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AN INVERSE PROBLEM FOR THE QUADRATIC PENCIL OF DIFFERENTIAL OPERATORS WITH ALMOST PERIODIC COEFFICIENTS

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In memory of dear M. G. Gasymov

Abstract. In this paper, the inverse spectral problem for the operator L_λ generated by the differential expression $\ell_\lambda(y) = y'' + p(x)y' + \left[\lambda^2 + i\lambda p(x) + q(x)\right] y$ is investigated in the space $L_2(\mathbb{R})$. Here the coefficients $p(x)$, $q(x)$ are almost periodic functions whose Fourier series are absolutely convergent and the sequence of Fourier exponents (which are positive) has a unique limit point at $+\infty$. The set of spectral data $\left(\left\{s_n^{(1)}\right\}, \left\{s_n^{(2)}\right\}\right)$ of the operator L_λ is defined and the problem of finding the coefficients $p(x)$, $q(x)$ from these sequences is considered.

1. Introduction

In this study, the inverse spectral problem is investigated for the maximal differential operator L_λ generated by the linear differential expression

$$\ell_\lambda(y) = y'' + p(x)y' + (\lambda^2 + i\lambda p(x) + q(x))y$$

in the space $L_2(-\infty, +\infty)$. Here λ is a complex parameter,

$$p(x) = \sum_{n=1}^{\infty} p_n e^{i\alpha_n x},\ q(x) = \sum_{n=1}^{\infty} q_n e^{i\alpha_n x} \tag{1.1}$$

with p_n, $q_n \in \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} \alpha_n |p_n| < +\infty,\ \sum_{n=1}^{\infty} |q_n| < +\infty, \tag{1.2}$$

$\{\alpha_n\}_{n\geq 1}$ is an increasing sequence of positive numbers with $\alpha_n \to +\infty$ and the set $G = \{\alpha_n : n \in \mathbb{N}\}$ is an additive semigroup.

Let AP^+ be the class of almost periodic functions $\varphi(x) = \sum\limits_{n=1}^{\infty} \varphi_n e^{i\alpha_n x}$ with $\|\varphi(x)\| = \sum\limits_{n=1}^{\infty} |\varphi_n| < +\infty$. AP^+ is a complex Banach space. For every $\varphi(x)$,

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$\psi(x) \in AP^+$ the relations $\varphi(x)\psi(x) \in AP^+$ and $\|\varphi(x)\psi(x)\| \leq \|\varphi(x)\| \cdot \|\psi(x)\|$ hold. In the case $\alpha_n = n$, $\forall n \in \mathbb{N}$, we denote this class by Q^+. Obviously, conditions (1.1), (1.2) mean that $p(x)$, $q(x)$, $p'(x) \in AP^+$ or $\{\alpha_n p_n\}_{n\geq 1} \in \ell_1$, $\{q_n\}_{n\geq 1} \in \ell_1$.

In [2], the Floquet solutions of equation $\ell_\lambda(y) = 0$ in the case $p(x) \equiv 0$, $q(x) \in Q^+$ have been constructed and using these solutions direct and inverse spectral problems have been investigated for the operator $L = -\frac{d^2}{dx^2} + q(x)$ in the space $L_2(\mathbb{R})$. The inverse problem of finding the potential $q(x)$ for the given spectral data sequence $\{s_n\}_{n\geq 1}$ of the operator L has been first considered in that work. Sufficient condition for the existence of solution of the inverse problem also obtained in [2]. Later, using some different methods, the inverse problem for the operator $L = -\frac{d^2}{dx^2} + q(x)$ with periodic potential $q(x) \in L_2(0, 2\pi)$ was investigated in [5]. The necessary and sufficient conditions for the sequence $\{s_n\}_{n\geq 1}$ to be the *set of spectral data* of the operator L have been found in that work. Then, the results of [5] have been generalized in [6] for the case of potential $q(x)$ which is an almost periodic function that belongs to Besicovitch class, and has only positive Fourier exponents. The pencil of the second order differential operators $L(\lambda) = -\frac{d^2}{dx^2} + 2\lambda p(x) + q(x) - \lambda^2$ with periodic coefficients $p(x)$, $q(x) \in L_2(0, 2\pi)$ has been investigated in [1], with the results of [5] generalized for operator $L(\lambda)$.

In this article, we prove the validity of another representation for the Floquet solutions of the equation $\ell_\lambda(y) = 0$ constructed in [3] and, using these solutions, we study the inverse problem for the operator L_λ. The uniqueness of solution of the inverse problem is proved and the sufficient condition for the existence of a solution to the inverse problem is obtained in case $p(x) \equiv 0$. In contrast to studies [1, 2, 5, 6], an algebraic method has been used for this purpose.

2. Floquet solutions of the equation $\ell_\lambda(y) = 0$

The system of linearly independent solutions of an equation of type $\ell_\lambda(y) = 0$ with almost periodic coefficient was investigated in [3]. According to Theorem 1 in [3], we can state the following theorem related to the equation

$$y'' + p(x)y' + [\lambda^2 + i\lambda p(x) + q(x)]y = 0, \quad -\infty < x < +\infty. \tag{2.1}$$

Theorem 2.1. *If $p(x)$, $q(x)$, $p'(x) \in AP^+$, then for $\forall\lambda \neq \pm\frac{\alpha_n}{2}$, $\forall n \in \mathbb{N}$ the differential equation (2.1) has the solutions*

$$f_1(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} U_n^{(1)}(\lambda)e^{i\alpha_n x}\right), \ f_2(x,\lambda) =$$

$$= e^{-i\lambda x}\left(1 + \sum_{n=1}^{\infty} U_n^{(2)}(\lambda)e^{i\alpha_n x}\right), \tag{2.2}$$

where the series

$$\sum_{n=1}^{\infty} \left|U_n^{(s)}(\lambda)\right| \alpha_n^2, \ s = 1, 2,$$

is uniformly convergent in each compact set $S \subseteq \mathbb{C}$ which doesn't contain the numbers $\lambda = -\frac{n}{2}$, $n \in N$ in case $s = 1$ and $\lambda = \frac{\alpha_n}{2}$, $n \in \mathbb{N}$ in case $s = 2$. Here $U_n^{(1)}(\lambda) = U_{0n}^{(1)} + \sum_{k=1}^{n} \frac{U_{kn}^{(1)}}{\alpha_k+2\lambda}$, $U_n^{(2)}(\lambda) = U_{0n}^{(2)} + \sum_{k=1}^{n} \frac{U_{kn}^{(2)}}{\alpha_k-2\lambda}$, $n \in \mathbb{N}$.

The solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ can be used for the investigation of the structure of the spectrum and the kernel of the resolvent operator, but they are not sufficient for studying the inverse problem for the operator L_λ. For this reason, it is convenient to use the Floquet solutions of the form

$$\begin{cases} f_1(x,\lambda) = e^{i\lambda x}\left(1 + \sum\limits_{n=1}^{\infty} U_{0n}^{(1)} e^{i\alpha_n x} + \sum\limits_{k=1}^{\infty} \dfrac{1}{\alpha_k + 2\lambda} \sum\limits_{n=k}^{\infty} U_{kn}^{(1)} e^{i\alpha_n x}\right), \\ f_2(x,\lambda) = e^{-i\lambda x}\left(1 + \sum\limits_{n=1}^{\infty} U_{0n}^{(2)} e^{i\alpha_n x} + \sum\limits_{k=1}^{\infty} \dfrac{1}{\alpha_k - 2\lambda} \sum\limits_{n=k}^{\infty} U_{kn}^{(2)} e^{i\alpha_n x}\right) \end{cases} \tag{2.3}$$

with conditions $\sum\limits_{n=1}^{\infty} \alpha_n^2 \left|U_{0n}^{(s)}\right| < +\infty$, $\sum\limits_{k=1}^{\infty} \dfrac{1}{\alpha_k} \sum\limits_{n=k}^{\infty} \alpha_n^2 \left|U_{kn}^{(s)}\right| < +\infty$, $s = 1, 2$. It is clear that these representations of the solutions are a modified form of formulas (2.2).

