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# ON BASICITY OF PERTURBED SYSTEM OF EXPONENTS IN GRAND-LEBESGUE SPACES

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**Abstract**. This work is dedicated to the study of basicity of perturbed system of exponents $\left\{e^{i(n-\beta sign n)t}\right\}_{n\in Z}$ in grand-Lebesgue spaces $L_{p)}(-\pi,\pi)$, where $\beta$ is a complex parameter. Some subspace $G_{p)}(-\pi,\pi)$ of the space $L_{p)}(-\pi,\pi)$ and the grand-Hardy classes $H^{+}_{p)}$ and ${}_{m}H^{-}_{p)}$ are considered. Using the analog of Cauchy representation theorem in grand-Hardy classes, the subspaces $GH^{+}_{p)}$ and ${}_{m}GH^{-}_{p)}$ are defined and the solvability of nonhomogeneous Riemann problem in the classes $GH^{+}_{p)}\times{}_{m}GH^{-}_{p)}$ is studied. The subspaces $G^{+}_{p)}(-\pi,\pi)$ and $G^{-}_{p)}(-\pi,\pi)$ of the space $G_{p)}(-\pi,\pi)$ are defined and the basicity of the systems $\left\{e^{int}\right\}_{n\in Z_+}$ and $\left\{e^{-int}\right\}_{n\in N}$ for these subspaces, respectively, is established. Finally, using boundary value problems method, the basicity of the system $\left\{e^{i(n-\beta sign n)t}\right\}_{n\in Z}$ for $G_{p)}(-\pi,\pi)$ is established.

## 1. Introduction

As is known, one of the methods for studying the basis properties of systems in Banach spaces is a boundary value problems method for analytic functions (see [3, 7, 19, 22], etc). Note that the Riemann boundary value problem was first stated by B. Riemann, and the existence of its solution was first treated by D. Hilbert by reducing it to integral equations. A lot of research has been later dedicated to this theory (see, e.g., N. Wiener, I.I. Privalov, E. Hopf, T. Carleman, etc). The theory of boundary value problems has grown rapidly after being largely applied in the problems of mechanics, hydro and aerodynamics, theory of pseudoanalytic functions, theory of elliptic equations, and approximation theory. Theory of boundary value problems for analytic functions has been studied by Y.V. Sokhotski, J. Plemelj, V. Volterra, Ch.E. Picard, T. Carleman, F.D. Gakhov, N.I. Muskhelishvili, I.N. Vekua, etc. For the theory of Riemann boundary value problems, we refer the readers to, e.g., [8, 12, 14, 21].

Since recently, there arose great interest in studying various problems of different branches of mathematics in so-called nonstandard Banach spaces. As examples to this kind of spaces, we can mention, for example, Lebesgue space with the variable summability index, Morrey space, grand-Lebesgue space, etc. Many

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classical facts about harmonic analysis have been extended to these spaces (for most detailed information about these matters see, e.g. [1, 9, 10, 13, 17, 18, 20, 23-25], etc).

Approximation problems in these spaces have been studied to varying degrees. In Lebesgue spaces with variable indices, these problems have been studied well enough ([4, 18, 25]). Situation is different in case of Morrey or grand-Lebesgue spaces, where a lot of problems have yet to be studied. In this direction, we can mention the works [5, 6, 11, 16]. Grand-Lebesgue spaces were first introduced by T. Iwaniec and C. Sbordone [17] in 1992. Due to the non-separability of grand-Lebesgue spaces, basicity matters in such spaces have not yet been considered. Therefore, there is a need to consider a subspace required in the theory of differential equations. This work is dedicated to the study of basicity of perturbed system of exponents $\left\{e^{i(n-\beta signn)t}\right\}_{n\in Z}$ in the grand-Lebesgue space $L_{p)}(-\pi,\pi)$, where $\beta$ is a complex parameter.

In this work, we define grand-Hardy classes $H^{+}_{p)}$ and ${}_mH^{-}_{p)}$, and then we study some properties of functions belonging to these spaces. Using the analog of Cauchy representation theorem in grand-Hardy classes, we define the subspaces $GH^{+}_{p)}$ and ${}_mGH^{-}_{p)}$. We study the solvability of nonhomogeneous Riemann problem in the classes $GH^{+}_{p)}\times {}_mGH^{-}_{p)}$. We define the subspaces $G^{+}_{p)}(-\pi,\pi)$ and $G^{-}_{p)}(-\pi,\pi)$ of the space $G_{p)}(-\pi,\pi)$ and establish the basicity of systems $\left\{e^{int}\right\}_{n\in Z_+}$ and $\left\{e^{-int}\right\}_{n\in N}$ for the subspaces $G^{+}_{p)}(-\pi,\pi)$ and $G^{-}_{p)}(-\pi,\pi)$, respectively. We find a necessary and sufficient condition on the parameter $\beta$, which provides the basicity of system $\left\{e^{i(n-\beta signn)t}\right\}_{n\in Z}$ for $G_{p)}(-\pi,\pi)$.

