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OPTIMIZATION OF THE NICOLETTI BOUNDARY VALUE PROBLEM FOR SECOND-ORDER DIFFERENTIAL INCLUSIONS

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Abstract. The paper is devoted to optimal control of the second-order Nicoletti boundary value problem (BVP) with differential inclusions (DFIs) and duality. First, we formulate the optimality conditions for the problem posed, and then, based on the concept of infimal convolution, the dual problems. It turns out that the Euler-Lagrange type inclusions are "duality relations" for both primal and dual problems, which means that a pair consisting of solutions to the primal and dual problems satisfies this extremal relation, and vice versa. Finally, as an application of the results obtained, we consider the second-order Nicoletti BVP with polyhedral DFIs.

1. Introduction

It is well known that numerous problems of optimal control theory, models of economic dynamics, differential games, etc., described in terms of differential inclusions, constitute an integral part of modern mathematical science [2, 8, 10, 11, 25, 29]. Much earlier [2] considered a control problem given in terms of an evolution differential inclusion and develop necessary conditions for a minimum in the problem. These conditions are given in terms of certain normal to arbitrary closed sets, and require no smoothness or convexity in the problem. The results subsume related works that incorporate convexity assumptions. The paper [25] is devoted to the study of a Mayer-type optimal control problem for semilinear unbounded evolution inclusions in reflexive and separable Banach spaces subject to endpoint constraints described by finitely many Lipschitzian equalities and inequalities. New necessary optimality conditions for continuous-time evolution inclusions are established by passing to the limit from discrete approximations. Recently, in the works [12, 13, 14], for optimal control problems of discrete processes and differential inclusions (DFIs) on the basis of locally adjoint mappings (LAMs), necessary and sufficient optimality conditions are obtained. Papers [13, 15] are mainly devoted to establishing sufficient optimality conditions for differential inclusions of the second and third orders. The work [13]

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studies a new class of problems of optimal control theory with Sturm-Liouville-type differential inclusions involving second-order linear self-adjoint differential operators. The main goal is to obtain optimality conditions for the Mayer problem for differential inclusions with initial point constraints. By using the discretization method guaranteeing transition to continuous problem, the discrete and discrete-approximate inclusions are investigated. Necessary and sufficient conditions, containing both the Euler-Lagrange and Hamiltonian-type inclusions and appropriate "transversality" conditions are derived. The work [15] deals for the first time with the Dirichlet problem for discrete, discrete-approximate problem on a uniform grid and differential inclusions of elliptic type. In the form of EulerLagrange inclusion necessary and sufficient conditions for optimality are derived for the problems under consideration on the basis of LAMs. The article [17] considers a convex programming problem with functional and non-functional constraints. Unlike previous works, the classical perturbation approach is not used in the study of convex optimization problems. In particular, thanks to a new representation of the indicator function on a convex set, the successful use of the infimal convolution method in this paper plays a key role in proving the duality results for the convex programming problem. The paper [16] deals with the optimal control problem described by second-order differential inclusions. Based on the infimal convolution concept of convex functions, dual problems for differential inclusions are constructed and the results of duality are proved. In particular, the linear second-order optimal control problem with the Mayer functional is considered. This problem shows that maximization in the dual problems is realized over the set of solutions of the adjoint equation. Finally, dual problems are constructed for the problem with a second-order polyhedral differential inclusion. The paper [14] studies the Mayer problem with higher order evolution differential inclusions and functional constraints of optimal control theory. Are proved necessary and sufficient conditions incorporating the Euler-Lagrange inclusion, the Hamiltonian inclusion, the transversality and complementary slackness conditions. The basic concept of obtaining optimal conditions is LAM, which is closely related to the coderivative concept [27]. More recently, a survey article [28] was published, devoted to some applications of topological degree theory for non-compact set-valued mappings to some control systems governed by differential inclusions in Banach spaces. In [32] for a convex differential inclusion depending on some parameter, the controllability in a separable Banach space is investigated. This result includes well-known results on the controllability of perturbed linear systems. The application of the decomposition method in controllability problems is also considered.

Recall that from different points of view, the theory of duality plays a fundamental role in the analysis of optimization and variational problems (see [4, 6, 15, 22, 24, 30] and references therein). A key player in any duality framework is the Legendre-Fenchel conjugate transform. Often, duality is associated with convex problems, yet it turns out that duality theory also has a fundamental impact even on the analysis of nonconvex problems. This covers well-known Fenchel and Lagrangian duality schemes, as well as minimax theorems for convex-concave functionals within a unified approach that emphasizes the importance of asymptotic functions. In the paper [24], an improvement of the recent duality theorem and a new result are presented that emphasize the fact that strong duality, without assumptions about the interior of the ordering cone, is associated with the normal cone. In the work [22], a general duality theorem was established in a generalized conjugacy system, which generalizes the classical result on the minimization of a convex function over a closed convex cone. The theorem gives two quasiconvex duality schemes; one of them belongs to the type of surrogate duality, while the other is applicable to problems with Lipschitz quasiconvex objective functions and gives duals, whose objective functions do not contain any surrogate constraints. The paper [4] gives assertions about φ-conjugate in connection with duality theorems and optimality conditions for a rather broad class of nonconvex optimization problems. A geometrical interpretation of the statements follows in a natural way by the projective extension of the Euclidean space. The work [30] provides a detailed framework for analyzing optimality conditions for rather abstract mathematical programming problems in terms of a dual program. The irreplacability of the concepts of Lagrange function, saddle point and saddle value are emphasized. Interesting examples are taken from nonlinear programming, approximation, stochastic programming, calculus of variations, and optimal control. For convex optimization problems, the duality gap, that is, the difference between the optimal values of the primal and dual problems, is zero under a constraint qualification condition. We also recall that, based on the dual operations of addition and infimal convolution of convex functions [10, 15, 30] the dual problems were constructed for different problems in the book of Mahmudov [10].