The special solutions of type (2.3) are used in [1, 2, 4, 5, 6], under various conditions on the coefficients of the considered equations. We use the following theorem about existence of the Floquet solutions of the equation (2.1).

Theorem 2.2. *If $p(x)$, $q(x)$, $p'(x) \in AP^+$, then for each $\lambda \neq -\frac{\alpha_n}{2}$, $\forall n \in \mathbb{N}$ the differential equation (2.1) has a solution*

$$f(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} u_n e^{i\alpha_n x} + \sum_{k=1}^{\infty} \frac{1}{\alpha_k + 2\lambda} \sum_{n=k}^{\infty} u_{kn} e^{i\alpha_n x}\right), \tag{2.4}$$

where the sequences $\{u_n\}, \{u_{kn}\}$ of complex numbers are uniquely defined by the system of equations

$$\alpha_n^2 u_n = -\alpha_n \sum_{k=1}^{n} u_{kn} + q_n + \sum_{\alpha_s+\alpha_m=\alpha_n} (i\alpha_m p_s + q_s) u_m + \sum_{\alpha_s+\alpha_m=\alpha_n} ip_s \sum_{k=1}^{m} u_{km}, \tag{2.5}$$

$$-\alpha_n u_n + ip_n + \sum_{\alpha_s+\alpha_m=\alpha_n} ip_s u_m = 0,\ n \in \mathbb{N}, \tag{2.6}$$

$$-\alpha_n(\alpha_n - \alpha_k) u_{kn} + \sum_{\substack{\alpha_s+\alpha_m=\alpha_n \\ m \geq k}} [i(\alpha_m - \alpha_k) p_s + q_s] u_{km} = 0,\ k, n \in \mathbb{N},\ n > k \tag{2.7}$$

and the series

$$\sum_{n=1}^{\infty} \alpha_n^2 |u_n| < +\infty,\ \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \sum_{n=k}^{\infty} \alpha_n^2 |u_{kn}| < +\infty \tag{2.8}$$

converge.

Remark 2.1. In what follows, we assume that sums of the form $\sum\limits_{\alpha_s+\alpha_m=\alpha_n} a_s b_m$, $\sum\limits_{m=1}^{n-1} a_m$ are equal to zero for $n = 1$. Also we set $u_{kn} = 0$ if $k > n$. If $\{u_{kn}\}_{n \geq k}$ is the solution of equation (2.7) and $\alpha_n - \alpha_k \notin G$ (in the special case $\alpha_n - \alpha_k < \alpha_1$), then $u_{kn} = 0$. From here it follows that $\alpha_1 |u_{kn}| \leq (\alpha_n - \alpha_k) |u_{kn}|$, $1 \leq k < n$, which we will use frequently. The sequences $\{u_n\}_{n\geq 1}$, $\{u_{kn}\}_{n\geq k}$, $k\in\mathbb{N}$ will be treated as the solution of the system of equations (2.5)-(2.7). We will denote this solution as $\{u_n\},\{u_{kn}\}$.

Remark 2.2. The above theorem was proved in [4] for periodic coefficients $p(x)$, $q(x)\in Q^+$. Here we will use the same proof method.

Proof. If we assume the existence of the solution of the equation (2.1) of the form (2.4), then according to (2.8), we find the derivatives of $f(x,\lambda)$ with respect to x and substitute these expressions in the equation (2.1) as in [4]. So, we obtain the system of equations (2.5)-(2.7) to determine the sequences $\{u_n\}$, $\{u_{kn}\}$.

On the contrary, if $\{u_n\}$ and $\{u_{kn}\}$ satisfy the system of equations (2.5)-(2.7) and the series (2.8) converges, then the function $f(x,\lambda)$ defined by the formula (2.4) is a solution of equation (2.1). Therefore to prove the theorem it is sufficient to show the solvability of the system (2.5)-(2.7) and convergence of the series (2.8). It is easy to see that the system of equations (2.5)-(2.7) has a unique solution. Indeed, the sequence $\{u_n\}$ is determined from the equation (2.6) by the recurrent manner uniquely. Furthermore, by the known sequence $\{u_n\}$, the sequence $\{u_{kn}\}$ is determined from the equations (2.5), (2.7) by the recurrent manner uniquely.

Now let us show that for the solution $\{u_n\},\{u_{kn}\}$ of the system (2.5)- (2.7) the series $\sum\limits_{n=1}^{\infty}\alpha_n^2|u_n|$ and $\sum\limits_{k=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2|u_{kn}|$ converge, therefore the function $f(x,\lambda)$ is a solution of equation (2.1) for $\forall\lambda\in\mathbb{C}$, $\lambda\neq-\frac{\alpha_n}{2}$, $\forall n\in\mathbb{N}$. For this reason, from equation (2.6) we have $\alpha_n^2|u_n|\leq\alpha_n|p_n|+\alpha_n\sum\limits_{\alpha_s+\alpha_m=\alpha_n}|p_s||u_m|$ and by summing with respect to n for $j>1$ we obtain

$$\sum_{n=1}^{j}\alpha_n^2|u_n|\leq\sum_{n=1}^{j}\alpha_n|p_n|+\sum_{n=2}^{j}\left(\sum_{\alpha_s+\alpha_m=\alpha_n}\alpha_n|p_s||u_m|\right)=$$

$$\sum_{n=1}^{j}\alpha_n|p_n|+\sum_{n=2}^{j}\left(\sum_{\alpha_s+\alpha_m=\alpha_n}\alpha_s|p_s|\alpha_m|u_m|\frac{\alpha_s+\alpha_m}{\alpha_s\alpha_m}\right)\leq$$

$$\sum_{n=1}^{j}\alpha_n|p_n|+\frac{2}{\alpha_1}\sum_{s=1}^{j-1}\alpha_s|p_s|\sum_{m=1}^{j-1}\alpha_m|u_m|\leq\left\|p'(x)\right\|+\frac{2}{\alpha_1}\left\|p'(x)\right\|\sum_{n=1}^{j-1}\alpha_n|u_n|$$

or

$$\sum_{n=1}^{j}\alpha_n^2|u_n|\leq A_0+A_1\sum_{n=1}^{j-1}\alpha_n|u_n|,\forall j\in\mathbb{N},\ A_0,A_1>0,$$

where

$$A_0=\left\|p'(x)\right\|,\ A_1=2\left\|p'(x)\right\|/\alpha_1.$$

Then, according to the Lemma* (see [4], page 363), the series $\sum\limits_{n=1}^{\infty}\alpha_n^2|u_n|$ converges.

Now let us show that if the series $\sum\limits_{n=1}^{\infty}\alpha_n^2|u_n|$ converges, then for the sequence $\{u_{kn}\}$ obtained from the system (2.5)-(2.7) the series $\sum\limits_{k=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2|u_{kn}|$ also converges.

Since the series $\sum_{n=1}^{\infty} \alpha_n^2 |u_n|$ converges, by setting $U_n = \sum_{n=1}^{n} u_{kn}$ from the equation (2.5) we have

$$\alpha_n U_n = -\alpha_n^2 u_n + q_n + \sum_{\alpha_s+\alpha_m=\alpha_n} (i\alpha_m p_s + q_s) u_m + \sum_{\alpha_s+\alpha_m=\alpha_n} i p_s U_m$$

which implies

$$\alpha_n |U_n| \leq \alpha_n^2 |u_n| + |q_n| + \sum_{\alpha_s+\alpha_m=\alpha_n} (\alpha_m |p_s| + |q_s|) |u_m| + \sum_{\alpha_s+\alpha_m=\alpha_n} |p_s| |U_m| \leq$$

$$\alpha_n^2 |u_n| + |q_n| + \sum_{\alpha_s+\alpha_m=\alpha_n} \left(|p_s| + \frac{|q_s|}{\alpha_1} \right) \alpha_m |u_m| + \sum_{\alpha_s+\alpha_m=\alpha_n} |p_s| |U_m| .$$