## 2. Some concepts and auxiliary facts

We will use the following notations. $N$ will denote the set of positive integers, $Z$ will be the set of integers, $Z_+=\{0\}\bigcup N$, $R$ will stand for the set of real numbers, $R_+$ will denote the set of positive real numbers, $|A|$ will be the Lebesgue measure of the set $A$ in $R$, $\delta_{nk}$ will stand for Kronecker symbol, and by $\overline{M}$ we will denote the closure of the set $M$ in corresponding space.

Let's state some concepts and facts from the theory of grand-Lebesgue spaces, theory of Hardy classes and their analogs for grand-Hardy classes which will be used in this work.

Let $L_{p)}(-\pi,\pi)$, $1<p<+\infty$, be a grand-Lebesgue space of measurable functions $f$ on $[-\pi,\pi]$, which satisfy the condition

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}\,dt\right)^{\frac{1}{p-\varepsilon}}<+\infty.$$

The space $L_{p)}(-\pi,\pi)$ is a complete normed space with the norm $\|f\|_{p)}$. The embeddings

$$L_p(-\pi,\pi)\subset L_{p)}(-\pi,\pi)\subset L_{p-\varepsilon}(-\pi,\pi)$$

and for $\forall f\in L_p(-\pi,\pi)$ the relations

$$\left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}}\|f\|_{p-\varepsilon}\le\|f\|_{p)}\le 2\pi(p-1)\,\|f\|_{p}\,,0<\varepsilon<p-1 \qquad (2.1)$$

hold. For $\forall\delta \in R$, consider the shift operator

$$T_\delta f(x) = \left\{ \begin{array}{l} f(x+\delta), x+\delta \in [-\pi,\pi] \\ 0, x+\delta \in R\backslash[-\pi,\pi] \end{array} \right. , \forall f \in L_{p)}(-\pi,\pi).$$

Denote by $\tilde{G}_{p)}(-\pi,\pi)$ the linear manifold of functions $f \in L_{p)}(-\pi,\pi)$ satisfying the condition

$$\|T_\delta f - f\|_{p)} \to 0, \delta \to 0.$$

Let $G_{p)}(-\pi,\pi)$ be a closure of $\tilde{G}_{p)}(-\pi,\pi)$ in $L_{p)}(-\pi,\pi)$. As $\forall f \in L_p(-\pi,\pi)$ we have $\|T_\delta f - f\|_p \to 0$, $\delta \to 0$, from (2.1) it follows that $L_p(-\pi,\pi) \subset G_{p)}(-\pi,\pi)$. The set $C_0^\infty([-\pi,\pi])$ of infinitely differentiable finite functions on $[-\pi,\pi]$ is dense in $G_{p)}(-\pi,\pi)$ ([26]).

Let $\rho : [-\pi,\pi] \to R_+$ be some weight function. By $A_p(-\pi,\pi)$, $1 < p < +\infty$, we denote the Muckenhoupt class, i.e. the class of weight functions $\rho(t)$ satisfying the condition

$$\frac{1}{|I|}\int_I \rho(t)dt \left(\frac{1}{|I|}\int_I \rho(t)^{-\frac{1}{p-1}}dt\right)^{p-1} < +\infty$$

for every interval $I \subset [-\pi,\pi]$. Let $L_{p),\rho}(-\pi,\pi)$ be a weighted grand-Lebesgue space of measurable functions $f$ on $[-\pi,\pi]$ such that

$$\|f\|_{p),\rho} = \|f\rho\|_{p)} < +\infty.$$

We will often use the following fact from concerning the boundedness of linear operator in the weighted grand-Lebesgue space $L_{p),\rho}(-\pi,\pi)$ ([18]).

**Theorem 2.1.** *Let the operator $T$ be bounded in the weighted spaces $L_{p,\rho}(-\pi,\pi)$ and $L_{p-\varepsilon_0,\rho}(-\pi,\pi)$ for some $\varepsilon_0 \in (0,p-1)$ and $\rho \in A_p \bigcap A_{p-\varepsilon_0}$. Then the operator $T$ is bounded in $L_{p),\rho}(-\pi,\pi)$.*

Denote by $\left(L_{p)}(-\pi,\pi)\right)'$, $p > 1$, a space of functions $g \in L_1(-\pi,\pi)$ such that

$$\|g\|_{p)'} = \sup_{f \in S_{p)}} \|fg\|_{L_1} < +\infty,$$

where $S_{p)} = \left\{f \in L_{p)}(-\pi,\pi) : \|f\|_{p)} \le 1\right\}$.

We will need the following fact ([15]).

**Theorem 2.2.** *Let $-\pi = s_0 < s_1 < s_2 < ..... < s_r < \pi$ be arbitrary points and*

$$\rho(t) = \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\alpha_k}.$$

*The inclusion $\rho(t) \in \left(L_{p)}(-\pi,\pi)\right)'$ is true if and only if the following conditions hold: $\alpha_k \in \left(-1+\frac{1}{p}, +\infty\right)$, $k = \overline{0,r}$.*

When establishing the basicity of perturbed system of exponents in the space $L_{p)}(-\pi,\pi)$, we will follow the case of Morrey space considered in [5]. Namely, the basicity of perturbed system of exponents will be reduced to the study of conjugate Riemann problem in corresponding Hardy classes. Let's state an analog of usual Hardy classes generated by a grand-Lebesgue space.

