMMUTATORS ON GENERALIZED VARIABLE EXPONENT MORREY SPACES

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Abstract. Let T be a Calderón-Zygmund operator of type ω with $\omega(t)$ being nondecreasing and satisfying a kind of Dini's type condition and let $T_{\vec{b}}$ be the multilinear commutators of T with BMO^m functions. In this paper, we study the boundedness of the operators T and $T_{\vec{b}}$ on generalized variable exponent Morrey spaces $M^{p(\cdot),\varphi}$. We find the sufficient conditions on the pair (φ_1,φ_2) with $\vec{b}\in BMO^m(\mathbb{R}^n)$ which ensures the boundedness of the operators T and $T_{\vec{b}}$ from $M^{p(\cdot),\varphi_1}$ to $M^{p(\cdot),\varphi_2}$.

1. Introduction

The theory of Calderón-Zygmund operators has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [3, 4, 6, 29, 30, 39]). In particular, Yabuta introduced certain ω-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators (see [43]). Let ω be a non-negative and non-decreasing function on $\mathbb{R}_+=(0,\infty)$. We say that ω satisfies the *Dini* condition and write $\omega\in Dini$, if

$$\int_0^\infty \frac{\omega(t)}{t}dt<\infty. \tag{1.1}$$

A measurable function $K(\cdot,\cdot)$ on $\mathbb{R}^n\times\mathbb{R}^n$ is said to be a ω-type Calderón-Zygmund kernel if it satisfies

$$|K(x,y)|\le C\,|x-y|^{-n}$$

and for all distinct $x,y\in\mathbb{R}^n$, and all z with $2|x-z|<|x-y|$, there exist positive constants C and γ such that

$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\le C\omega\Big(\frac{|x-z|}{|x-y|}\Big)\,|x-y|^{-n}.$$

Definition 1.1. Let T be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class. One can say that T is a ω-type Calderón-Zygmund operator if it satisfies the following conditions:

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i) T can be extended to be a bounded linear operator on $L_2(\mathbb{R}^n)$;
ii) there is a ω-type Calderón-Zygmund kernel $K(\cdot,\cdot)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \text{ as } f \in C_c^\infty \text{ and } x \notin \operatorname{supp} f. \tag{1.2}$$

It is easy to see that the classical Calderón-Zygmund operator with standard kernel is a special case of ω-type operator T as $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \le 1$. Given a locally integrable function b, the commutator generated by T and b is defined by

$$[b,T]f(x) := b(x)Tf(x) - T(bf)(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) K(x,y) f(y) dy.$$

Let $\vec{b} = (b_1, ..., b_m)$ and b_j, $1 \le j \le m$ be locally integrable functions when we consider multilinear commutators as defined by

$$T_{\vec{b}} f(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x,y) f(y) dy.$$

Furthermore, if we take $b_i = b$, , $i = 1, \dots, m$, then we define the following integral equation

$$T_{\vec{b}} f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x,y) f(y) dy = [b,T]^m f(x).$$

Integral operators of Calderón-Zygmund kind appear in the representation formulas of the solutions of various PDEs. Thus the continuity of the Calderón-Zygmund integral in certain functional space permit to study the regularity of the solutions of boundary value problems for linear PDEs in the corresponding space, see, for example, [5, 14, 12, 35].

The classical Morrey spaces were introduced by Morrey [35] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. The first author, Mizuhara and Nakai [15, 33, 36] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [16, 38]).

Variable exponent function spaces (see [8]) received considerable attentions in recent decades. They are important not only in theory as generalizations of classical function spaces, but also for their wide applications in the fields of fluid dynamics, elasticity dynamics, the differential equations with nonstandard growth. The rich development can be found in many research works of the theory of variable exponent function spaces. We refer to [2, 8, 10, 23, 26, 42] for the details. For example, Lebesgue spaces with variable exponent were studied in [7, 9, 24, 27, 40], Morrey spaces with variable exponent were studied in [2, 13, 34] and generalized Morrey spaces with variable exponent were studied in [1, 11, 19, 20, 21].