Various qualitative problems of the Nicoletti differential inclusions, including classical existence results, have been considered by many authors (see [1, 3, 5, 7, 9, 23, 31, 33] and references therein). In the mathematical literature from the point of view of applicability, Nicoletti differential inclusions with a special boundary condition attracts the attention of many mathematicians and, therefore, is intensively studied. In [9] are presented results concerning the uniqueness and existence of solutions of the Nicoletti BVP for the first order ordinary differential equation. Similar results for the Floquet BVP for the first order ordinary differential equation were obtained in [7]. In the work [5] the existence theorems for the Floquet and Nicoletti BVPs for the n-th order nonconvex differential inclusions, the Picard BVP for the second order differential inclusion, and some BVPs for hyperbolic partial differential inclusions are obtained. In the paper [31] are derived existence, uniqueness and comparison results for the functional differential equation $x'(t) = f(t, x)$, a.e. $t \in I$, with classical Nicoletti boundary condition $x_i(t_i) = v_i \in X, i \in A$, where I is a real interval, A is a nonempty set and X is a Banach space. In [3] a singular BVP of the Cauchy-Nicoletti type is studied, the existence theorems are proved. The paper [23] deals with a system of quasilinear fractional differential equations, subjected to the Cauchy-Nicoletti type boundary conditions, is suggested a suitable numerical-analytic technique that allows to construct an approximate solution of the studied fractional boundary value problem with high precision.

In the present work, the optimality conditions for a first order Nicoletti's discrete and differential inclusions with BVP were considered for the first time. As an auxiliary problem, a new problem is defined for the so-called Nicoletti discrete inclusion. The construction of the optimality and duality condition for the first order Nicoletti DFI is accompanied with the optimal condition and the duality of certain new Nicoletti first order discrete inclusions (DSI). So, this paper is devoted to one of the complex and interesting areas of constructing optimality conditions and dual problems of first order Nicoletti DSIs and DFIs. At the same time, although the construction of the dual problem is based on the concept of infimal convolution, the specificity of the problem posed both in technical and ideological terms makes it much more difficult to prove the results of strong duality.

The article is organized in the following order:

In Section 2, the needed definitions and results from the book [10] are given; Hamiltonian function and argmaximum sets of a set-valued mapping F, the LAM, infimal convolution of proper convex functions, for second-order DFIs with the Mayer functional are formulated.

In Section 3 we establish optimality conditions for second-order Nicoletti's DFIs. Although a discrete approximation of the problem posed is used to formulate these conditions in the continuous problem, the cumbersome calculations associated with the discrete problem are not carried out here and only the final result is given for the continuous problem.

In Section 4 on the basis of the infimal convolution of convex functions, a dual problem for convex problems with Nicoletti DSIs is formulated and dual theorem is proved. It appears that if v and v^* are the values of primal and dual problems, respectively, then there is a weak duality, i.e., $v \geq v^*$ for all feasible solutions. In addition, if the standard condition for convex analysis on the existence of interior point is satisfied, then both problems have solutions and there is a strong duality $v = v^*$.

Section 4 discusses an important class of Nicoletti control problems with second-order polyhedral DFIs. The specificity of the problem allows one to obtain the optimal conditions and the dual problem in terms of transposed matrices defining the polyhedral map and the nonnegative dual variable.

2. Needed Facts and Problem Statement

For the convenience of the reader, all the necessary concepts related to convex analysis can be found in the book of Mahmudov [10]; let $\langle x, y\rangle$ and $z = (x, y)$ be the inner product and a pair of elements $x, y \in \mathbb{R}^m$, where $\mathbb{R}^m$ is an m-dimensional Euclidean space. Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ be a set-valued mapping from $\mathbb{R}^m$ into the set of subsets of $\mathbb{R}^m$. F is convex closed if its graph is a convex closed set in $\mathbb{R}^{2m}$. Further, $\operatorname{gph} F = \{(x, y) : y \in F(x)\}, \operatorname{dom} F = \{x : F(x) \neq \varnothing\}$. The Hamiltonian function and argmaximum set for a set-valued mapping F are defined as follows

$$H_F(x, y^*) = \sup_y \{\langle y, y^*\rangle : y \in F(x)\}, y^* \in \mathbb{R}^m,$$

$$F_A(x, y^*) = \{y \in F(x) : \langle y, y^*\rangle = H_F(x, y^*)\},$$

respectively. If $F(x) = \varnothing$ for a convex F we put $H_F(x, y^*) = -\infty$, so that Hamiltonian $H(\cdot, y^*)$ is concave. As usual, $W_A(x^*), x^* \in \mathbb{R}^m$ is a support function of the set $A \subset \mathbb{R}^m$ i.e., $W_A(x^*) = \sup_x \{\langle x, x^*\rangle : x \in A\}$.