Let $\omega = \{z \in C : |z| < 1\}$ be a unit disk, and $\gamma = \{z \in C : |z| = 1\}$ be a unit circumference. Denote by $H_p^+$, $p > 0$, a usual Hardy class of functions $f$ analytic in $\omega$, with

$$\sup_{0<r<1} \int_0^{2\pi} |f_r(t)|^p \, dt < +\infty,$$

where $f_r(t) = f(re^{it})$. Let the function $f(z)$ be analytic outside unit disk $\omega$ and have finite order at infinitely remote point, i.e. let $f(z)$ have a Laurent expansion of the form

$$f(z) = \sum_{k=-\infty}^{m} a_k z^k, z \to \infty,$$

in a neighborhood of the infinitely remote point. If $\overline{f_0\left(\frac{1}{\bar{z}}\right)} \in H_p^+$, $p > 1$, then say that $f$ belongs to the class ${}_mH_p^-$, $p > 1$, where $f_0(z) = \sum_{k=-\infty}^{-1} a_k z^k$.

We will also use the following uniqueness theorem for analytic function ([12]).

**Theorem 2.3.** *Let $f = (f^+, f^-) \in H_1^+ \times_m H_1^-$ and $f^+(\tau) = f^-(\tau)$ for almost every $\tau \in \gamma$, where $f^{\pm}$ are non-tangential boundary values of the functions $f^{\pm}(z)$ as $r \to 1$. Then $f$ is a polynomial of degree $k \leq m$ (zero polynomial for $m < 0$).*

For more details on these and other facts from the theory of Hardy classes we refer the readers to [19]. Denote the grand-Hardy space $H_{p)}^+$, $p > 1$, of analytic functions $f$ in $\omega$ satisfying the condition

$$\|f\|_{H_{p)}^+} = \sup_{0<r<1} \|f_r\|_{p)} < +\infty.$$

It is easy to show that the following inclusions are true:

$$H_p^+ \subseteq H_{p)}^+ \subseteq H_{p-\varepsilon}^+, p > 1, 0 < \varepsilon < p - 1.$$

It is clear that every function $f \in H_{p)}^+$, $p > 1$, has non-tangential boundary values $f^+(e^{it})$ almost everywhere on $\gamma$ as $r \to 1$. Denote by ${}_mH_{p)}^-$, $p > 1$ the class of functions $f \in_m H_{p-\varepsilon}^-$, $0 < \varepsilon < p - 1$: $\overline{f_0(\frac{1}{z})} \in H_{p)}^+$.

The following analog of the theorem on relationship between function and its boundary function in grand-Hardy spaces is true ([15]).

**Theorem 2.4.** *The following assertions are true:*

*i) If $f \in H_{p)}^+$, $1 < p < +\infty$, then the Cauchy formula holds:*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\xi)}{\xi - z} d\xi, z \in \omega. \tag{2.2}$$

*ii) If $f^+ \in L_{p)}(0, 2\pi)$, $1 < p < +\infty$, then the function $f$, defined by the Cauchy formula (2.2), belongs to the class $H_{p)}^+$.*

Let the function $f(z)$ be analytic outside unit disk $\omega$ and have finite order at infinitely remote point, i.e. Laurent expansion of $f(z)$ in a neighborhood of the infinitely remote point has the following form:

$$f(z) = \sum_{k=-\infty}^{m} a_k z^k, z \to \infty.$$

If the principal part $f_0(z)=\sum_{k=-\infty}^{-1} a_k z^k$ is such that $\overline{f_0\left(\frac{1}{\bar{z}}\right)} \in H_{p)}^{+}$, $p>1$, then we will say that $f$ belongs to the class ${}_mH_{p)}^{-}$, $p>1$.

Let $G(\tau)$ and $f(\tau)$ be the given functions on the unit circumference $\gamma$ such that

$$f \in L_{p)}(\gamma), p>1, G^{\pm 1}(\tau) \in L_{\infty}(\gamma).$$

Consider the following nonhomogeneous Riemann problem in the classes $H_{p)}^{+} \times {}_mH_{p)}^{-}$, $p>1$:

$$F^{+}(\tau)-G(\tau)F^{-}(\tau)=f(\tau), \tau \in \gamma. \tag{2.3}$$

By the solution of this problem we mean any pair of functions $F^{+}(z)$ and $F^{-}(z)$ belonging to the classes $H_{p)}^{+}$ and ${}_mH_{p)}^{-}$, respectively, whose boundary values $F^{\pm}(\tau)$ on unit circumference $\gamma$ satisfy (2.3) almost everywhere. For $f(\tau)=0$, we obtain the corresponding homogeneous problem

$$F^{+}(\tau)-G(\tau)F^{-}(\tau)=0, \tau \in \gamma. \tag{2.4}$$

Introduce the following piecewise analytic functions in $C\backslash\gamma$:

$$\mathrm{Z}_1(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G(e^{is})\right| \frac{e^{is}+z}{e^{is}-z} ds\right\},$$

$$\mathrm{Z}_2(z)=\exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(s) \frac{e^{is}+z}{e^{is}-z} ds\right\}, z \notin \gamma,$$