The main purpose of this paper is to establish a number of results concerning variable exponent Morrey boundedness of Calderón-Zygmund operators with kernels of mild regularity. Let T be a linear Calderón-Zygmund operator of type $\omega(t)$ with ω being nondecreasing and $\omega \in Dini$, but without assuming to be concave. We show that the ω-type Calderón-Zygmund operators T and their multilinear commutators $T_{\vec{b}}$ are bounded from one generalized variable exponent

Morrey space $M^{p(\cdot),\varphi_1}$ to another $M^{p(\cdot),\varphi_2}$. We find the sufficient conditions on the pair (φ_1,φ_2) with $\vec{b}\in BMO^m(\mathbb{R}^n)$ which ensures the boundedness of the operators T and $T_{\vec{b}}$ from $M^{p(\cdot),\varphi_1}$ to $M^{p(\cdot),\varphi_2}$.

By $A\lesssim B$ we mean that $A\le CB$ with some positive constant C independent of appropriate quantities. If $A\lesssim B$ and $B\lesssim A$, we write $A\approx B$ and say that A and B are equivalent.

2. Some definitions and auxiliary results

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space of points $x=(x_1,...,x_n)$ with norm $|x|=(\sum_{i=1}^n x_i^2)^{1/2}$. For $x\in\mathbb{R}^n$ and $r>0$, denote $B(x,r)$ the open ball centered at x of radius r. Let ${}^{\complement}B(x,r)$ be the complement of the ball $B(x,r)$, and $|B(x,r)|$ be the Lebesgue measure of $B(x,r)$. For a measurable set E, we define the Lebesgue measure of E by $|E|$, and the characteristic function of E by χ_E.

Given an open set $E\subset\mathbb{R}^n$, and a measurable function $p(\cdot):E\to[1,\infty)$, $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot)=p(\cdot)/(p(\cdot)-1)$. For a measurable subset $E\subset\mathbb{R}^n$, we denote $p^-(E)=\operatorname{ess\,inf}\{p(x):x\in E\}$, $p^+(E)=\operatorname{ess\,sup}\{p(x):x\in E\}$. Especially, we denote $p^-=p^-(\mathbb{R}^n)$ and $p^+=p^+(\mathbb{R}^n)$. The set $\mathcal{P}(E)$ consists of all $p(\cdot):E\to[1,\infty)$ satisfying $p^-(E)>1$, $p^+(E)<\infty$. Similarly we denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(1,\infty)$ such that $1<p^-\le p(x)\le p^+<\infty$. Denote by $\mathcal{P}_0(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(0,\infty)$ such that $0<p^-\le p(x)\le p^+<\infty$, $x\in\mathbb{R}^n$. Let $\mathcal{P}_1(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot):\mathbb{R}^n\to[1,\infty)$ such that $1\le p^-\le p(x)\le p^+<\infty$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(E)$ as the set of real-valued measurable functions f on E such that, for some $\varepsilon>0$, $\int_E\left(\varepsilon|f(x)|\right)^{p(x)}dx<\infty$. This is a Banach function space with respect to the Luxemburg-Nakano norm,

$$\|f\|_{L^{p(\cdot)}(E)}=\inf\Big\{\lambda>0:\ \int_E\Big(\frac{f(x)|}{\lambda}\Big)^{p(x)}dx\le 1\Big\}.$$

The space $L^{p(\cdot)}_{\rm loc}(E)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\Omega):=\big\{f \text{ is measurable}:\ f\in L^{p(\cdot)}(E) \text{ for all compact subsets } E\subset\Omega\big\}.$$

Next we define some classes of variable exponent functions. The set $\mathcal{B}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

An important subset of $\mathcal{B}(\mathbb{R}^n)$ is the class of globally log-Hölder continuous functions $p(\cdot)\in LH(\mathbb{R}^n)$, with $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Recall that $p(\cdot)\in LH(\mathbb{R}^n)$, if $p(\cdot)$ satisfies

$$|p(x)-p(y)|\le\frac{C}{-\log(|x-y|)},\ |x-y|\le 1/2 \text{ and}$$
$$|p(x)-p(y)|\le\frac{C}{\log(e+|x|)},\ |y|\ge|x|.$$

We will also make use of the estimate provided by the following inequality (see [20, Lemma 2.2], [21, Theorem 3.2]).

$$\|\chi_{B(x,r)}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C r^{\theta_p(x,r)}, \qquad x\in\mathbb{R}^n,\ p\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n), \tag{2.1}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r\leq 1, \\ \frac{n}{p(\infty)}, & r\geq 1 \end{cases}$.