$f : \mathbb{R}^{2m} \to \mathbb{R}^1$ is said to be proper function, if it does not assume the value $-\infty$

and is not identically equal to $+\infty$. Clearly, f is proper function if and only if $\operatorname{dom} f \neq \varnothing$ and $f(x,y)$ is finite for $(x,y) \in \operatorname{dom} f = \{(x,y) : f(x,y) < +\infty\}$. Usually, we denote the dual cone to the cone of tangent directions $K_F(x,y) \equiv K_{gphF}(x,y)$ by $K_F^*(x,y)$. Here $K_F(z) = \{\bar{z} : \bar{z} = \gamma\,(z_1 - z)\,, z_1 \in \operatorname{gph} F\} = \{\bar{z} : \bar{z} + \gamma z \in \operatorname{gph}\, F$ for a sufficiently small $\gamma > 0\}$. Then, a set-valued mapping $F^*(\cdot, x, y) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined by

$$F^*\left(y^*; (x,y)\right) := \{x^* : (x^*, -y^*) \in K_F^*(x,y)\}\,,$$

is called the LAM to a set-valued F at a point $(x,y) \in \operatorname{gph} F$. We provide another definition of LAM to mapping F which is more relevant for further development

$$F^*\left(y^*; (x,y)\right) := \{x^* : H_F\left(x_1, y^*\right) - H_F\left(x, y^*\right) \leq \left\langle x^*, x_1 - x\right\rangle, \forall x_1 \in \mathbb{R}^m\}\,,$$
$$(x,y) \in gphF, \quad y \in F_A\left(x, y^*\right).$$

Note that the notion of coderivativity [25, 26, 27], in spite of outward similarities, differs substantially from LAM for nonconvex mappings. Since the Hamiltonian $H\left(\cdot, y^*\right)$ is concave in the convex case, the latter and previous definitions of LAMs coincide.

Definition 2.1. A function $f : \mathbb{R}^{2m} \to \mathbb{R}^1$ is called closed if its epigraph epi $f = \left\{\left(x^0, x, y\right) : x^0 \geq f(x,y)\right\}$ is a closed set.

Definition 2.2. The function f^* defined as $f^*\left(x^*, y^*\right) = \sup\left\{\langle x, x^*\rangle + \langle y, y^*\rangle - f(x,y)\right\}$ is called conjugate to f. It is clear that the conjugate function f^* is, by definition, closed and convex.

In what follows, the function defined as

$$M_F\left(x^*, y^*\right) = \inf_{x,y}\left\{\langle x, x^*\rangle - \langle y, y^*\rangle : (x,y) \in \operatorname{gph} F\right\} = \inf_{x}\left\{\langle x, x^*\rangle - H_F\left(x, y^*\right)\right\},$$

will often be used. It easily seen that the function M_F is a support function of gphF taken with a minus sign.

Definition 2.3. The operation of infimal convolution $(g_1 \oplus g_2)$ of functions $g_i, i = 1, 2$ is defined as follows

$$\left(g_1 \oplus g_2\right)(u) = \inf_{u^1, u^2}\left\{g_1\left(u^1\right) + g_2\left(u^2\right) : u^1 + u^2 = u\right\}, u^i \in \mathbb{R}^m, i = 1, 2.$$

If $g_i, i = 1, 2$ are not identically equal to $+\infty$, then $\left(g_1 \oplus g_2\right)^* = g_1^* + g_2^*$. Thus, the operations $+$ and $\oplus$ are thus dual to each other with respect to taking conjugates.

In Section 3 we deal with so-called second-order Nicoletti problem (PNC) with evolution DFIs

$$\text{infimum } f\left(x(T), x'(T)\right), \tag{2.1}$$
$$\text{(PNC)} \qquad x''(t) \in F(x(t), t), \ \text{a.e. } t \in [0, T], \tag{2.2}$$
$$x_k^{(j)}(t_k^j) = v_k^j, \quad k = 1, \ldots, m;\ j = 0, 1,\ 0 \leq t_1^j \leq t_2^j \leq \ldots .. t_m^j \leq T, \tag{2.3}$$
$$x(t) = \left(x_1(t), \ldots, x_m(t)\right),$$