where $\theta(t)=\arg G(e^{it})$, $t\in[-\pi,\pi]$. Let

$$\mathrm{Z}_\theta(z)=\mathrm{Z}_1(z)\mathrm{Z}_2(z), z \notin \gamma.$$

It is absolutely clear that the function $\mathrm{Z}_\theta(z)$ depends on the choice of argument $\theta(\cdot)$. It is known (see [12]) that the boundary values of the functions $\mathrm{Z}_i(z)$ are expressed as follows:

$$\mathrm{Z}_1^{\pm}(e^{it})=\exp\left\{\pm\frac{1}{2}\ln\left|G(e^{it})\right|+\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\left|G(e^{is})\right| \frac{e^{is}+e^{it}}{e^{is}-e^{it}} ds\right\},$$

$$\mathrm{Z}_2^{\pm}(e^{it})=\exp\left\{\pm\frac{i}{2}\theta(t)+\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(s) \frac{e^{is}+e^{it}}{e^{is}-e^{it}} ds\right\}, t\in[-\pi,\pi].$$

Hence,

$$\frac{\mathrm{Z}_1^{+}(e^{it})}{\mathrm{Z}_1^{-}(e^{it})}=\left|G(e^{it})\right|, \frac{\mathrm{Z}_2^{+}(e^{it})}{\mathrm{Z}_2^{-}(e^{it})}=e^{i\theta(t)}, \text{ a.e. } t\in[-\pi,\pi].$$

Consequently,

$$\frac{\mathrm{Z}_\theta^{+}(e^{it})}{\mathrm{Z}_\theta^{-}(e^{it})}=\frac{\mathrm{Z}_1^{+}(e^{it})\mathrm{Z}_2^{+}(e^{it})}{\mathrm{Z}_1^{-}(e^{it})\mathrm{Z}_2^{-}(e^{it})}=\left|G(e^{it})\right| e^{i\theta(t)}=G(e^{it})$$

or

$$\mathrm{Z}_\theta^{+}(\tau)-G(\tau)\mathrm{Z}_\theta^{-}(\tau)=0, \text{ a.e. } \tau\in\gamma,$$

i.e. the function $\mathrm{Z}_\theta(z)$ satisfies the relation (2.4).

We will call it the canonical solution of homogeneous problem (2.4).

Theorem below is true concerning the solvability of Riemann problem ([15]).

**Theorem 2.5.** *Let the following conditions hold:*

*1)* $G^{\pm 1}(\tau) \in L_\infty(\gamma)$, $\theta(t) = \arg G(e^{it})$ *is piecewise Holder on* $[-\pi, \pi]$, $\theta(t) = \theta_0(t) + \theta_1(t)$,*where* $\theta_0(t)$ *is a continuous part of* $\theta(t)$, $\theta_1(t)$ *is a jump function of* $\theta(t)$ *at discontinuity points* $-\pi < s_1 < s_2 < ...... < s_r < \pi$, *i.e.* $\theta_1(-\pi) = 0$, $\theta_1(t) = \sum_{k:t<s_k} h_k$, $t \in (-\pi, \pi]$, *where* $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$.

*2 ) the relations*

$$-1 + \frac{1}{p} < \frac{h_k}{2\pi} < \frac{1}{p}, k = \overline{0, r},$$

*are satisfied for the sequence* $\{h_k\}_0^r$, $h_0 = \theta(-\pi) - \theta(\pi)$.

*Then the following assertions concerning the solvability of the problem (1.3) in the classes* $H^+_{p)} \times {}_m H^-_{p)}$, $p > 1$, *are true:*

$\alpha$) *for* $m \geq -1$, *the problem (2.4) has a general solution of the form*

$$F(z) = Z_\theta(z) P_k(z) + F_1(z),$$

*where* $Z_\theta(z)$ *is a canonical solution of the problem (2.4),* $P_k(z)$ *is a polynomial of degree* $k \leq m$ $(P_{-1}(z) \equiv 0)$, *and* $F_1(z)$ *is a function defined by the formula*

$$F_1(z) = \frac{Z_\theta(z)}{2\pi i} \int_\gamma \frac{f(\xi)[Z^+_\theta(\xi)]^{-1}}{\xi - z} d\xi, z \notin \gamma; \tag{2.5}$$

$\beta$) *for* $m < -1$, *the problem (2.3) is solvable if and only if the right-hand side* $f(\tau) \in L_{p)}(-\pi, \pi)$, $p > 1$, *satisfies the orthogonality condition*

$$\int_{-o}^{\pi} \frac{f(e^{it})}{Z^+_\theta(e^{it})} e^{ikt} dt = 0, k = \overline{1, -m-1}, \tag{2.6}$$

*and the problem (2.3) has a unique solution* $F(z) = F_1(z)$.