Lemma 2.1. *Let $q(\cdot), q_1(\cdot)\ldots, q_m(\cdot)\in\mathcal{P}_0(\mathbb{R}^n)$ so that $1/q(\cdot) = 1/q_1(\cdot)+\cdots+1/q_m(\cdot)$. Then, the inequality*

$$\|f_1\cdots f_m\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f_1\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\cdots\|f_m\|_{L^{q_m(\cdot)}(\mathbb{R}^n)}$$

holds for any $f_j\in L^{q_j(\cdot)}(\mathbb{R}^n)$, $j=1,\ldots,m$.

We define the generalized variable exponent Morrey spaces as follows.

Definition 2.1. Let $p\in\mathcal{P}_1(\mathbb{R}^n)$, φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$. We denote by $M^{p(\cdot),\varphi}\equiv M^{p(\cdot),\varphi}(\mathbb{R}^n)$ the generalized variable exponent Morrey space, the space of all functions $f\in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M^{p(\cdot),\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, r^{-\theta_p(x,r)}\, \|f\|_{L^{p(\cdot)}(B(x,r))},$$

where $\|f\|_{L^{p(\cdot)}(B(x,r))}\equiv\|f\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Remark 2.1. Generalized variable exponent Morrey space $M^{p(\cdot),\varphi}$ was introduced and studied by Guliyev, Hasanov and Samko in [19], see also [20].

(1) If $\varphi(x,r) = r^{\frac{\lambda-n}{p(x)}}$ with $0<\lambda<n$, then $M^{p(\cdot),\varphi} = L^{p(\cdot),\lambda}(\mathbb{R}^n)$ is the variable exponent Morrey space introduced by Almeida, Hasanov and Samko in [2].

(2) If $\varphi(x,r)\equiv r^{-\theta_p(x,r)}$, then $M^{p(\cdot),\varphi} = L^{p(\cdot)}(\mathbb{R}^n)$ is the variable exponent Lebesgue space.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds,\ \ H_w^* g(t) := \int_t^\infty \Big(1+\frac{s}{t}\Big)\, g(s)\, w(s)\, ds,\ \ 0<t<\infty,$$

where w is a weight. The following theorem was proved in [18].

Theorem 2.1. [18] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \leq C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.2. [17] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \leq C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^{\infty} \Big(1+\frac{s}{t}\Big) \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3. ω-type Calderón-Zygmund operators in the spaces $M^{p(\cdot),\varphi}$

The following theorem was proved in [32].

Theorem 3.1. [32] *Let $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and T be an ω-type Calderón-Zygmund operator defined by* (1.2) *with ω satisfying* (1.1)*. Then, the operator T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.*

The following local estimates are valid (see [16, 20]).

Theorem 3.2. *Let $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and T be an ω-type Calderón-Zygmund operator defined by* (1.2) *with ω satisfying* (1.1)*. Then the inequality*

$$\|Tf\|_{L^{p(\cdot)}(B)} \lesssim r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))}\, t^{-\theta_p(x_0,t)} \frac{dt}{t} \tag{3.1}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$.

Proof. Let $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r>0. \tag{3.2}$$

For all $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ we define

$$Tf(x) := T_0 f_1(x) + \int_{\mathbb{R}^n} K(x,y) f_2(y) dy,$$

here T_0 denotes a bounded linear operator on $L^{p(\cdot)}$ with $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ (see [32, 44]). It is easy to check that the definition of $Tf(x)$ does not depend on the choice of the ball B. First we show that $Tf(x)$ is well-defined *a.e.* x and independent of the choice B containing x. As T_0 is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ provided by Theorem 3.1 and $f_1 \in L^{p(\cdot)}(\mathbb{R}^n)$, $T_0 f_1$ is well-defined. Next, we show that the second-term of the right-hand side defining $Tf(x)$ converges absolutely for any $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$.