where $F(\cdot, t) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a time dependent set-valued mapping, $v_j = (v_1^j, v_2^j, \ldots, v_m^j) \in \mathbb{R}^m$, and $(t_1^j, t_2^j, \ldots, t_m^j) \in [0, T]^m (j = 0, 1)$. A map $x : [0, T] \to \mathbb{R}^m$ given by $x(t) = \left(x_1(t), \ldots, x_m(t)\right)$ for $t \in [0, T]$, with the absolutely continuous derivatives $x_k' : [0, T] \to \mathbb{R}^1 (k = 1, \ldots, m)$ satisfying the above system (PNC), we call a feasible solution of the Nicoletti BVP for higher-order DFIs (2.2), (2.3), $f:\mathbb{R}^{2m}\to\mathbb{R}^1$ is continuous function, $v_k^j, k=1,\dots,m$ are fixed real numbers. The problem is to find an arc $\tilde{x}(\cdot)$ of problem (2.1) - (2.3) that satisfies (2.2) almost everywhere (a.e.) on $[0,T]$ and minimizes the Mayer functional $f\left(x(T),x'(T)\right)$.

For further development, we define the discrete analogue of the continuous problem (PNC) First of all, we derive necessary and sufficient conditions of optimality for problems (PND), which is both an independent problem and an integral part of the proof of a sufficient condition of optimality for a problem (PNC).

3. The Sufficient Conditions of Optimality for Second-Order Nicoletti's DFIs

Although we use a discretized method to formulate sufficient optimality conditions for the problem (PNC), the cumbersome calculations associated with the discrete problem [18, 19, 20, 21] are not carried out here, and only the final result for the continuous problem is given. Thus, in Theorem 3.1, sufficient optimality conditions are formulated for the basic Nicoletti problem (PNC) with convex second-order DFIs that include the second-order adjoint Euler-Lagrange type inclusions. We formulate the second-order Euler-Lagrange type adjoint inclusion and transversality conditions for the problem (PNC):
(1) $x^{*\prime\prime}(t)\in F^*\left(x^*(t);\left(\tilde{x}(t),\tilde{x}''(t)\right),t\right)$, a.e. $t\in[0,T](t\neq t_k^j)$,
$x^{*\prime\prime}(t_k^j)-\psi^*(t_k^j)\in F^*(x^*(t_k^j);(\tilde{x}(t_k^j),\tilde{x}''(t_k^j)),t_k^j), k=1,\dots,m, j=0,1$, where
(2) $\tilde{x}''(t)\in F_A\left(\tilde{x}(t);x^*(t),t\right)$, a.e. $t\in[0,T]$;
The transversality conditions at the endpoints $t=0,T$ consist of the following
(3) $\left(x^{*\prime}(T),-x^*(T)\right)\in\partial f\left(\tilde{x}(T),\tilde{x}'(T)\right)$, $x^{*\prime}(0)=x^*(0)=0$.
A feasible solution is a pair of functions $\{x^*(t),\psi^*(t)\}$, where the map $x^*:[0,T]\to\mathbb{R}^m$ is defined as $x^*(t)=(x_1^*(t),\dots,x_m^*(t))$ for $t\in[0,T]$, with absolutely continuous functions $x_k^{*\prime}:[0,T]\to\mathbb{R}^1$ $(k=1,\dots,m)$, whose second-order derivative is defined at the points t_k^j and satisfy the above Euler-Lagrange type adjoint inclusions.

Theorem 3.1. *Let $f:\mathbb{R}^{2m}\to\mathbb{R}^1$ be a continuous and proper convex function, $F(\cdot,t):\mathbb{R}^m\rightrightarrows\mathbb{R}^m$ be a convex set-valued mapping. Then for optimality of the feasible trajectory $\tilde{x}(t)$ in the problem (PNC) it is sufficient that there exists a pair of functions $\{x^*(t),\psi^*(t)\}, t\in[0,T]$ $(\psi^*(t_k^j)=0, t\neq t_k^j)$ satisfying a.e. the second-order DFIs (1), (2) and the transversality conditions (3) at the endpoint $t=0,T$, respectively.*

Proof. Setting $\psi^*(t)\equiv 0, t\neq t_k^j(j=0,1)$ from the definition of LAM, and from condition (1), (2) we have the following inequality

$$H_F\left(x(t),x^*(t)\right)-H_F\left(\tilde{x}(t),x^*(t)\right)\leq\left\langle x^{*\prime\prime}(t)-\psi^*(t),x(t)-\tilde{x}(t)\right\rangle$$

whence

$$\left\langle x''(t),x^*(t)\right\rangle-\left\langle\tilde{x}''(t),x^*(t)\right\rangle\leq\left\langle x^{*\prime\prime}(t)-\psi^*(t),x(t)-\tilde{x}(t)\right\rangle$$

or

$$\left\langle x''(t)-\tilde{x}''(t),x^*(t)\right\rangle-\left\langle x^{*\prime\prime}(t),x(t)-\tilde{x}(t)\right\rangle\leq\left\langle-\psi^*(t),x(t)-\tilde{x}(t)\right\rangle, \qquad (3.1)$$