By summing with respect to n, we obtain for any $j > 1$

$$\sum_{n=1}^{j} \alpha_n |U_n| \leq \sum_{n=1}^{j} (\alpha_n^2 |u_n| + |q_n|) + \sum_{n=2}^{j} \left(\sum_{\alpha_s+\alpha_m=\alpha_n} \left(|p_s| + \frac{|q_s|}{\alpha_1} \right) \alpha_m |u_m| \right) +$$

$$+ \sum_{n=2}^{j} \left(\sum_{\alpha_s+\alpha_m=\alpha_n} |p_s| |U_m| \right) \leq \sum_{n=1}^{\infty} (\alpha_n^2 |u_n| + |q_n|) +$$

$$+ \sum_{s=1}^{j-1} \left(|p_s| + \frac{|q_s|}{\alpha_1} \right) \sum_{m=1}^{j-1} \alpha_m |u_m| + \sum_{s=1}^{j-1} |p_s| \sum_{m=1}^{j-1} |U_m|$$

or

$$\sum_{n=1}^{j} \alpha_n |U_n| \leq \sum_{n=1}^{\infty} (\alpha_n^2 |u_n| + |q_n|) + \sum_{s=1}^{\infty} \left(|p_s| + \frac{|q_s|}{\alpha_1} \right) \sum_{m=1}^{\infty} \alpha_m |u_m| + \sum_{s=1}^{\infty} |p_s| \sum_{n=1}^{j-1} |U_n| .$$

If we set

$$B_0 = \sum_{n=1}^{\infty} (\alpha_n^2 |u_n| + |q_n|) + (\|p(x)\| + \|q(x)\| / \alpha_1) \sum_{m=1}^{\infty} \alpha_m |u_m| , \; B_1 = \|p(x)\| ,$$

then we obtain

$$\sum_{n=1}^{j} \alpha_n |U_n| \leq B_0 + B_1 \sum_{n=1}^{j-1} |U_n| , \, \forall j \in \mathbb{N}.$$

Then, according to Lemma*, the series $\sum_{n=1}^{\infty} \alpha_n |U_n| = \sum_{n=1}^{\infty} \alpha_n \left| \sum_{k=1}^{n} u_{kn} \right|$ converges. From the equation $\sum_{k=1}^{n} u_{kn} = U_n$ we have

$$u_{nn} = U_n - \sum_{k=1}^{n-1} u_{kn} \Rightarrow |u_{nn}| \leq |U_n| + \sum_{k=1}^{n-1} |u_{kn}| , \; n \in \mathbb{N}.$$

Considering this, from the equation (2.7), we get for $1 \leq k < n$

$$\alpha_n (\alpha_n - \alpha_k) |u_{kn}| \leq \sum_{\alpha_s+\alpha_m=\alpha_n} [(\alpha_m - \alpha_k) |p_s| + |q_s|] |u_{km}|$$

or by summing with respect to k

$$\alpha_n \sum_{k=1}^{n-1} (\alpha_n - \alpha_k) |u_{kn}| \leq \sum_{k=1}^{n-1} \left(\sum_{\alpha_s+\alpha_m=\alpha_n} [(\alpha_m - \alpha_k) |p_s| + |q_s|] |u_{km}| \right) =$$

$$\sum_{\alpha_s+\alpha_m=\alpha_n} \left(\sum_{k=1}^{m-1} [(\alpha_m - \alpha_k) |p_s| + |q_s|] |u_{km}| \right) + \sum_{\alpha_s+\alpha_m=\alpha_n} |q_s| |u_{mm}| \leq$$

$$\sum_{\alpha_s+\alpha_m=\alpha_n} \sum_{k=1}^{m-1} [(\alpha_m - \alpha_k) |p_s| + 2 |q_s|] |u_{km}| + \sum_{\alpha_s+\alpha_m=\alpha_n} |q_s| |U_m|$$

or

$$\alpha_n \sum_{k=1}^{n-1} (\alpha_n - \alpha_k) |u_{kn}| \leq \sum_{\alpha_s+\alpha_m=\alpha_n} \sum_{k=1}^{m-1} \left(|p_s| + \frac{2 |q_s|}{\alpha_1} \right) (\alpha_m - \alpha_k) |u_{km}| + \sum_{\alpha_s+\alpha_m=\alpha_n} |q_s| |U_m| .$$

If we set $V_1 = 0$, $V_n = \sum\limits_{k=1}^{n-1} (\alpha_n - \alpha_k) |u_{kn}|$, by summing the last inequality term by term, we have

$$\sum_{n=2}^{j} \alpha_n V_n \leq \sum_{n=2}^{j} \sum_{\alpha_s+\alpha_m=\alpha_n} \left(|p_s| + \frac{2 |q_s|}{\alpha_1} \right) V_m + \sum_{n=2}^{j} \sum_{\alpha_s+\alpha_m=\alpha_n} |q_s| |U_m| \leq$$

$$\sum_{s=1}^{j-1} \left(|p_s| + \frac{2 |q_s|}{\alpha_1} \right) \sum_{m=2}^{j-1} V_m + \sum_{s=1}^{j-1} |q_s| \sum_{m=1}^{j-1} |U_m| \leq$$

$$\sum_{s=1}^{\infty} |q_s| \sum_{m=1}^{\infty} |U_m| + \sum_{n=1}^{\infty} \left(|p_s| + \frac{2 |q_s|}{\alpha_1} \right) \sum_{m=2}^{j-1} V_m, \ \forall j \geq 2.$$

As a result, we prove the inequality

$$\sum_{n=2}^{j} \alpha_n V_n \leq C_0 + C_1 \sum_{n=2}^{j-1} V_n, \ \forall j \geq 2,$$

where $C_0 = \|q(x)\| \sum\limits_{m=1}^{\infty} |U_m|$, $C_1 = \|p(x)\| + 2 \|q(x)\| / \alpha_1$. From here, according to Lemma*, it follows that the series $\sum\limits_{n=2}^{\infty} \alpha_n V_n$ converges. Consequently the series $\sum\limits_{n=2}^{\infty} \sum\limits_{k=1}^{n-1} \alpha_n (\alpha_n - \alpha_k) |u_{kn}|$ also converges. Then, because of the inequalities

$$\alpha_1 |u_{kn}| \leq (\alpha_n - \alpha_k) |u_{kn}| , \ \forall k, n \in \mathbb{N}, \ n > k \Rightarrow$$

$$\sum_{n=2}^{\infty} \alpha_n \sum_{k=1}^{n-1} |u_{kn}| < \frac{1}{\alpha_1} \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \alpha_n (\alpha_n - \alpha_k) |u_{kn}| ,$$

the series $\sum\limits_{n=2}^{\infty}\alpha_n\sum\limits_{k=1}^{n-1}|u_{kn}|$ also converges. On the other hand, taking into account the inequality

$$|u_{nn}|\leq|U_n|+\sum_{k=1}^{n-1}|u_{kn}|\,,\ n\in\mathbb{N},$$

we have

$$\alpha_n\,|u_{nn}|\leq\alpha_n|U_n|+\alpha_n\sum_{k=1}^{n-1}|u_{kn}|\Rightarrow\sum_{n=1}^{\infty}\alpha_n\,|u_{nn}|\leq\sum_{n=1}^{\infty}\alpha_n|U_n|+\sum_{n=2}^{\infty}\alpha_n\sum_{k=1}^{n-1}|u_{kn}|\,.$$

Consequently, the series $\sum\limits_{n=1}^{\infty}\alpha_n\,|u_{nn}|$ and $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{n}\alpha_n\,|u_{kn}|$ converge.

Now let's show that the series $\sum\limits_{n=2}^{\infty}\alpha_n^2\sum\limits_{k=1}^{n}\frac{|u_{kn}|}{\alpha_k}$ converges. For this reason, we can write for $j>1$

$$\sum_{n=1}^{j}\alpha_n^2\sum_{k=1}^{n}\frac{|u_{kn}|}{\alpha_k}=\sum_{n=1}^{j}\alpha_n\sum_{k=1}^{n}\frac{(\alpha_n-\alpha_k+\alpha_k)\,|u_{kn}|}{\alpha_k}\leq\sum_{n=2}^{j}\alpha_n\sum_{k=1}^{n-1}\frac{(\alpha_n-\alpha_k)\,|u_{kn}|}{\alpha_k}+$$

$$\sum_{n=1}^{j}\alpha_n\sum_{k=1}^{n}|u_{kn}|\leq\frac{1}{\alpha_1}\sum_{n=2}^{\infty}\alpha_n\sum_{k=1}^{n-1}(\alpha_n-\alpha_k)\,|u_{kn}|+\sum_{n=1}^{\infty}\alpha_n\sum_{k=1}^{n}|u_{kn}|<+\infty.$$

From here it follows that the series $\sum\limits_{n=1}^{\infty}\alpha_n^2\sum\limits_{k=1}^{n}\frac{|u_{kn}|}{\alpha_k}=\sum\limits_{k=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2\,|u_{kn}|$ converges. Therefore, for the solution $\{u_n\}$, $\{u_{kn}\}$ of the system (2.5)-(2.7) the series (2.8) converges. Then the function $f(x,\lambda)$ is a solution of equation (2.1). The proof is completed.