We have

$$\|Tf\|_{L^{p(\cdot)}(B)} \le \|Tf_1\|_{L^{p(\cdot)}(B)} + \|Tf_2\|_{L^{p(\cdot)}(B)}.$$

Since $f_1 \in L^{p(\cdot)}(\mathbb{R}^n)$, $Tf_1 \in L^{p(\cdot)}(\mathbb{R}^n)$ and from the boundedness of T in $L^{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 3.1) it follows that

$$\|Tf_1\|_{L^{p(\cdot)}(B)} \le \|Tf_1\|_{L^{p(\cdot)}} \le C\|f_1\|_{L^{p(\cdot)}} = C\|f\|_{L^{p(\cdot)}(2B)},$$

where constant $C>0$ is independent of f.

On the other hand,

$$\begin{aligned}\|f\|_{L^{p(\cdot)}(2B)} &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq r^{\theta_p(x_0,r)} \|1\|_{L^{p'}(B)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \|1\|_{L^{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} t^{-\theta_p(x_0,t)} \frac{dt}{t}.\end{aligned} \tag{3.3}$$

It is clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We have

$$|Tf_2(x)| \leq 2^n c_0 \int_{^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality and from (2.1), we get

$$\int_{^{\complement}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} t^{-\theta_p(x_0,t)} \frac{dt}{t}. \tag{3.4}$$

Moreover, for all $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ the inequality

$$\|Tf_2\|_{L^{p(\cdot)}(B)} \lesssim r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} t^{-\theta_p(x_0,t)} \frac{dt}{t}$$

is valid. Thus from (3.3) we have

$$\begin{aligned}\|Tf\|_{L^{p(\cdot)}(B)} &\lesssim \|f\|_{L^{p(\cdot)}(2B)} + r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} t^{-\theta_p(x_0,t)} \frac{dt}{t} \\ &\lesssim r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \|f\|_{L^{p(\cdot)}(B(x_0,t))} t^{-\theta_p(x_0,t)} \frac{dt}{t}.\end{aligned}$$

Then we get the inequality (3.1). □

Theorem 3.3. *Let $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, T be an ω-type Calderón-Zygmund operator defined by* (1.2) *with ω satisfying* (1.1)*, and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\theta_p(x,s)}}{t^{\theta_p(x,t)}} \frac{dt}{t} \leq C \varphi_2(x,r),$$

where C does not depend on x and r. Then the operator T is bounded from $M^{p(\cdot),\varphi_1}$ to $M^{p(\cdot),\varphi_2}$.

Proof. For $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ from Theorem 2.1 and Theorem 3.2 we get

$$\begin{aligned}\|Tf\|_{M^{p(\cdot),\varphi_2}} &\lesssim \sup_{x \in \mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L^{p(\cdot)}(B(x,t))} t^{-\theta_p(x,t)} \frac{dt}{t} \\ &\lesssim \sup_{x \in \mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} r^{-\theta_p(x,r)} \|f\|_{L^{p(\cdot)}(B)} = \|f\|_{M^{p(\cdot),\varphi_1}}.\end{aligned}$$

□

In the case $p(x) \equiv const$ from Theorem 3.3 we get the following corollary, which was proved in [25].

Corollary 3.1. [25] *Let T be an ω-type Calderón-Zygmund operator defined by* (1.2) *with ω satisfying* (1.1)*. Let also $1 < p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \le C\varphi_2(x,r),$$

where C does not depend on x and r. Then the operator T is bounded from M^{p,φ_1} to M^{p,φ_2}.

Remark 3.1. Let $0 \le \kappa < 1$. Assume that ψ is a positive increasing function defined in $(0, \infty)$ and satisfies the following $\mathcal{D}_\kappa$ condition :

$$\frac{\psi(t_2)}{t_2^\kappa} \le C \frac{\psi(t_1)}{t_1^\kappa}, \text{ for any } 0 < t_1 < t_2 < \infty,$$

where $C > 0$ is a constant independent of t_1 and t_2. If $p(x) \equiv const$, $\varphi_1(x,r) = \varphi_2(x,r) = \psi(w(x,r))$ and ψ satisfy the $\mathcal{D}_\kappa$ condition, Theorems 3.2 and 3.3 were proved in [41]. Also, in the case $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \le 1$, Theorems 3.2 and 3.3 were proved in [19].