where it can be easily seen that

$$\langle \psi^*(t), x(t) - \tilde{x}(t) \rangle = 0, t \in [0, T]. \tag{3.2}$$

Obviously, there can be two cases here, either $t \neq t_k^j$ or $t = t_k^j$. In the first case since $\psi^*(t) \equiv 0$ the equality (3.2) is satisfied trivially. In the second case, $\psi^*(t_k^j) = (0, \ldots, 0, \psi_k^*(t_k^j), 0, \ldots, 0)$, and since $x(\cdot), \tilde{x}(\cdot)$ are feasible trajectories, that is, $x_k^{(j)}(t_k^j) = \tilde{x}_k^{(j)}(t_k^j) = v_k^j$, $k = 1, \ldots, m$; $j = 0, 1$ we have $\langle \psi^*(t), x(t) - \tilde{x}(t) \rangle = \psi_k^*(t_k^j)(x_k(t_k^j) - \tilde{x}_k(t_k^j)) = 0$ and the scalar product is zero.
Now, considering (3.2) in the inequality (3.1) we have

$$\left\langle x''(t) - \tilde{x}''(t), x^*(t) \right\rangle - \left\langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t) \right\rangle \leq 0.$$

It can be shown by direct verification that

$$\begin{aligned} &\frac{d}{dt} \left\langle x'(t) - \tilde{x}'(t), x^*(t) \right\rangle - \frac{d}{dt} \left\langle x(t) - \tilde{x}(t), x^{*\prime}(t) \right\rangle \\ &= \left\langle x''(t) - \tilde{x}''(t), x^*(t) \right\rangle - \left\langle x^{*\prime\prime}(t), x(t) - \tilde{x}(t) \right\rangle \leq 0. \end{aligned} \tag{3.3}$$

Also it is easy to see that, integrating the inequality (3.3) over the intervals (t_k^j, t_{k+1}^j) we have

$$\begin{aligned} &\langle x'(t_{k+1}^j) - \tilde{x}'(t_{k+1}^j), x^*(t_{k+1}^j) \rangle - \langle x'(t_k^j) - \tilde{x}'(t_k^j), x^*(t_k^j) \rangle \\ -&\langle x(t_{k+1}^j) - \tilde{x}(t_{k+1}^j), x^{*\prime}(t_{k+1}^j) \rangle + \langle x(t_k^j) - \tilde{x}(t_k^j), x^{*\prime}(t_k^j) \rangle \leq 0, \ k = 0, 1, \ldots, m \end{aligned}$$

or

$$\begin{aligned} &\langle x'(t_{m+1}^j) - \tilde{x}'(t_{m+1}^j), x^*(t_{m+1}^j) \rangle - \langle x'(t_0^j) - \tilde{x}'(t_0^j), x^*(t_0^j) \rangle \\ -&\langle x(t_{m+1}^j) - \tilde{x}(t_{m+1}^j), x^{*\prime}(t_{m+1}^j) \rangle + \langle x(t_0^j) - \tilde{x}(t_0^j), x^{*\prime}(t_0^j) \rangle \leq 0, t_{m+1} = T, t_0 = 0. \end{aligned}$$

Then, considering $x^*(t_0) = x^{*\prime}(t_0) = x^*(0) = x^{*\prime}(0) = 0$, we have

$$\left\langle x'(T) - \tilde{x}'(T), x^*(T) \right\rangle - \left\langle x(T) - \tilde{x}(T), x^{*\prime}(T) \right\rangle \leq 0$$

or

$$-\left\langle x^*(T), x'(T) - \tilde{x}'(T) \right\rangle + \left\langle x^{*\prime}(T), x(T) - \tilde{x}(T) \right\rangle \geq 0. \tag{3.4}$$

Now using the classical definition of the subdifferential for all feasible trajectories $x(\cdot)$, the first transversality condition of conditions (3) can be rewritten as follows

$$f(x(T), x'(T)) - f(\tilde{x}(T), \tilde{x}'(T)) \geq -\langle x^*(T), x'(T) - \tilde{x}'(T) \rangle + \langle x^{*\prime}(T), x(T) - \tilde{x}(T) \rangle. \tag{3.5}$$

Hence, from the inequalities (3.4), (3.5) for all feasible trajectories $x(\cdot)$ we have

$$f\left(x(T), x'(T)\right) - f\left(\tilde{x}(T), \tilde{x}'(T)\right) \geq 0 \text{ or } f\left(x(T), x'(T)\right) \geq f\left(\tilde{x}(T), \tilde{x}'(T)\right).$$