## 3. Solvability of Riemann problem in classes $GH^+_{p)} \times {}_m GH^-_{p)}$

Let $H^+_{p)}$ be a grand-Hardy space. Denote by $L^+_{p)}$ the subspace of $L_{p)}$ generated by the restrictions of the functions from $H^+_{p)}$. From Uniqueness Theorem 2.3 and Theorem 2.4 it follows that the operator $J_+ : H^+_{p)} \to L^+_{p)}$, defined by the formula $J_+ f(\xi) = f^+(\xi)$, $\xi \in \gamma$, performs isomorphism of the spaces $L^+_{p)}$ and $H^+_{p)}$. Let $G^+_{p)} = G_{p)} \bigcap L^+_{p)}$. Obviously, $G^+_{p)}$ is a subspace of the space $L^+_{p)}$. Let $GH^+_{p)} = J_+^{-1}(G^+_{p)})$. Now let ${}_m L^-_{p)}$ be a subspace of $L_{p)}$, generated by the restrictions of the functions from ${}_m H^-_{p)}$. Denote ${}_m G^-_{p)} = G_{p)} \bigcap_m L^-_{p)}$ and ${}_m GH^-_{p)} = J_-^{-1}({}_m G^-_{p)})$, where the isomorphism $J_- :_m H^-_{p)} \to_m L^-_{p)}$ is defined by the formula $J_- f(\xi) = f^-(\xi)$, $\xi \in \gamma$.

Consider the following nonhomogeneous Riemann problem in the classes $GH^+_{p)} \times {}_m GH^-_{p)}$:

$$F^+(\tau) - G(\tau) F^-(\tau) = f(\tau), \tau \in \gamma, \tag{3.1}$$

where $G(\tau)$ and $f(\tau)$ are the given functions on the unit circumference $\gamma$.

Denote $G_{p),\rho}(-\pi, \pi) = \{f \in L_{p),\rho}(-\pi, \pi) : \rho f \in G_{p)}(-\pi, \pi)\}$. It is clear that the operator $T : L_{p)}(\gamma) \to L_{p)}(-\pi, \pi)$, defined by the formula $Tf(t) = f(e^{it})$, $t \in [-\pi, \pi]$, is an isomorphism. Let $G_{p)}(\gamma)$ and $G_{p),\rho}(\gamma)$ be the images of the spaces $G_{p)}(-\pi,\pi)$ and $G_{p),\rho}(-\pi,\pi)$, respectively, during the mapping $T^{-1}$. To study the solvability of the problem (3.1) in the classes $GH^{+}_{p)}\times {}_{m}GH^{-}_{p)}$, we need the boundedness in $G_{p),\rho}(\gamma)$ of the singular operator $S_\gamma$:

$$S_\gamma(f)(\tau)=\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-\tau}d\xi, \tau\in\gamma.$$

The following lemma is true.

**Lemma 3.1.** *Let the singular operator $S_\gamma$ be bounded in the weighted spaces $L_{p),\rho}(\gamma)$, $p>1$ and $L_{p,\rho}(\gamma)$. Then $S_\gamma$ acts boundedly in $G_{p),\rho}(\gamma)$, $p>1$.*

*Proof.* Take $\forall f\in G_{p),\rho}(\gamma)$ and $\forall\varepsilon>0$. Then, due to the density of $L_{p,\rho}(\gamma)$ in the space $L_{p),\rho}(\gamma)$, it follows that there exists $g\in L_{p,\rho}(\gamma)$ such that

$$\|f-g\|_{p),\rho}<\varepsilon.$$

Therefore, from the boundedness of the operator $S_\gamma$ in $L_{p),\rho}(\gamma)$ it follows that

$$\|S_\gamma(f)-S_\gamma(g)\|_{p),\rho}\le\|S_\gamma\|\,\|f-g\|_{p),\rho}<\|S_\gamma\|\,\varepsilon.$$

Hence, taking into account the arbitrariness of $\forall\varepsilon>0$ and $S_\gamma(g)\in L_{p,\rho}(\gamma)$, we obtain that $S_\gamma(f)$ belongs to the closure of $L_{p,\rho}(\gamma)$ in $L_{p),\rho}(\gamma)$, i.e. $S_\gamma(f)\in G_{p),\rho}(\gamma)$. Lemma is proved.

Now let's consider the solvability of the problem (3.1). The following theorem is true.

**Theorem 3.1.** *Let the coefficient of the problem (3.1) satisfy the conditions of Theorem 2.5, and $f\in G_{p)}(\gamma)$. Then the following assertions concerning the solvability of the problem (3.1) in the classes $GH^{+}_{p)}\times {}_{m}GH^{-}_{p)}$, $p>1$, are true:*

*$\alpha$) for $m\ge-1$, the problem (3.1) has a general solution of the form*

$$F(z)=Z_\theta(z)P_k(z)+F_1(z),$$

*where $Z_\theta(z)$ is a canonical solution of corresponding homogenous problem, $P_k(z)$ is a polynomial of degree $k\le m$ ($P_{-1}(z)\equiv0$), and $F_1(z)$ is a function defined by the formula (2.5).*

*$\beta$) for $m<-1$, the problem (3.1) is solvable if and only if the function $f(\tau)$, $p>1$, satisfies the orthogonality condition (2.6), and the problem (3.1) has a unique solution $F(z)=F_1(z)$.*