4. Commutators of ω-type Calderón-Zygmund operators in the spaces $M^{p(\cdot),\varphi}$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 4.1. Suppose that $b \in L_1^{\rm loc}(\mathbb{R}^n)$, and let

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)dy$. Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

By the generalized Hölder's inequality in Orlicz spaces (see [37, page 58]) and John-Nirenberg's inequality, we get (see also [28, (2.14)])

$$\frac{1}{|B|} \int_B \left|b_1(x) - \left(b_1\right)_B\right| \dots \left|b_m(x) - \left(b_m\right)_B\right| |g(x)| dx \lesssim \prod_{j=1}^m \|b_j\|_{BMO} \, \|g\|_{L(\log L)^m, B}. \tag{4.1}$$

Definition 4.2. The $BMO_{p(\cdot)}(\mathbb{R}^n)$ space is the set of all locally integrable functions f with finite norm

$$\|b\|_{BMO_{p(\cdot)}} = \sup_B \frac{\|(b(\cdot) - b_B)\chi_B\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}}.$$

Theorem 4.1. [23] *Let $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Then, the norms $\|\cdot\|_{BMO_{p(\cdot)}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Since linear commutator has a greater degree of singularity than the corresponding ω-type Calderón-Zygmund operator, we need a slightly stronger version of condition

$$\int_0^1 \frac{\omega(t)}{t}\Big(1+\log\frac{1}{t}\Big)^m dt < \infty. \tag{4.2}$$

The following weighted endpoint estimate for commutator $T_{\vec{b}}$ of the ω-type Calderón-Zygmund operator was established in [44] under a stronger version of condition (4.2) imposed on ω, if $\vec{b} \in BMO^m(\mathbb{R}^n)$ (for the unweighted case, see [31]).

The following theorem was proved in [40, 44].

Theorem 4.2. [40, 44] *Let T be a linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. If ω satisfies condition (4.2) and $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$\|T_{\vec{b}} f\|_{L^{p(\cdot)}} \le C\, \|\vec{b}\|_*\, \|f\|_{L^{p(\cdot)}},$$

where $\|\vec{b}\|_* = \prod_{j=1}^m \|b_j\|_*$.

The following local estimates are valid (see [17]).

Theorem 4.3. *Let T be a linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfying condition (4.2) and $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. Then*

$$\|T_{\vec{b}} f\|_{L^{p(\cdot)}(B)} \le C\, \|\vec{b}\|_*\, r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \ln^m\Big(e+\frac{t}{r}\Big)\, \|f\|_{L^{p(\cdot)}(B(x_0,t))}\, t^{-\theta_p(x_0,t)}\, \frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$, where C does not depend on f, $x_0 \in \mathbb{R}^n$ and $r > 0$.

Proof. Let $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$, set $B = B(x_0,r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. For all $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ we define

$$T_{\vec{b}} f(x) := T_{\vec{b},0} f_1(x) + \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x)-b_j(y)) K(x,y) f_2(y)dy,$$

here $T_{\vec{b},0}$ denotes the commutator as a bounded linear operator on $L^{p(\cdot)}$ with $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ (see [44]). It is easy to check that the definition of $T_{\vec{b}} f(x)$ does not depend on the choice of the ball B. First we show that $T_{\vec{b}} f(x)$ is well-defined *a.e.* x and independent of the choice B containing x. As $T_{\vec{b},0}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ provided by Theorem 4.2 and $f_1 \in L^{p(\cdot)}(\mathbb{R}^n)$, $T_{\vec{b},0} f_1$ is well-defined. Next, we show that the second-term of the right-hand side defining $Tf(x)$ converges absolutely for any $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$.