The proof of theorem is completed. □

4. The Dual Problem for Second-Order Nicoletti's DFIs

Using Definitions 2.3 and the method of constructing dual problems for differential inclusions given in M, we can construct the following dual problem for the primal continuous convex problem (PNC):

$$\sup_{x^*(\cdot),\psi^*(\cdot)} \left\{ -f^*\left(x^{*\prime}(T), -x^*(T)\right) + \int_0^T M_F\left(x^{*\prime\prime}(t), x^*(t)\right) dt \right.$$

$$\text{(PNC}^*\text{)} \qquad \left. - \sum_{k=1}^m [W_k(\psi^*(t_k^0)) + W_k(\psi^*(t_k^1))] + \sum_{k=1}^m \langle v_0, \psi^*(t_k^0)\rangle + \sum_{k=1}^m \langle v_1, \psi^*(t_k^1)\rangle \right\}$$

$$v_j = \left(v_1^j, v_2^j, \dots, v_m^j\right) \in \mathbb{R}^m.$$

Here, the components-functions $x_k^* : [0,T] \to \mathbb{R}^1$, $x_k^{*\prime} : [0,T] \to \mathbb{R}^1 (k = 1,\dots,m)$ of the map $x^* : [0,T] \to \mathbb{R}^m$, given by $x^*(t) = (x_1^*(t),\dots,x_m^*(t))$ for $t \in [0,T]$, are absolutely continuous functions.

Theorem 4.1. *Let $f : \mathbb{R}^{2m} \to \mathbb{R}^1$ be a continuous and proper convex function, $F(\cdot,t) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ be a convex set-valued mapping and let $\tilde{x}(t)$ be an optimal solution of the primal second-order problem (PNC). Then, for the optimality of a pair of functions $\{\tilde{x}^*(\cdot), \tilde{\psi}^*(\cdot)\}$ in the problem (PNC*), it is necessary and sufficient that the conditions (1)-(3) of Theorem* 3.1 *are satisfied. Moreover, the optimal values in the primal and dual problems are equal.*

Proof. We prove that for all feasible solutions $x(\cdot)$ and dual variables $\{x^*(\cdot), \psi^*(\cdot)\}$ of the primal and dual problems, respectively, the following inequality holds:

$$\begin{aligned} f(x(T), x'(T)) \geq -f*(x^{*\prime}(T), -x^*(T)) + \int_0^T M_F(x^{*\prime\prime}(t), x^*(t))dt \\ - \sum_{k=1}^m [W_k(\psi^*(t_k^0)) + W_k(\psi^*(t_k^1))] + \sum_{k=1}^m \langle v_0, \psi^*(t_k^0)\rangle + \sum_{k=1}^m \langle v_1, \psi^*(t_k^1)\rangle \\ v_j = (v_1^j, v_2^j, \dots, v_m^j) \in \mathbb{R}^m, \quad j = 0,1. \end{aligned} \tag{4.1}$$

In fact, using the conjugate f^*, definitions of functions H_F, W_k, the fact that $\psi^*(t) = 0, t \neq t_k^j$, and formula

$$\frac{d}{dt}\left\langle x(t), x^{*\prime}(t)\right\rangle - \frac{d}{dt}\left\langle x'(t), x^*(t)\right\rangle = \left\langle x^{*\prime\prime}(t), x(t)\right\rangle - \left\langle x''(t), x^*(t)\right\rangle$$

it can be proved that

$$\begin{aligned} -f^*(x^{*\prime}(T), -x^*(T)) + \int_0^T M_F(x^{*\prime\prime}(t), x^*(t))dt \\ - \sum_{k=1}^m [W_k(\psi^*(t_k^0)) + W_k(\psi^*(t_k^1))] + \sum_{k=1}^m \langle v_0, \psi^*(t_k^0)\rangle + \sum_{k=1}^m \langle v_1, \psi^*(t_k^1)\rangle \\ \leq f(x(T), x'(T)) - \langle x(T), x^{*\prime}(T)\rangle + \langle x'(T), x^*(T)\rangle \\ + \int_0^T [\langle x(t), x^{*\prime\prime}(t)\rangle - \langle x''(t), x^*(t)\rangle]dt - \sum_{k=1}^m [W_k(\psi^*(t_k^0)) + W_k(\psi^*(t_k^1))] \\ + \sum_{k=1}^m \langle v_0, \psi^*(t_k^0)\rangle + \sum_{k=1}^m \langle v_1, \psi^*(t_k^1)\rangle = f(x(T), x'(T)) \\ -\langle x(T), x^{*\prime}(T)\rangle + \langle x'(T), x^*(T)\rangle + \int_0^T d\langle x(t), x^{*\prime}(t)\rangle - \int_0^T d\langle x'(t), x^*(t)\rangle \end{aligned}$$

$$-\sum_{k=1}^{m}\langle x(t_k^0),\psi^*(t_k^0)\rangle-\sum_{k=1}^{m}\langle x(t_k^1),\psi^*(t_k^1)\rangle+\sum_{k=1}^{m}\langle v_0,\psi^*(t_k^0)\rangle+\sum_{k=1}^{m}\langle v_1,\psi^*(t_k^1)\rangle$$

$$=f(x(T),x'(T))-\langle x(T),x^{*\prime}(T)\rangle+\langle x'(T),x^*(T)\rangle+\langle x(T),x^{*\prime}(T)\rangle-\langle x(0),x^{*\prime}(0)\rangle$$
$$-\langle x'(T),x^*(T)\rangle+\langle x'(0),x^*(0)\rangle=f(x(T),x'(T))+\langle x'(0),x^*(0)\rangle-\langle x(0),x^{*\prime}(0)\rangle,$$