*Proof.* Using Theorem 2.5, we can see that the problem (3.1) is solvable in the classes $H^{+}_{p)}\times {}_{m}H^{-}_{p)}$, $p>1$, and the assertions $\alpha$) and $\beta$) are true. It only remains to show the validity of the inclusions $F_1^+\in G^{+}_{p)}$ and $F_1^-\in_m G^{-}_{p)}$. Using known transformations, the boundary values of the function $F_1(z)$ are expressed in the form

$$F_1^{\pm}(\tau)=Z_\theta^{\pm}(\tau)\left(\pm\frac{1}{2}f(\tau)[Z_\theta^{+}(\tau)]^{-1}+S_\gamma(f(\xi)[Z_\theta^{+}(\xi)]^{-1})(\tau)\right), \text{a.e.}\tau\in\gamma. \quad (3.2)$$

Let's first verify that $F_1^+\in G^{+}_{p)}$. As $F_1\in H^{+}_{p)}$, it suffices to show the validity of the inclusion $F_1^+\in G_{p)}$. From (3.2) we obtain

$$F_1^+(\tau) = \frac{1}{2}f(\tau) + \mathrm{Z}_\theta^+(\tau)S_\gamma(f(\xi)[\mathrm{Z}_\theta^+(\xi)]^{-1})(\tau), \text{ a.e. } \tau \in \gamma.$$

It follows that $F_1^+ \in G_{p)}$ if $\mathrm{Z}_\theta^+ S_\gamma(f[\mathrm{Z}_\theta^+]^{-1}) \in G_{p)}$. The results obtained in [12] imply

$$\left|\mathrm{Z}_\theta^+(e^{it})\right| \sim \rho(t) = \left|t^2 - \pi^2\right|^{-\frac{h_0}{2\pi}} \prod_{k=1}^{r} |t - s_k|^{-\frac{h_k}{2\pi}}, t \in [-\pi, \pi].$$

Thus, the inclusion $\mathrm{Z}_\theta^+ S_\gamma(f[\mathrm{Z}_\theta^+]^{-1}) \in G_{p)}$ is equivalent to the inclusion $\rho(t)S_\gamma(f\rho^{-1})(e^{it}) \in G_{p)}$. From the condition 2) of Theorem 2.5 it follows that

$$\exists\varepsilon_0 \in (0, p-1) : \rho \in A_{p-\varepsilon}, 0 \le \varepsilon \le \varepsilon_0. \tag{3.3}$$

It is known [12] that the singular operator $S_\gamma$ is bounded in the weighted space $L_{p,\rho}(-\pi, \pi)$ if and only if $\rho \in A_p$. Therefore it follows from (3.3) that $S_\gamma \in L(L_{p-\varepsilon,\rho})$, $0 \le \varepsilon \le \varepsilon_0$. Then, by Theorem 2.1, the operator $S_\gamma$ is bounded in the space $L_{p),\rho}(-\pi, \pi)$. Consequently, by Lemma 3.1, from $f\rho^{-1} \in G_{p),\rho}$ it follows that $S_\gamma(f\rho^{-1}) \in G_{p),\rho}$, and, consequently, $\rho S_\gamma(f\rho^{-1}) \in G_{p)}$. The validity of the inclusion $F_1^- \in_m G_{p)}^-$ can be shown similarly. Theorem is proved.

Theorem 3.1 has the following immediate corollary.

**Corollary 3.1.** *Let all conditions of Theorem 3.1 be fulfilled. Then for every $f \in G_{p)}(\gamma)$, $p > 1$, the problem (3.1) has a unique solution in the classes $GH_{p)}^+ \times {}_{-1}GH_{p)}^-$, defined by the formula (2.5).*

## 4. Basis properties of the system $\left\{e^{i(n-\beta signn)t}\right\}_{n\in Z}$ in the space $L_{p)}(-\pi, \pi)$

Consider the basicity of perturbed system of exponents of the form

$$E_\beta = \left\{e^{i(n-\beta signn)t}\right\}_{n\in Z} \tag{4.1}$$

in the space $L_{p)}(-\pi, \pi)$, where $\beta$ is some complex parameter. Rewrite (4.1) in the form of following double system:

$$E_\beta = \left\{x_n^+; x_k^-\right\}_{n\in Z_+, k\in N}, x_n^\pm = e^{\pm i(n-\beta)t}.$$

First we will study the basicity of the systems $\left\{e^{int}\right\}_{n\in Z_+}$ and $\left\{e^{-int}\right\}_{n\in N}$ in $G_{p)}(-\pi, \pi)$. The following theorem is true.

**Theorem 4.1.** *A system of exponents $\left\{e^{int}\right\}_{n\in Z_+}$ forms a basis for the space $G_{p)}^+(-\pi, \pi)$, $1 < p < +\infty$.*

*Proof.* Consider an arbitrary function $f \in G_{p)}^+(-\pi, \pi)$. Due to the basicity of $\left\{e^{int}\right\}_{n\in Z}$ for $G_{p)}(-\pi, \pi)$, the function$f$ has an expansion

$$f(t) = \sum_{n=-\infty}^{+\infty} a_n e^{int} \tag{4.2}$$

in $G_{p)}(-\pi,\pi)$. From $f \in L^+_{p)}(-\pi,\pi)$ it follows that the expansion (4.2) holds in the space $L^+_{p-\varepsilon}(-\pi,\pi)$ for every $\varepsilon \in (0, p-1)$. Therefore, $a_n = 0$ for $n < 0$. Consequently, the expansion of (4.2) has the form

$$f(t) = \sum_{n=0}^{+\infty} a_n e^{int}.$$

The uniqueness of such expansion follows from the existence of a system biorthogonal to $\left\{e^{int}\right\}_{n\in Z}$ in $G_{p)}(-\pi,\pi)$. Consequently, an arbitrary function $f \in G^+_{p)}(-\pi,\pi)$ can be expanded uniquely in a series with respect to the system $\left\{e^{int}\right\}_{n\in Z_+}$, i.e. the system $\left\{e^{int}\right\}_{n\in Z_+}$ forms a basis for the space $G^+_{p)}(-\pi,\pi)$. Theorem is proved.