Hence

$$\|T_{\vec{b}} f\|_{L^{p(\cdot)}(B)} \le \|T_{\vec{b}} f_1\|_{L^{p(\cdot)}(B)} + \|T_{\vec{b}} f_2\|_{L^{p(\cdot)}(B)}.$$

From the boundedness of $T_{\vec{b}}$ in $L^{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 4.2) it follows that:

$$\|T_{\vec{b}} f_1\|_{L^{p(\cdot)}(B)} \le \|T_{\vec{b}} f_1\|_{L^{p(\cdot)}} \lesssim \|\vec{b}\|_*\, \|f_1\|_{L^{p(\cdot)}} = \|\vec{b}\|_*\, \|f\|_{L^{p(\cdot)}(2B)}.$$

For the term $\|T_{\vec{b}}f_2\|_{L^{p(\cdot)}(B)}$, without loss of generality, we can assume $m = 2$. Thus, the operator $T_{\vec{b}}f_2$ can be divided into four parts

$$\begin{aligned}
T_{\vec{b}}f_2(x) &= \big(b_1(x) - (b_1)_B\big)\big(b_2(x) - (b_2)_B\big)\int_{\mathbb{R}^n} K(x,y)f_2(y)dy \\
&+ \int_{\mathbb{R}^n} K(x,y)\big(b_1(y) - (b_1)_B\big)\big(b_2(y) - (b_2)_B\big)f_2(y)dy \\
&- \big(b_1(x) - (b_1)_B\big)\int_{\mathbb{R}^n} K(x,y)\big(b_2(y) - (b_2)_B\big)f_2(y)dy \\
&- \big(b_2(x) - (b_2)_B\big)\int_{\mathbb{R}^n} K(x,y)\big(b_1(y) - (b_1)_B\big)f_2(y)dy \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}
|T_{\vec{b}}f_2(x)| &\leq |I_1(x) + |I_2(x)| + |I_3(x)| + |I_4(x)| \\
&\lesssim \big|b_1(x) - (b_1)_B\big|\big|b_2(x) - (b_2)_B\big| \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n}dy \\
&+ \int_{\complement(2B)} \big|b_1(y) - (b_1)_B\big|\big|b_2(y) - (b_2)_B\big| \frac{|f(y)|}{|x_0 - y|^n}dy \\
&+ \big|b_1(x) - (b_1)_B\big| \int_{\complement(2B)} \big|b_2(y) - (b_2)_B\big| \frac{|f(y)|}{|x_0 - y|^n}dy \\
&+ \big|b_2(x) - (b_2)_B\big| \int_{\complement(2B)} \big|b_1(y) - (b_1)_B\big| \frac{|f(y)|}{|x_0 - y|^n}dy.
\end{aligned}$$

Then

$$\begin{aligned}
\|T_{\vec{b}}f_2\|_{L^{p(\cdot)}(B)} &\lesssim \Big\| \int_{\complement(2B)} \frac{\prod_{j=1}^{2} \big|b_i(y) - (b_i)_B\big|}{|x_0 - y|^n} |f(y)|dy \Big\|_{L^{p(\cdot)}(B)} \\
&+ \Big\| \big|b_1(x) - (b_1)_B\big| \int_{\complement(2B)} \frac{\big|b_2(y) - (b_2)_B\big|}{|x_0 - y|^n} |f(y)|dy \Big\|_{L^{p(\cdot)}(B)} \\
&+ \Big\| \big|b_2(x) - (b_2)_B\big| \int_{\complement(2B)} \frac{\big|b_1(y) - (b_1)_B\big|}{|x_0 - y|^n} |f(y)|dy \Big\|_{L^{p(\cdot)}(B)} \\
&+ \Big\| \int_{\complement(2B)} \frac{\prod_{j=1}^{2} \big|b_i(x) - (b_i)_B\big|}{|x_0 - y|^n} |f(y)|dy \Big\|_{L^{p(\cdot)}(B)} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}
I_1 &= r^{\theta_p(x_0,r)} \int_{\complement_{(2B)}} \frac{\prod\limits_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right|}{|x_0 - y|^n} |f(y)| dy \\
&\approx r^{\theta_p(x_0,r)} \int_{\complement_{(2B)}} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \int_{2r \le |x_0-y| \le t} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right| |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \int_{B(x_0,t)} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_B\right| |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality (4.1), by Lemma 2.1 and Theorem 4.1, we get