where we considered that $\langle x(t_k^0),\psi^*(t_k^0)\rangle = v_k^0\psi_k^*(t_k^0), \langle x(t_k^1),\psi^*(t_k^1)\rangle = v_k^1\psi_k^*(t_k^1)$ and $x^*(0) = x^{*\prime}(0) = 0$. This proves the inequality (4.1). Now, suppose that a pair $\{\tilde{x}^*(\cdot),\tilde{\psi}^*(t_k^j)\}$ $(k=1,\ldots,m, j=0,1)$ satisfies the conditions (1)-(3) of Theorem 3.1 Then by Lemma 2.6[10, p.64] we can write

$$\langle\tilde{x}^{*\prime\prime}(t),\tilde{x}(t)\rangle-H_F(\tilde{x}(t),\tilde{x}^*(t))=M_F(\tilde{x}^{*\prime\prime}(t),\tilde{x}^*(t)),$$
$$\langle\tilde{x}^{*\prime\prime}(t_k^j)-\tilde{\psi}^*(t_k^j),\tilde{x}(t_k)\rangle-H_F(\tilde{x}(t_k^j),\tilde{x}^*(t_k^j)) \tag{4.2}$$
$$=M_F(\tilde{x}^{*\prime\prime}(t_k^j)-\tilde{\psi}^*(t_k^j),\tilde{x}^*(t_k^j)), k=1,\ldots,m, j=0,1.$$

Moreover, by Theorem 1.27 [10] the first transversality condition (3) is equivalent to the equality

$$f^*\left(\tilde{x}^{*\prime}(T),-\tilde{x}^*(T)\right)=\left\langle\left(\tilde{x}(T),\tilde{x}^{*\prime}(T)\right\rangle-\left\langle\tilde{x}'(T),\tilde{x}^*(T)\right\rangle-f\left(\tilde{x}(T),\tilde{x}'(T)\right)\right). \tag{4.3}$$

As a result, considering the relationships (4.2)-(4.3) in (4.1) the inequality is replaced by equality and for $\tilde{x}(\cdot)$ and $\{\tilde{x}^*(\cdot),\tilde{\psi}^*(t_k^j)\}(k=1,\ldots,m, j=0,1)$ the equality of values of the primal and dual problems is provided. Moreover, $\tilde{x}(\cdot)$ and $\{\tilde{x}^*(\cdot),\tilde{\psi}^*(t_k^j)\}(k=1,\ldots,m, j=0,1)$ are satisfy the conditions (1)-(3) of Theorem 3.1 and the collection (1)-(3) is a dual relation for the primal and dual problems. □

5. The Nicoletti's Control Problem with Second-Order Polyhedral DFIs

In this section, as an example, we construct the dual problem to the problem with the following second-order Nicoletti's problem:

$$\text{infimum } f\left(\tilde{x}(T),\tilde{x}'(T)\right),$$
$$\text{(PL)} \qquad x''(t)\in F(x(t)), \text{ a.e. } t\in[0,T],$$
$$x_k^{(j)}(t_k^j)=v_k^j,\ k=1,\ldots,m; j=0,1, \quad x(t)=(x_1(t),\ldots,x_m(t)),$$
$$0\le t_1^j\le t_2^j\le\ldots\le t_m^j\le T, \quad F(x)=\{y: Nx-Ky\le d\}, \tag{5.1}$$

where F is a polyhedral set-valued mapping, N, K are $s\times m$ dimensional matrices, d is a s -dimensional column-vector, $f:\mathbb{R}^{2m}\to\mathbb{R}^1$ is a convex function, v_k^0, v_k^1 are fixed numbers. According to the dual problem we calculate $M_F(x^*,y^*)$:

$$M_F(x^*,y^*)=\inf\{\langle x,x^*\rangle-\langle y,y^*\rangle:(x,y)\in\operatorname{gph}F\}. \tag{5.2}$$

Denoting $w=(x,y)\in\mathbb{R}^{2m}, w^*=(x^*,-y^*)\in\mathbb{R}^{2m}$ we have a linear programming problem

$$\inf\{\langle w,w^*\rangle: Sw\le d\}, \tag{5.3}$$

where $S=[N\vdots-K\vdots]$ is $s\times 2m$ block matrix. Hence, according to classical theory of linear programming, $\tilde{w}=(\tilde{x},\tilde{y})$ is a solution of (5.3), if and only if there exists s-dimensional vector $\lambda \geq 0$ such that $w^* = -S^*\lambda$, $\langle N\tilde{x} - K\tilde{y} - d, \lambda\rangle = 0$. Therefore, $w^* = -S^*\lambda$ means that $x^* = -N^*\lambda, \quad y^* = -K^*\lambda, \lambda \geq 0$. Thus, from (5.2) we find that

$$M_F\left(x^*, y^*\right) = \langle \tilde{x}, -N^*\lambda\rangle - \langle \tilde{y}, -K^*\lambda\rangle = -\langle N\tilde{x}, \lambda\rangle + \langle K\tilde{y}, \lambda\rangle = -\langle d, \lambda\rangle. \quad (5.4)$$