If by the same method, we search for the second solution of equation (2.1) in the form

$$\widetilde{f}(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{n=1}^{\infty}v_ne^{i\alpha_nx}+\sum_{k=1}^{\infty}\frac{1}{\alpha_k-2\lambda}\sum_{n=k}^{\infty}v_{kn}e^{i\alpha_nx}\right)$$

which is linearly independent with the solution $f(x,\lambda)$, then we obtain $v_n=0$, $\forall n\in\mathbb{N}$ and the coefficients v_{kn} are found uniquely from the equations

$$\alpha_n\sum_{k=1}^{n}v_{kn}=q_n,\ n\in\mathbb{N},\tag{2.9}$$

$$-\alpha_n(\alpha_n-\alpha_k)v_{kn}+\sum_{\substack{\alpha_s+\alpha_m=\alpha_n\\ m\geq k}}(i\alpha_mp_s+q_s)\,v_{km}=0,\ k,n\in\mathbb{N},\ n>k\tag{2.10}$$

by the recurrent manner. Thus, the second solution of equation (2.1) has the form

$$\widetilde{f}(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{k=1}^{\infty}\frac{1}{\alpha_k-2\lambda}\sum_{n=k}^{\infty}v_{kn}e^{i\alpha_nx}\right).\tag{2.11}$$

For the solution $\{v_{kn}\}$, of the system of equations (2.9)-(2.10), it is necessary to show the convergence of the series $\sum\limits_{k=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2\,|v_{kn}|$. Since this cannot be proven with the method we used above, we will use another method.

If we put $\mu=-\lambda$, then the equation (2.1) is written as

$$y''+p(x)y'+[\mu^2-i\mu p(x)+q(x)]y=0,\quad -\infty<x<+\infty. \tag{2.12}$$

The substitution $y(x)=e^{-\int p(x)dx}z(x)$ (where $\int p(x)dx\in AP^+$) in the equation (2.12) after some simplifications gives

$$z''-p(x)z'+[\mu^2-i\mu p(x)+q(x)-p'(x)]z=0,\ -\infty<x<+\infty. \tag{2.13}$$

According to the above proved, the equation (2.13) for each $\mu\neq-\frac{\alpha_n}{2}$, $\forall n\in\mathbb{N}$ has a solution $z(x,\mu)$ in the form

$$z(x,\mu)=e^{i\mu x}\left(1+\sum_{n=1}^{\infty}z_ne^{i\alpha_nx}+\sum_{k=1}^{\infty}\frac{1}{\alpha_k+2\mu}\sum_{n=k}^{\infty}z_{kn}e^{i\alpha_nx}\right).$$

Therefore, for each $\lambda\neq\frac{\alpha_n}{2}$, $\forall n\in\mathbb{N}$, the function

$$y(x,\lambda)=z(x,-\lambda)e^{-\int p(x)dx}=$$

$$e^{-i\lambda x-\int p(x)dx}\left(1+\sum_{n=1}^{\infty}z_ne^{i\alpha_nx}+\sum_{k=1}^{\infty}\frac{1}{\alpha_k-2\lambda}\sum_{n=k}^{\infty}z_{kn}e^{i\alpha_nx}\right),$$

where the series $\sum\limits_{n=1}^{\infty}|z_n|\,\alpha_n^2$ and $\sum\limits_{k=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2\,|z_{kn}|$ converge, is the second solution of equation (2.1). For each $\varphi(x)\in AP^+$, using the convergence of the series $e^{\varphi(x)}-1=\sum\limits_{n=1}^{\infty}\frac{[\varphi(x)]^n}{n!}$ with respect to the norm of the space AP^+, one can easily prove that $e^{\varphi(x)}-1\in AP^+$. Since $\sum\limits_{n=1}^{\infty}\frac{p_n}{i\alpha_n}e^{i\alpha_nx}=\int p(x)dx$, $p(x)$, $p'(x)\in AP^+$, there exists a function $q_0(x)\in AP^+$ such that $e^{-\int p(x)dx}=1+q_0(x)$ and $q_0'(x)$, $q_0''(x)\in AP^+$. Consequently, we have the representation

$$y(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{n=1}^{\infty}v_ne^{i\alpha_nx}+\sum_{k=1}^{\infty}\frac{1}{\alpha_k-2\lambda}\sum_{n=k}^{\infty}v_{kn}e^{i\alpha_nx}\right),$$

for which the series $\sum\limits_{n=1}^{\infty}|v_n|\,\alpha_n^2$ and $\sum\limits_{k=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2\,|v_{kn}|$ converge. The condition $v_n=0$, $\forall n\in\mathbb{N}$ and the uniqueness of the solution in the form (2.11) imply the representation

$$\widetilde{f}(x,\lambda)=y(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{k=1}^{\infty}\frac{1}{\alpha_k-2\lambda}\sum_{n=k}^{\infty}v_{kn}e^{i\alpha_nx}\right).$$

In what follows, we will denote the solutions $f(x,\lambda)$, $\widetilde{f}(x,\lambda)$ with representations of type (2.3) by $e_1(x,\lambda)$, $e_2(x,\lambda)$, respectively.

Corollary 2.1. *If $p(x), p'(x), q(x) \in AP^+$, then for $\forall\lambda \neq \pm\frac{\alpha_n}{2}$, $\forall n \in \mathbb{N}$, the equation (2.1) has the Flouqet solutions*

$$e_1(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} u_n e^{i\alpha_n x} + \sum_{k=1}^{\infty} \frac{1}{\alpha_k + 2\lambda} \sum_{n=k}^{\infty} u_{kn} e^{i\alpha_n x}\right),$$
$$e_2(x,\lambda) = e^{-i\lambda x}\left(1 + \sum_{k=1}^{\infty} \frac{1}{\alpha_k - 2\lambda} \sum_{n=k}^{\infty} v_{kn} e^{i\alpha_n x}\right)$$

in $\mathbb{R}$ for which the series of type (2.8) converge.

Corollary 2.2. *For $\forall x \in \mathbb{R}$ the functions $e_j(x,\lambda)$, $j = 1, 2$, and their derivatives $e_j'(x,\lambda)$, $e_j''(x,\lambda)$ with respect to x are meromorphic functions with respect to λ. Moreover they may have only simple poles $\lambda = (-1)^j \alpha_n/2$, $\forall n \in \mathbb{N}$ and they are continuous functions of (x,λ) for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\lambda \neq (-1)^j \alpha_n/2$, $n \in \mathbb{N}$. (see* [3], *page 199)*

Note that if $p(x)$ and $q(x)$ are distinct from zero, then the functions $e_j(x,\lambda)$, $j = 1, 2$ have at most one pole. Namely, if $u_{kk} \neq 0$, then $\lambda = -\frac{\alpha_k}{2}$ is the pole of the function $e_1(x,\lambda)$. Similarly, if $v_{kk} \neq 0$, then $\lambda = \frac{\alpha_k}{2}$ is the pole of the function $e_2(x,\lambda)$.

Wronskian of the functions $e_1(x,\lambda)$ and $e_2(x,\lambda)$ is obtained as $W[e_1, e_2](x,\lambda) = -2i\lambda e^{-\int p(x)dx}$ for each $\lambda \neq \mp\frac{\alpha_k}{2}$, $\forall k \in \mathbb{N}$, where $\int p(x)dx = \sum_{n=1}^{\infty} \frac{p_n}{i\alpha_n} e^{i\alpha_n x}$ (see [3], page 200). Therefore, for each $\lambda \neq 0, \pm\dfrac{\alpha_k}{2}$, $\forall k \in \mathbb{N}$, the functions $e_1(x,\lambda)$ and $e_2(x,\lambda)$ are linearly independent in $\mathbb{R}$.