$$\begin{aligned}
I_1 &\lesssim r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \Big\| \prod_{j=1}^{2} \left|b_i(\cdot) - \left(b_i\right)_B\right| \Big\|_{L^{p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim r^{\theta_p(x_0,r)} \Big(\int_{2r}^{\infty} \Big\| \prod_{j=1}^{2} \left|b_i(\cdot) - \left(b_i\right)_{B(x_0,t)}\right| \Big\|_{L^{p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&+ \int_{2r}^{\infty} \left|\left(b_1\right)_{B(x_0,t)} - \left(b_1\right)_B\right| \left\|b_2(\cdot) - \left(b_2\right)_{B(x_0,t)}\right\|_{L^{p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&+ \int_{2r}^{\infty} \left|\left(b_2\right)_{B(x_0,t)} - \left(b_2\right)_B\right| \left\|b_1(\cdot) - \left(b_1\right)_{B(x_0,t)}\right\|_{L^{p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&+ \int_{2r}^{\infty} \prod_{j=1}^{2} \left|\left(b_i\right)_{B(x_0,t)} - \left(b_i\right)_B\right| \|1\|_{L^{p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \Big) \\
&\lesssim r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \prod_{j=1}^{2} \left\|b_i(\cdot) - \left(b_i\right)_{B(x_0,t)}\right\|_{L^{2p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&+ \|\vec{b}\|_* \, r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \ln \frac{t}{r} \, \|1\|_{L^{2p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&+ r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \prod_{j=1}^{2} \left|\left(b_i\right)_{B(x_0,t)} - \left(b_i\right)_B\right| \|1\|_{L^{p'(\cdot)}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|\vec{b}\|_* \, r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \ln^2 \left(e + \frac{t}{r}\right) \|1\|_{L^{p'}(B(x_0,t))} \|f\|_{L^{p(\cdot)}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|\vec{b}\|_* \, r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \ln^2 \left(e + \frac{t}{r}\right) \|f\|_{L^{p(\cdot)}(B(x_0,t))} \, t^{-\theta_p(x_0,t)} \frac{dt}{t}.
\end{aligned}$$

Let us estimate I_2.

$$
\begin{aligned}
I_2 &= \left\|b_1(\cdot)-\left(b_1\right)_B\right\|_{L^{p(\cdot)}(B(x_0,t))}\int_{\complement_{(2B)}}\frac{\left|b_2(y)-\left(b_2\right)_B\right|}{|x_0-y|^n}|f(y)|dy\\
&\lesssim \|b_1\|_*\, r^{\theta_p(x_0,r)}\int_{\complement_{(2B)}}\left|b_2(y)-\left(b_2\right)_B\right||f(y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n+1}}dy\\
&\approx \|b_1\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\int_{2r\le|x_0-y|\le t}\left|b_2(y)-\left(b_2\right)_B\right||f(y)|dy\frac{dt}{t^{n+1}}\\
&\lesssim \|b_1\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\int_{B(x_0,t)}\left|b_2(y)-\left(b_2\right)_B\right||f(y)|dy\frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality (4.1) and by Theorem 4.1, we get

$$
\begin{aligned}
I_2 &\lesssim \|b_1\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\left\|b_2(\cdot)-\left(b_2\right)_B\right\|_{L^{p'(\cdot)}(B(x_0,t))}\|f\|_{L^{p(\cdot)}(B(x_0,t))}\frac{dt}{t^{n+1}}\\
&\lesssim \prod_{j=1}^{2}\|b_j\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|1\|_{L^{p'}(B(x_0,t))}\|f\|_{L^{p(\cdot)}(B(x_0,t))}\frac{dt}{t^{n+1}}\\
&\lesssim \|\vec{b}\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\ln^2\left(e+\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\,t^{-\theta_p(x_0,t)}\,\frac{dt}{t}.
\end{aligned}
$$

In the same way, we shall get the result of I_3

$$
I_3 \lesssim \|\vec{b}\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\ln^2\left(e+\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\,t^{-\theta_p(x_0,t)}\,\frac{dt}{t}.
$$