Then by Theorem 3.1 we derive that

$$\begin{gathered} x^{*\prime\prime}(t) = -N^*\lambda(t), x^*(t) = -K^*\lambda(t), \lambda(t) \geq 0, \\ x^{*\prime\prime}(t_k^j) - \psi^*(t_k^j) = -N^*\lambda(t_k^j), \quad \lambda(t_k^j) \geq 0, \quad k = 1, \ldots, m, j = 0, 1. \end{gathered} \quad (5.5)$$

Now, substituting $x^*(t)$ in (5.5) into first equation, we obtain

$$\begin{gathered} N^*\lambda(t) - K^*\lambda''(t) = 0, \ \lambda(t) \geq 0, \\ \psi^*(t_k^j) = N^*\lambda(t_k^j) - K^*\lambda'(t_k^j), \ \lambda(t_k^j) \geq 0. \end{gathered} \quad (5.6)$$

As a result, taking into account (5.4)-(5.6) and Theorem 4.1 we have the following dual problem:

$$\begin{aligned} \sup_{\lambda(\cdot)\geq 0} \Bigg\{ & -f^*\left(-K^*\lambda'(T), K^*\lambda(T)\right) - \int_0^T \langle d, \lambda(t)\rangle dt \\ & - \sum_{k=1}^m \left[W_k\left(N^*\lambda\left(t_k^0\right) - K^*\lambda'\left(t_k^0\right)\right) + W_k\left(N^*\lambda\left(t_k^1\right) - K^*\lambda'\left(t_k^1\right)\right)\right] \\ & + \sum_{k=1}^m \left\langle v_0, N^*\lambda\left(t_k^0\right) - K^*\lambda'\left(t_k^0\right)\right\rangle + \sum_{k=1}^m \left\langle v_1, N * \lambda\left(t_k^1\right) - K^*\lambda'\left(t_k^1\right)\right\rangle \Bigg\}. \end{aligned}$$

First, we give a sufficient optimality condition for the problem (PL).

Theorem 5.1. *Suppose that $f : \mathbb{R}^{2m} \to \mathbb{R}^1$ is a continuous proper convex function and that $F(\cdot, t) : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a polyhedral set-valued mapping given in problem (12). Then for the optimality of the trajectory $\tilde{x}(\cdot)$ in problem (PL) with second-order Nicoletti's polyhedral DFIs, it is sufficient that there exists a function $\lambda(t) \geq 0, t \in [0, T]$ satisfying a.e. the following second-order Euler-Lagrange type polyhedral differential inclusion and transversality condition at the endpoint $t = T$:*
(1) $N^*\lambda(t) - K^*\lambda''(t) = 0, \lambda(t) \geq 0; \langle N\tilde{x}(t) - K\tilde{x}''(t) - d, \lambda(t)\rangle = 0$, a.e. $t \in [0, T]$,
$W_k(N^*\lambda(t_k^j) - K^*\lambda'(t_k^j)) = \langle \tilde{x}(t_k), N^*\lambda(t_k^j) - K^*\lambda'(t_k^j)\rangle, k = 1, \ldots, m, j = 0, 1$,
(2) $(-K^*\lambda'(T), K^*\lambda(T)) \in \partial f(\tilde{x}(T), \tilde{x}'(T)), \ K^*\lambda(0) = K^*\lambda'(0) = 0$.

Proof. Indeed, the condition (1) of theorem is the formula (5.6). Then from the second relation (5.6) and condition $x^*(0) = x^{*\prime}(0) = 0$ we obtain the transversality condition (2) at the endpoints $t = 0, T$. The proof of theorem is completed. ☐

Hence, based on Theorems 3.1 and 4.1 for problem (PL), we have obtained the following duality theorem.

Theorem 5.2. *Suppose that the conditions of Theorem 3.1 are satisfied and that $\tilde{x}(\cdot)$ is an optimal solution of the primal Nicoletti problem (PL). Then, in order for $\lambda(t) \geq 0, t \in [0, T]$ to be an optimal solution to the dual problem* (PL*) *it is necessary that the conditions of Theorem 5.1 are satisfied. In addition, the optimal values in the primal (*PL) *and dual (*PL**) problems are equal.*

6. Conclusions

This article proposes an approach to solving an optimization problem using the so-called second-order Nicoletti DFI with special boundary conditions, which can be used to study various processes in the science of engineering and economics. For example, for $d = 0$ in the problem (PL) we get a problem with second-order Nicoletti DFIs whose graph is a polyhedral cone. And such problems with cones occupy a special place in the study of economic dynamics called the Neumann-Gayl model. Nevertheless, for such problems it is very important to construct dual problems, as well as to prove duality theorems. The arising second-order adjoint DFIs are called Euler - Lagrange type inclusions. It should be noted that the justification of sufficient conditions for the higher-order primal problem, as well as for the higher-order duality problem can be obtained.

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