Obviously that function

$$e_{1m}(x) = \lim_{\lambda \to -\frac{\alpha_m}{2}} e_1(x,\lambda)(\alpha_m + 2\lambda) = e^{-\frac{\alpha_m}{2}x} \sum_{n=m}^{\infty} u_{mn} e^{i\alpha_n x} \tag{2.14}$$

is a solution of equation (2.1) for $\lambda = -\frac{\alpha_m}{2}$, $m \in \mathbb{N}$. If $u_{mm} = 0$, then $e_{1m}(x) \equiv 0$ and $e_1(x, -\frac{\alpha_m}{2})$ is a solution of (2.1). Moreover, the functions $e_1(x, -\frac{\alpha_m}{2})$, $e_2(x, -\frac{\alpha_m}{2})$ are linearly independent. If $u_{mm} \neq 0$, then $e_{1m}(x)$ is not zero and solutions $e_{1m}(x)$, $e_2(x, -\frac{\alpha_m}{2})$ are linearly dependent because of $W[e_{1m}(x), e_2(x, -\frac{\alpha_m}{2})] \equiv 0$. Then there exists $s_m^{(1)}$ such that

$$e_{1m}(x) = s_m^{(1)} e_2(x, -\tfrac{\alpha_m}{2}) \Leftrightarrow \sum_{n=m}^{\infty} u_{mn} e^{i\alpha_n x} =$$
$$s_m^{(1)} e^{i\alpha_m x}\left(1 + \sum_{s=1}^{\infty}\sum_{k=1}^{s} \frac{v_{ks}}{\alpha_k + \alpha_m} e^{i\alpha_s x}\right). \tag{2.15}$$

If we compare the analytical expressions at both sides of this identity, we get $s_m^{(1)} = u_{mm}$, $m \in \mathbb{N}$. Similarly, the function

$$e_{2m}(x) = \lim_{\lambda \to \frac{\alpha_m}{2}} e_2(x,\lambda)(\alpha_m - 2\lambda) = e^{-\frac{\alpha_m}{2}x} \sum_{n=m}^{\infty} v_{mn} e^{i\alpha_n x} \tag{2.16}$$

is a solution to equation (2.1) for $\lambda = \frac{\alpha_m}{2}$, $m \in \mathbb{N}$ and solutions $e_1(x, \frac{\alpha_m}{2})$, $e_{2m}(x)$ are linearly dependent. Hence there exists $s_m^{(2)}$ such that

$$e_{2m}(x) = s_m^{(2)} e_1(x, \tfrac{\alpha_m}{2}) \Leftrightarrow \sum_{n=m}^{\infty} v_{mn} e^{i\alpha_n x} =$$

$$s_m^{(2)} e^{i\alpha_m x} \left(1 + \sum_{s=1}^{\infty} \left(u_s + \sum_{k=1}^{s} \frac{u_{ks}}{\alpha_k + \alpha_m} \right) e^{i\alpha_s x} \right). \tag{2.17}$$

Moreover, $s_m^{(2)} = v_{mm}$, $m \in \mathbb{N}$. Thus, we get sequences $\left\{ s_n^{(1)} \right\}_{n \geq 1}$ and $\left\{ s_n^{(2)} \right\}_{n \geq 1}$ of complex numbers, which we will call the *sequences of the spectral data* of the operator L_λ.

Definition 2.1. A pair $\left(\left\{ s_n^{(1)} \right\}_{n \geq 1}, \left\{ s_n^{(2)} \right\}_{n \geq 1} \right)$ constructed on the basis of a solution $e_1(x, \lambda)$, $e_2(x, \lambda)$ with the help of (2.14)-(2.17) is called a set of spectral data of the operator L_λ with coefficients $p(x)$, $q(x) \in AP^+$.

3. The inverse problem for the operator L_λ

In this section, we consider the inverse problem of determining the coefficients $p(x), q(x) \in AP^+$ of operator L_λ for the given sequences of spectral data. This type of inverse problem in case $p(x) \equiv 0$, $q(x) \in Q^+$, was considered in [2], where the inverse problem of determining the potential $q(x)$ from the *set of spectral data* $\{s_n\}_{n \geq 1}$ has been analyzed for the operator $L = -\frac{d^2}{dx^2} + q(x)$ with $q(x) \in Q^+$ defined in the space $L_2(\mathbb{R})$. Later this problem has been investigated for periodic potential $q(x) \in L_2(0, 2\pi)$ in [5] and for almost periodic potential $q(x)$ in [6].

One sequence of spectral data is not enough to determine the coefficients $p(x), q(x)$ of the operator L_λ, when $p(x)$, $q(x)$ is not zero. Therefore we will study the problem of finding the coefficients $p(x)$, $q(x) \in AP^+$ of the operator L_λ by *a set of spectral data* $\left(\left\{ s_n^{(1)} \right\}_{n \geq 1}, \left\{ s_n^{(2)} \right\}_{n \geq 1} \right)$. An analogous inverse problem for the quadratic pencil of differential operators with periodic coefficients has been investigated in [1]. Unlike [1, 2, 5, 6], we will apply here the algebraic method that does not involve integral equations.

We will analyze the uniqueness of the solution. In particular, in case $p(x) \equiv 0$ we obtain the sufficient condition for the existence of solution of the inverse problem. For this purpose, we will use the system of equations

$$\begin{cases} u_{mm} = s_m^{(1)},\ u_{mn} = 0 \text{ if } \alpha_n - \alpha_m \notin G \\ u_{mn} = s_m^{(1)} \sum\limits_{k=1}^{s} \dfrac{v_{ks}}{\alpha_k + \alpha_m}, \text{ if } \alpha_m + \alpha_s = \alpha_n \\ v_{mm} = s_m^{(2)},\ v_{mn} = 0 \text{ if } \alpha_n - \alpha_m \notin G \\ v_{mn} = s_m^{(2)} \left(u_s + \sum\limits_{k=1}^{s} \dfrac{u_{ks}}{\alpha_k + \alpha_m} \right), \text{ if } \alpha_s + \alpha_m = \alpha_n \\ \qquad n, m \in N,\ n > m \end{cases} \tag{3.1}$$

obtained from identities (2.15) and (2.17) by equating the coefficients of $e^{i\alpha_n x}$ on both sides. The system of equations (3.1) expresses the relationship between the sequences $\left\{s_n^{(1)}\right\}_{n\geq 1}$, $\left\{s_n^{(2)}\right\}_{n\geq 1}$ and the trinity of sequences $\{u_n\}$, $\{u_{kn}\}$ and $\{v_{kn}\}$ which define the solutions $e_1(x,\lambda)$ and $e_2(x,\lambda)$ of equation $\ell_\lambda(y)=0$.

First let us show that, there is a one-to-one correspondence between the *set of spectral data* $\left(\left\{s_n^{(1)}\right\}_{n\geq 1}, \left\{s_n^{(2)}\right\}_{n\geq 1}\right)$ and the pair of coefficients $(p(x), q(x))$ of operator L_λ with $p(x)$, $q(x)$, $p'(x)\in AP^+$. Since the sequences $\{u_n\}$, $\{u_{kn}\}$ and $\{v_{kn}\}$ which define the solutions $e_1(x,\lambda)$ and $e_2(x,\lambda)$ of equation (2.1) are the unique solution of the systems (2.5)-(2.7) and (2.9)-(2.10), respectively, the sequences $\{u_{nn}\}=\left\{s_n^{(1)}\right\}$, $\{v_{nn}\}=\left\{s_n^{(1)}\right\}$ are also defined uniquely. Thus, for any operator L_λ, there is the only *set of spectral data* $\left(\left\{s_n^{(1)}\right\}_{n\geq 1}, \left\{s_n^{(2)}\right\}_{n\geq 1}\right)$.