In order to estimate I_4 we note that

$$
\begin{aligned}
I_4 &= \left\|\prod_{j=1}^{2}|b_i(x)-\left(b_i\right)_B\right\|_{L^{p(\cdot)}(B(x_0,t))}\int_{\complement_{(2B)}}\frac{|f(y)|}{|x_0-y|^n}dy\\
&\le \prod_{j=1}^{2}\left\|b_i(x)-\left(b_i\right)_B\right\|_{L^{2p(\cdot)}(B(x_0,t))}\int_{\complement_{(2B)}}\frac{|f(y)|}{|x_0-y|^n}dy.
\end{aligned}
$$

By Theorem 4.1, we get

$$
I_4 \lesssim \|\vec{b}\|_*\, r^{\theta_p(x_0,r)}\int_{\complement_{(2B)}}\frac{|f(y)|}{|x_0-y|^n}dy.
$$

Thus, by (3.4)

$$
I_4 \lesssim \|\vec{b}\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\|f\|_{L^{p(\cdot)}(B(x_0,t))}\,t^{-\theta_p(x_0,t)}\,\frac{dt}{t}.
$$

Summing up I_1 and I_4, for all $p\in LH(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ we get

$$
\|T_{\vec{b}}f_2\|_{L^{p(\cdot)}(B)} \lesssim \|\vec{b}\|_*\, r^{\theta_p(x_0,r)}\int_{2r}^{\infty}\ln^2\left(e+\frac{t}{r}\right)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\,t^{-\theta_p(x_0,t)}\,\frac{dt}{t}.
$$

Finally,

$$\|T_{\vec{b}}f\|_{L^{p(\cdot)}(B)} \lesssim \|\vec{b}\|_* \, \|f\|_{L^{p(\cdot)}(2B)} + \|\vec{b}\|_* \, r^{\theta_p(x_0,r)} \int_{2r}^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \|f\|_{L^{p(\cdot)}(B(x_0,t))} \, t^{-\theta_p(x_0,t)} \, \frac{dt}{t},$$

and the statement of Theorem 4.3 follows by (3.3). □

Theorem 4.4. *Let T be a linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfying condition (4.2), $p \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\theta_p(x,s)}}{t^{\theta_p(x,t)}} \frac{dt}{t} \le C\varphi_2(x,r),$$

where C does not depend on x and r. Then the operator $T_{\vec{b}}$ is bounded from $M^{p(\cdot),\varphi_1}$ to $M^{p(\cdot),\varphi_2}$. Moreover,

$$\|T_{\vec{b}}f\|_{M^{p(\cdot),\varphi_2}} \lesssim \|\vec{b}\|_* \, \|f\|_{M^{p(\cdot),\varphi_1}}.$$

Proof. Using Theorem 2.2 and Theorem 4.3 we have

$$\begin{aligned}\|T_{\vec{b}}f\|_{M^{p(\cdot),\varphi_2}} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} r^{-\theta_p(x,r)} \|T_{\vec{b}}f\|_{L^{p(\cdot)}B(x,r)} \\ &\lesssim \|\vec{b}\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \|f\|_{L^{p(\cdot)}(B(x,t))} \, t^{-\theta_p(x,t)} \, \frac{dt}{t} \\ &\lesssim \|\vec{b}\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} r^{-\theta_p(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))} = \|\vec{b}\|_* \, \|f\|_{M^{p(\cdot),\varphi_1}}.\end{aligned}$$

□

In the case $p(x) \equiv const$ from Theorem 4.4 we get the following corollary, which was proved in [25].

Corollary 4.1. [25] *Let T be a linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfy condition (4.2), $1 < p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \le C\varphi_2(x,r),$$

where C does not depend on x and r. Then the operator $T_{\vec{b}}$ is bounded from M^{p,φ_1} to M^{p,φ_2}.

Remark 4.1. Note that, if $p(x) \equiv const$, $\varphi_1(x,r) = \varphi_2(x,r) = \psi(w(x,r))$ and ψ satisfy the $\mathcal{D}_\kappa$ condition, Corollary 4.1 were proved in [41]. Also, in the case $\omega(t) = t^{\varepsilon}$ with $0 < \varepsilon \le 1$, Theorems 4.3 and 4.4 were proved in [11], and in the case $m = 1$ in [22].

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