Theorem below can be proved similarly.

**Theorem 4.2.** *A system of exponents $\left\{e^{-int}\right\}_{n\in N}$ forms a basis for the space ${}_{-1}G^-_{p)}(-\pi,\pi)$, $1 < p < +\infty$.*

Now we proceed to study the basis properties of the system (4.1). We will need the following lemma ([5]).

**Lemma 4.1.** *Let the inequality $|Re\beta| < \frac{1}{2}$ hold. Assume*

$$h^+_n(t) = \frac{e^{i\beta t}}{2\pi}(e^{it}+1)^{-2\beta}_- \sum_{k=0}^{n} C^{n-k}_{2\beta} e^{-ikt}, n \in Z_+,$$

$$h^-_n(t) = \frac{e^{i\beta t}}{2\pi}(e^{it}+1)^{-2\beta}_- \sum_{k=1}^{n} C^{n-k}_{2\beta} e^{-ikt}, n \in N,$$

*where $(e^{it}+1)^{-2\beta}_-$ is some single-valued branch of multivalued function $(e^{it}+1)^{-2\beta}$ and $C^k_n = \frac{n(n-1)(n-2).....(n-k+1)}{k!}$ are binomial coefficients. Then the equalities*

$$(x^+_n, h^+_k) = (x^-_{n+1}, h^-_{k+1}) = \delta_{nk},$$

$$(x^-_{n+1}, h^+_k) = (x^+_n, h^-_{k+}) = 0, n, k \in Z_+,$$

*hold and $(x,y) = \int_{-\pi}^{\pi} x(t)\overline{y(t)}dt$.*

Theorem below establishes the minimality of the system (4.1) in $L_{p)}(-\pi,\pi)$.

**Theorem 4.3.** *Let the following inequality hold:*

$$-1 < 2Re\beta < -\frac{1}{p} + 1. \tag{4.3}$$

*Then the system $E_\beta$ is minimal in the space $L_{p)}(-\pi,\pi)$, $1 < p < +\infty$.*

*Proof.* Lemma 4.1 implies that if the inclusion $\left\{h^+_n; h^-_k\right\}_{n\in Z_+, k\in N} \subset \left(L_{p)}\right)$ holds, then the system $\left\{h^+_n; h^-_k\right\}_{n\in Z_+, k\in N}$ is biorthogonal to $E_\beta = \left\{x^+_n; x^-_k\right\}_{n\in Z_+, k\in N}$. From

$$(e^{it}+1)^{-2\beta} = \exp\left\{-2\beta(\ln\left|2\sin\frac{\pi-t}{2}\right| + i\frac{t}{2} + 2\pi ki)\right\}, k \in Z,$$

it follows that

$$\left|h^\pm_n(t)\right| \sim \left|\sin\frac{\pi-t}{2}\right|^{-2Re\beta}, t \in [-\pi,\pi].$$

Therefore, by Theorem 2.2, the inclusion $\left\{h_n^+; h_k^-\right\}_{n\in Z_+, k\in N} \subset \left(L_{p)}\right)'$ is true if and only if $2Re\beta < -\frac{1}{p} + 1$. Thus, by inequality (4.3), the system $E_\beta$ has a conjugate system $\left\{h_n^+; h_k^-\right\}_{n\in Z_+, k\in N}$ in $\left(L_{p)}\right)^*$, and, consequently, is minimal in $L_{p)}(-\pi, \pi)$. Theorem is proved.

Now let's prove the main theorem on basicity of the system (4.1) for the space $G_{p)}(-\pi, \pi)$.

**Theorem 4.4.** *Let $2Re\beta + \frac{1}{p} \notin Z$, $1 < p < +\infty$. Then the system $E_\beta$ forms a basis for $G_{p)}(-\pi, \pi)$, $1 < p < +\infty$, if and only if $\left[2Re\beta + \frac{1}{p}\right] = 0$ ($[\alpha]$ is an integer part of $\alpha \in R$), $1 < p < +\infty$. The defect of the system $E_\beta$ is equal to $d(E_\beta) = \left[2Re\beta + \frac{1}{p}\right]$. In other words, when $d(E_\beta) < 0$, the system $E_\beta$ is not complete, but minimal in $G_{p)}(-\pi, \pi)$; when $d(E_\beta) > 0$, the system $E_\beta$ is complete, but not minimal in $G_{p)}(-\pi, \pi)$.*

*Proof.* Take an arbitrary function $f \in G_{p)}(-\pi, \pi)$. Consider the following nonhomogeneous Riemann problem in the class $GH_{p)}^+ \times {}_{-1}GH_{p)}^-$:

$$F^+(e^{it}) - e^{i2\beta t}F^-(e^{it}) = e^{i\beta t}f(t), \text{ a.e. } t \in [-\pi, \pi]. \tag{4.4}$$