On the other hand, it can be shown that any pair $\left(\left\{s_n^{(1)}\right\}_{n\geq 1}, \left\{s_n^{(2)}\right\}_{n\geq 1}\right)$ can be the *set of spectral data* of only one operator L_λ. Let the operators L_λ and $\widehat{L}_\lambda$ with their coefficients being the pairs $(p(x), q(x))$ and $(\widehat{p}(x), \widehat{q}(x))$, respectively, have the same *set of spectral data* $\left(\left\{s_n^{(1)}\right\}_{n\geq 1}, \left\{s_n^{(2)}\right\}_{n\geq 1}\right)$. Then, according to Corollary 2.1, apart from sequences $\{u_n\}$, $\{u_{kn}\}$ and $\{v_{kn}\}$ that define the solutions $e_1(x,\lambda)$ and $e_2(x,\lambda)$ of equation $\ell_\lambda(y)=0$, there are also sequences $\{\widehat{u}_n\}$, $\{\widehat{u}_{kn}\}$ and $\{\widehat{v}_{kn}\}$ which define the solutions $\widehat{e}_1(x,\lambda)$ and $\widehat{e}_2(x,\lambda)$ of equation $\widehat{\ell}_\lambda(y)=y''+\widehat{p}(x)y'+(\lambda^2+i\lambda\widehat{p}(x)+\widehat{q}(x))y=0$ and satisfy the system of equations

$$\alpha_n^2\widehat{u}_n=-\alpha_n\sum_{k=1}^{n}\widehat{u}_{kn}+\widehat{q}_n+\sum_{\alpha_s+\alpha_m=\alpha_n}(i\alpha_m\widehat{p}_s+\widehat{q}_s)\widehat{u}_m+$$

$$+\sum_{\alpha_s+\alpha_m=\alpha_n}i\widehat{p}_s\sum_{k=1}^{m}\widehat{u}_{km} \tag{3.2}$$

$$-\alpha_n\widehat{u}_n+i\widehat{p}_n+\sum_{\alpha_s+\alpha_m=\alpha_n}i\widehat{p}_s\widehat{u}_m=0,\ n\in\mathbb{N}, \tag{3.3}$$

$$-\alpha_n(\alpha_n-\alpha_k)\widehat{u}_{kn}+\sum_{\substack{\alpha_s+\alpha_m=\alpha_n\\ m\geq k}}[i(\alpha_m-\alpha_k)\widehat{p}_s+\widehat{q}_s]\widehat{u}_{km}=0,\ k,n\in\mathbb{N},\ n>k$$

and

$$\alpha_n\sum_{k=1}^{n}\widehat{v}_{kn}=\widehat{q}_n,\ n\in\mathbb{N} \tag{3.4}$$

$$-\alpha_n(\alpha_n-\alpha_k)\widehat{v}_{kn}+\sum_{\substack{\alpha_s+\alpha_m=\alpha_n\\ m\geq k}}(i\alpha_m\widehat{p}_s+\widehat{q}_s)\,\widehat{v}_{km}=0,\ k,n\in\mathbb{N},\ n>k,$$

respectively. Also, the following relations are satisfied:

$$\begin{cases} \widehat{u}_{mm} = s_m^{(1)},\ \widehat{u}_{mn} = 0 \text{ if } \alpha_n - \alpha_m \notin G \\ \widehat{u}_{mn} = s_m^{(1)} \sum\limits_{k=1}^{s} \dfrac{\widehat{v}_{ks}}{\alpha_k + \alpha_m},\ \text{ if } \alpha_m + \alpha_s = \alpha_n \\ \widehat{v}_{mm} = s_m^{(2)},\ \widehat{v}_{mn} = 0 \text{ if } \alpha_n - \alpha_m \notin G \\ \widehat{v}_{mn} = s_m^{(2)} \left(\widehat{u}_s + \sum\limits_{k=1}^{s} \dfrac{\widehat{u}_{ks}}{\alpha_k + \alpha_m} \right),\ \text{ if } \alpha_s + \alpha_m = \alpha_n \\ \qquad n, m \in N,\ n > m. \end{cases} \tag{3.5}$$

In this case, considering $\widehat{q}_1 = \alpha_1 \widehat{v}_{11} = \alpha_1 s_1^{(2)}$ and $q_1 = \alpha_1 v_{11} = \alpha_1 s_1^{(2)}$ for $n = 1$, from equations (3.4) and (2.9), respectively, we have $\widehat{q}_1 = q_1$. Then, taking into account the values $\widehat{u}_{11} = u_{11}$ and $\widehat{q}_1 = q_1$ in equations (3.2) and (2.5), we get $\widehat{u}_1 = u_1$. Subsequently, from equations (3.3) and (2.6) we obtain $\widehat{p}_1 = -i\alpha_1 \widehat{u}_1 = -i\alpha_1 u_1 = p_1$.

If the equations

$$\widehat{p}_n = p_n,\ \widehat{q}_n = q_n,\ n = 1, 2, ..., j$$

and

$$\widehat{u}_n = u_n,\ \widehat{u}_{kn} = u_{kn},\ \widehat{v}_{kn} = v_{kn},\ k = 1, 2, ..., n;\ n = 1, 2, ..., j$$

are satisfied for any $j \geq 1$, then from equations (3.1) and (3.5) we obtain

$$\widehat{u}_{kn} = u_{kn},\ \widehat{v}_{kn} = v_{kn},\ k = 1, 2, ..., n;\ n = 1, 2, ..., j + 1.$$

Further for $n = j+1$ from equations (3.4), (2.9) and (3.2), (2.5) we get $\widehat{q}_{j+1} = q_{j+1}$ and $\widehat{u}_{j+1} = u_{j+1}$, respectively. Then from equations (3.3) and (2.6) we obtain $\widehat{p}_{j+1} = p_{j+1}$. Thus, $\widehat{p}_n = p_n$, $\widehat{q}_n = q_n$ and $u_n = \widehat{u}_n$, $\widehat{u}_{kn} = u_{kn}$, $\widehat{v}_{kn} = v_{kn}$ $(k = 1, 2, ..., n)$ for $n = 1, 2, ..., j+1$ are also satisfied. Then, by induction principle the equalities $\widehat{p}_n = p_n$, $\widehat{q}_n = q_n$, $u_n = \widehat{u}_n$, $\widehat{u}_{kn} = u_{kn}$, $\widehat{v}_{kn} = v_{kn}$ $(k = 1, 2, ..., n)$ are true for $\forall n \in \mathbb{N}$. Consequently, $\widehat{p}(x) \equiv p(x)$, $\widehat{q}(x) \equiv q(x)$. Hence, the one-to-one correspondence between $(p(x), q(x))$ and $\left(\left\{ s_n^{(1)} \right\}_{n \geq 1}, \left\{ s_n^{(2)} \right\}_{n \geq 1} \right)$ is proven.

Theorem 3.1. *Any pair* $\left(\left\{ s_n^{(1)} \right\}_{n \geq 1}, \left\{ s_n^{(2)} \right\}_{n \geq 1} \right)$ *of sequences of complex numbers can be a set of spectral data of only one operator* L_λ*, with coefficients* $p(x)$*,* $q(x) \in AP^+$*.*

If the pair $\left(\left\{ s_n^{(1)} \right\}_{n \geq 1}, \left\{ s_n^{(2)} \right\}_{n \geq 1} \right)$ is the set of spectral data of operator L_λ then the coefficients $p(x)$, $q(x)$ can be determined by the following algorithm:

1. If $n = 1$ then $q_1 = \alpha_1 s_1^{(2)}$ is determined by the value $v_{11} = s_1^{(2)}$ from equations (2.9). Consequently, $u_1 = \left(s_1^{(2)} - s_1^{(1)} \right) / \alpha_1$ is found by taking $n = 1$ and $u_{11} = s_1^{(1)}$, $q_1 = \alpha_1 s_1^{(2)}$ from the equation (2.5). Similarly, $p_1 = i \left(s_1^{(1)} - s_1^{(2)} \right)$ is found by taking $u_1 = \left(s_1^{(2)} - s_1^{(1)} \right) / \alpha_1$ from the equation (2.6).

2. If the numbers p_n, q_n, u_n and u_{kn}, v_{kn} $(k = 1, 2, ..., n)$ are known for $n = 1, 2, ..., j$, where $j \in \mathbb{N}$ is arbitrary, then the numbers $u_{k,j+1}$, $v_{k,j+1}$ $(k = 1, 2, ..., j + 1)$ are uniquely determined from equations (3.1).

3. Further, according to the known values p_n, q_n, u_n $(n = 1, 2, ..., j)$ and u_{kn}, v_{kn} $(k = 1, 2, ..., n;\ n = 1, 2, ..., j + 1)$, the numbers q_{j+1}, u_{j+1} and p_{j+1} are uniquely determined from equations (2.9), (2.5) and (2.6), respectively, in a recurrent manner.

Thus, the sequences $\{p_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$, hence the coefficients $p(x)$ and $q(x)$, can be determined uniquely.

The equations (2.5), (2.6), (2.9) and (3.1) express a complex relationship between pairs $\left(\left\{s_n^{(1)}\right\}_{n\geq 1}, \left\{s_n^{(2)}\right\}_{n\geq 1}\right)$ and $(p(x), q(x))$. Proving the convergence of series $\sum\limits_{n=1}^{\infty} \alpha_n |p_n|$ and $\sum\limits_{n=1}^{\infty} |q_n|$ by using these equations is not an easy task. But in the particular case $p(x) \equiv 0$ this is possible and it is also possible to obtain a sufficient condition for the given numerical sequence $\{s_n\}_{n\geq 1}$ to be *a set of spectral data* of the operator L_λ.

When $p(x) \equiv 0$, the operator L_λ turns into the maximal operator L_λ^0 generated by the linear expression $l_\lambda^0(y) = y'' + \lambda^2 y + q(x)y$ in $L_2(\mathbb{R})$. Since $s_n^{(1)} = s_n^{(2)} = s_n$, $\forall n \in \mathbb{N}$, for the operator L_λ^0 we will examine the inverse problem of finding the potential $q(x) \in AP^+$ by the *set of spectral data* $\{s_n\}_{n\geq 1}$. An analogous inverse problem for the case $q(x) \in Q^+$ was first investigated in [2].

To solve the inverse problem, we will make use of the *Floquet* solutions

$$e_1(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{k=1}^{\infty} \frac{1}{\alpha_k + 2\lambda} \sum_{n=k}^{\infty} u_{kn} e^{i\alpha_n x}\right),$$

$$e_2(x,\lambda) = e^{-i\lambda x}\left(1 + \sum_{k=1}^{\infty} \frac{1}{\alpha_k - 2\lambda} \sum_{n=k}^{\infty} u_{kn} e^{i\alpha_n x}\right)$$

of the equation $\ell_\lambda^0(y) = y'' + \lambda^2 y + q(x)y = 0$ (since $\{p_n\}_{n\geq 1} = \{0\} \Leftrightarrow \{u_n\}_{n\geq 1} = \{0\}$). It is obvious that $e_2(x,\lambda) = e_1(x,-\lambda)$, $u_n = 0$, $v_{kn} = u_{kn}$, $u_{nn} = s_n^{(2)} = s_n^{(1)} = s_n$, $\forall n \in \mathbb{N}$, $1 \leq k \leq n$. The sequence $\{u_{kn}\}$ defining the solution $e_1(x,\lambda)$ is a unique solution of the system of equations

$$\alpha_n \sum_{k=1}^{n} u_{kn} = q_n,\ n \in \mathbb{N}, \tag{3.6}$$

$$\alpha_n(\alpha_n - \alpha_k)u_{kn} = \sum_{\substack{\alpha_s + \alpha_m = \alpha_n \\ m \geq k}} q_s u_{km} = 0,\ n, k \in \mathbb{N},\ n > k$$

which satisfies the condition $\sum\limits_{k=1}^{\infty} \dfrac{1}{\alpha_k} \sum\limits_{n=k}^{\infty} \alpha_n^2 |u_{kn}| < +\infty$ for $\{q_n\}_{n\geq 1} \in \ell_1$. According to the *set of spectral data* $\{s_n\}_{n\geq 1}$, the sequence $\{u_{kn}\}$ is found from equations

$$\begin{cases} u_{mm} = s_m,\ u_{mn} = 0 \text{ if } \alpha_n - \alpha_m \notin G \\ u_{mn} = s_m \sum\limits_{k=1}^{s} \dfrac{u_{ks}}{\alpha_k + \alpha_m}, \text{ if } \alpha_m + \alpha_s = \alpha_n \\ \qquad n, m \in N,\ n > m, \end{cases} \tag{3.7}$$

while $\{q_n\}_{n\geq 1}$ and the potential $q(x)$ are defined from the equation (3.6).

Theorem 3.2. *For the given sequence of the complex numbers* $\{s_n\}_{n\geq 1}$ *to be a set of spectral data of the operator* L_λ^0 *with the coefficient* $q(x) \in AP^+$, *it is sufficient that the conditions*

1) $\sum\limits_{n=1}^{\infty} \alpha_n |s_n| = S < +\infty$

2) $\sum\limits_{n=1}^{\infty} \dfrac{|s_n|}{\alpha_1 + \alpha_n} = \sigma < 1$

hold, and it is necessary that the condition 1) above holds.

Proof. For the given sequence $\{s_n\}_{n\geq 1}$, the sequence $\{u_{kn}\}$ is defined by recurrence method from the system of equations (3.7). Further, by the found sequence $\{u_{kn}\}$ the sequence $\{q_n\}_{n\geq 1}$ and the potential $q(x) = \sum\limits_{n=1}^{\infty} q_n e^{i\alpha_n x}$ are defined from the formula (3.6). We will show the convergence of the series $\|q(x)\| = \sum_{n=1}^{\infty} |q_n|$ for the found sequence $\{q_n\}_{n\geq 1}$. For this purpose, first let us show the convergence of the series $\sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{n} \alpha_n |u_{kn}|$ based on the conditions 1) and 2).

For every $m > 1$ we have

$$\sum_{n=1}^{m} \sum_{k=1}^{n} |u_{kn}| = \sum_{n=1}^{m} |u_{nn}| + \sum_{n=2}^{m} \sum_{k=1}^{n-1} |u_{kn}| \leq \sum_{n=1}^{m} |s_n| +$$

$$\sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} |s_k| \left| \sum_{r=1}^{p} \frac{u_{rp}}{\alpha_r + \alpha_k} \right| \leq \sum_{n=1}^{m} |s_n| + \sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} \frac{|s_k|}{\alpha_1 + \alpha_k} \sum_{r=1}^{p} |u_{rp}| \leq$$

$$\sum_{n=1}^{m} |s_n| + \sum_{k=1}^{m-1} \frac{|s_k|}{\alpha_1 + \alpha_k} \sum_{p=1}^{m-1} \sum_{r=1}^{p} |u_{rp}| \leq \frac{1}{\alpha_1} \sum_{n=1}^{\infty} \alpha_n |s_n| + \sum_{k=1}^{\infty} \frac{|s_k|}{\alpha_1 + \alpha_k} \sum_{p=1}^{m} \sum_{r=1}^{p} |u_{rp}| \Rightarrow$$

$$\sum_{n=1}^{m} \sum_{k=1}^{n} |u_{kn}| \leq \frac{S}{\alpha_1} + \sigma \sum_{n=1}^{m} \sum_{k=1}^{n} |u_{kn}| \Rightarrow \sum_{n=1}^{m} \sum_{k=1}^{n} |u_{kn}| \leq \frac{S}{\alpha_1 (1 - \sigma)},$$

hence $R_0 = \sum\limits_{n=1}^{\infty} \sum\limits_{k=1}^{n} |u_{kn}| < +\infty$ is obtained. Similarly, for $\forall m > 1$

$$\sum_{n=1}^{m} \alpha_n \sum_{k=1}^{n} |u_{kn}| \leq \sum_{n=1}^{m} \alpha_n |u_{nn}| + \sum_{n=2}^{m} \alpha_n \sum_{k=1}^{n-1} |u_{kn}| \leq \sum_{n=1}^{m} \alpha_n |s_n| +$$

$$\sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} (\alpha_k + \alpha_p) |s_k| \left| \sum_{r=1}^{p} \frac{u_{rp}}{\alpha_r + \alpha_k} \right| \leq \sum_{n=1}^{m} \alpha_n |s_n| +$$