Let the inequality

$$0 < 2Re\beta + \frac{1}{p} < 1$$

hold. Hence, $-\frac{1}{p} < 2Re\beta < -\frac{1}{p} + 1$, and therefore, by Corollary 3.1, the problem (4.4) has a unique solution in the class $GH_{p)}^+ \times {}_{-1}GH_{p)}^-$, which can be represented in the form of Cauchy type integral

$$F_1(z) = \frac{\mathrm{Z}(z)}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\beta t} f(t)}{\mathrm{Z}^+(e^{it})} K_z(t) dt, z \notin \gamma,$$

where $\mathrm{Z}(z)$ is a canonical solution of corresponding homogeneous problem and $K_z(t) = \frac{e^{it}}{e^{it} - z}$. Then the inclusions $F_1^+(e^{it}) \in G_{p)}^+$ and $F_1^-(e^{it}) \in_- G_{p)}^-$ are true. By Theorems 4.2 and 4.3, the functions $F_1^{\pm}(e^{it})$ have the following unique expansions:

$$F_1^+(e^{it}) = \sum_{n=0}^{\infty} a_n^+ e^{int},$$

$$F_1^-(e^{it}) = \sum_{n=1}^{\infty} a_n^- e^{-int}.$$

Taking into account these expansions in (4.4), we obtain the expansion of the function $f$ with respect to the system $E_\beta$ in $G_{p)}(-\pi, \pi)$. This expansion in unique, because, by Theorem 4.3, the system $E_\beta$ is minimal in $L_{p)}(-\pi, \pi)$. Thus, the system $E_\beta$ forms a basis for $G_{p)}(-\pi, \pi)$.

Conversely, let $\left[2Re\beta + \frac{1}{p}\right] \neq 0$. First consider the case $-1 < 2Re\beta + \frac{1}{p} < 0$. We have $-\frac{1}{p} < 2(Re\beta + \frac{1}{2}) < -\frac{1}{p} + 1$. Then, according to the facts proved above, the system $E_{\beta+\frac{1}{2}}$ forms a basis for $G_{p)}(-\pi,\pi)$. Let's make the following transformation:

$$E_{\beta+\frac{1}{2}}=\left\{e^{-i(\beta+\frac{1}{2})t}e^{int};e^{i(\beta+\frac{1}{2})t}e^{-ikt}\right\}_{n\in Z_+,k\in N}=$$

$$=\left\{e^{i\frac{t}{2}}e^{-i\beta t}e^{i(n-1)t};e^{i\frac{t}{2}}e^{i\beta t}e^{-ikt}\right\}_{n\in Z_+,k\in N}=$$

$$=e^{i\frac{t}{2}}\left\{e^{-i\beta t}e^{int};e^{i\frac{t}{2}}e^{i\beta t}e^{-ikt}\right\}_{n=-1,+\infty,k\in N}=e^{i\frac{t}{2}}E_{\beta}^{-}.$$

Hence, the system $E_{\beta}^{-}$ forms a basis for $G_{p)}(-\pi,\pi)$. Consequently, from

$$E_{\beta}^{-}=\left\{e^{-i(\beta+1)t}\right\}\bigcup E_{\beta}$$

it follows that the system $E_{\beta}$ is not complete, but minimal in $G_{p)}(-\pi,\pi)$, with its defect equal to 1. Absolutely similar to the previous case we obtain that if the inequality $-n_0<2Re\beta+\frac{1}{p}<-n_0+1$ holds for some $n_0\in N$, then the system $E_{\beta}$ is not complete, but minimal in $G_{p)}(-\pi,\pi)$ and its defect is equal to $n_0$.

Now let the inequality $1<2Re\beta+\frac{1}{p}<2$ holds. Then we have the inequality $-\frac{1}{p}<2(Re\beta-\frac{1}{2})<-\frac{1}{p}+1$. Consequently, the system $E_{\beta-\frac{1}{2}}$ forms a basis for $G_{p)}(-\pi,\pi)$. Similar to the above equalities, we obtain

$$E_{\beta-\frac{1}{2}}=e^{-i\frac{t}{2}}\left\{e^{-i\beta t}e^{int};e^{i\frac{t}{2}}e^{i\beta t}e^{-ikt}\right\}_{n,k\in N}=e^{-i\frac{t}{2}}E_{\beta}^{+},$$

$$E_{\beta}=\left\{e^{-i\frac{t}{2}}\right\}\bigcup E_{\beta}^{+}.$$

Therefore, the system $E_{\beta}^{+}$ forms a basis for $G_{p)}(-\pi,\pi)$, and the system $E_{\beta}$ is complete, but not minimal in $G_{p)}(-\pi,\pi)$. Consequently, the defect of the system $E_{\beta}$ is equal to 1. Continuing in this way, we establish that if the inequality $n_0<2Re\beta+\frac{1}{p}<n_0+1$ holds for some $n_0\in N$, then the system $E_{\beta}$ is complete, but not minimal in $G_{p)}(-\pi,\pi)$ and its defect is equal to $n_0$. Theorem is proved.

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