$$\sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} \frac{\alpha_k |s_k|}{\alpha_1 + \alpha_k} \sum_{r=1}^{p} |u_{rp}| + \sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} \frac{|s_k| \alpha_p}{\alpha_1 + \alpha_k} \sum_{r=1}^{p} |u_{rp}| \leq$$

$$\sum_{n=1}^{m} \alpha_n |s_n| + \sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} |s_k| \sum_{r=1}^{p} |u_{rp}| + \sum_{n=2}^{m} \sum_{\alpha_k+\alpha_p=\alpha_n} \frac{|s_k| \alpha_p}{\alpha_1 + \alpha_k} \sum_{r=1}^{p} |u_{rp}| \leq$$

$$S+\sum_{k=1}^{m-1}|s_k|\sum_{p=1}^{m-1}\sum_{r=1}^{p}|u_{rp}|+\sum_{k=1}^{m-1}\frac{|s_k|}{\alpha_1+\alpha_k}\sum_{p=1}^{m-1}\alpha_p\sum_{r=1}^{p}|u_{rp}|$$

or

$$\sum_{n=1}^{m}\alpha_n\sum_{k=1}^{n}|u_{kn}|\le S+\sum_{k=1}^{\infty}|s_k|\sum_{p=1}^{\infty}\sum_{r=1}^{p}|u_{rp}|+\sum_{k=1}^{\infty}\frac{|s_k|}{\alpha_1+\alpha_k}\sum_{n=1}^{m}\alpha_n\sum_{k=1}^{n}|u_{kn}|\Rightarrow$$

$$\sum_{n=1}^{m}\alpha_n\sum_{k=1}^{n}|u_{kn}|\le\left(S+\frac{SR_0}{\alpha_1}\right)/\left(1-\sigma\right)\Rightarrow\sum_{n=1}^{\infty}\alpha_n\sum_{k=1}^{n}|u_{kn}|<+\infty$$

is obtained. Consequently we have

$$\sum_{n=1}^{\infty}|q_n|=\sum_{n=1}^{\infty}\alpha_n\left|\sum_{k=1}^{n}u_{kn}\right|\le\sum_{n=1}^{\infty}\alpha_n\sum_{k=1}^{n}|u_{kn}|<+\infty$$

that is $q(x)=\sum\limits_{n=1}^{\infty}q_n e^{i\alpha_n x}\in AP^{+}$.

To complete the proof, let us show that the series $\sum\limits_{n=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2|u_{kn}|$ converges and the function

$$e_1(x,\lambda)=e^{i\lambda x}\left(1+\sum_{k=1}^{\infty}\frac{1}{\alpha_k+2\lambda}\sum_{n=k}^{\infty}u_{kn}e^{i\alpha_n x}\right)$$

is a solution of the equation $y''+\lambda^2 y+q(x)y=0$. We will use Corollary 2.1.

For the reconstructed function $q(x)=\sum\limits_{n=1}^{\infty}q_n e^{i\alpha_n x}$, according to the Corollary 2.1, the equation $y''+\lambda^2 y+q(x)y=0$ has the Floquet solution

$$\widetilde{e}_1(x,\lambda)=e^{i\lambda x}\left(1+\sum_{k=1}^{\infty}\frac{1}{\alpha_k+2\lambda}\sum_{n=k}^{\infty}\widetilde{u}_{kn}e^{i\alpha_n x}\right),$$

where the sequence $\{\widetilde{u}_{kn}\}$ satisfies the equations

$$\alpha_n\sum_{k=1}^{n}\widetilde{u}_{kn}=q_n,\ n\in\mathbb{N}\tag{3.8}$$

and

$$\begin{cases}\widetilde{u}_{mm}=\widetilde{s}_m,\ \widetilde{u}_{mn}=0\text{ if }\alpha_n-\alpha_m\notin G\\ \widetilde{u}_{mn}=\widetilde{s}_m\sum\limits_{k=1}^{s}\dfrac{\widetilde{u}_{ks}}{\alpha_k+\alpha_m},\ \text{if }\alpha_m+\alpha_s=\alpha_n\\ \qquad m,n\in\mathbb{N},\ n>m.\end{cases}\tag{3.9}$$

Moreover, the series $\sum\limits_{n=1}^{\infty}\frac{1}{\alpha_k}\sum\limits_{n=k}^{\infty}\alpha_n^2|\widetilde{u}_{kn}|$ is convergent.

Let's show that $\widetilde{u}_{mn}=u_{mn}$, $\forall m,n\in\mathbb{N}$, $m\le n$. When $m=n=1$, we get $\widetilde{u}_{11}=u_{11}$ or $\widetilde{s}_1=s_1$ from the equation $\alpha_1\widetilde{u}_{11}=q_1=\alpha_1 u_{11}$. For $m=1$, $n=2$, from the equations (3.9) and (3.7) we have $\widetilde{u}_{12}=u_{12}=0$ if $\alpha_2-\alpha_1\notin G$ or $\widetilde{u}_{12}=\widetilde{s}_1\widetilde{u}_{11}/2\alpha_1=s_1u_{11}/2\alpha_1=u_{12}$ if $\alpha_2-\alpha_1=\alpha_1$. On the other hand, the equations (3.8) and (3.6) imply $\alpha_2\left(\widetilde{u}_{12}+\widetilde{u}_{22}\right)=q_2=\alpha_2\left(u_{12}+u_{22}\right)$. From here we obtain $\widetilde{u}_{22} = u_{22}$ or $\widetilde{s}_2 = s_2$. If $\widetilde{u}_{mn} = u_{mn}$, $1 \leq m \leq n \leq j$ for $j > 1$ is true then from equations (3.9) and (3.7)

$$\widetilde{u}_{m,j+1} = \widetilde{s}_m \sum_{k=1}^{s} \frac{\widetilde{u}_{ks}}{\alpha_k + \alpha_m} = s_m \sum_{k=1}^{s} \frac{u_{ks}}{\alpha_k + \alpha_m} = u_{m,j+1}$$

if $\alpha_m + \alpha_s = \alpha_{j+1}$ or $\widetilde{u}_{m,j+1} = u_{m,j+1} = 0$ if $\alpha_{j+1} - \alpha_m \notin G$ is obtained for all m where $1 \leq m \leq j$. On other hand, for $n = j+1$ from equations (3.8), (3.6), $\widetilde{u}_{j+1,j+1} = u_{j+1,j+1}$ or $\widetilde{s}_{j+1} = s_{j+1}$, hence $\widetilde{u}_{mn} = u_{mn}$, $1 \leq m \leq n \leq j+1$ is obtained. Then, according to the induction principle, $\widetilde{u}_{mn} = u_{mn}$, $1 \leq m \leq n$, $\forall n \in \mathbb{N}$ is valid. Consequently, the series $\sum\limits_{n=1}^{\infty} \frac{1}{\alpha_k} \sum\limits_{n=k}^{\infty} \alpha_n^2 |u_{kn}|$ is convergent and

$$e_1(x,\lambda) = e^{i\lambda x} \left(1 + \sum_{k=1}^{\infty} \frac{1}{\alpha_k + 2\lambda} \sum_{n=k}^{\infty} u_{kn} e^{i\alpha_n x} \right) = \widetilde{e}_1(x,\lambda)$$

is the solution of the differential equation $y'' + \lambda^2 y + q(x)y = 0$ and the sequence $\{s_n\}_{n\geq 1}$ is the *set of spectral data* of the operator L_λ^0. Hence, $q(x) = \sum\limits_{n=1}^{\infty} q_n e^{i\alpha_n x}$ is the unique solution of the inverse problem. The proof is completed. □

Remark 3.1. Note that Theorem 3.2. is a generalization of Theorem 2 of [2]. Condition 2) of the theorem is not necessary. But the example of $\{s_n\}_{n\geq 1}$ defined as $s_1 = s \geq 2$, $s_n = 0$ for $n > 1$, given in [2], for $q(x) \in Q^+$, shows that condition 2) is important. Then the condition 2) of Theorem 3.2. is also important (see [2], page 14-15).

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