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# GLOBAL BIFURCATION FROM ZERO IN NONDIFFERENTIABLE PERTURBATIONS OF HALF-LINEAR FOURTH-ORDER EIGENVALUE PROBLEMS

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**Abstract**. In this paper, we consider nondifferentiable perturbations of a half-linear eigenvalue problems for ordinary differential equations of fourth order. The global bifurcation of these non-linear problems from the line of trivial solutions is studied. We prove the existence of two families of unbounded connected components of the solution set of these problems, bifurcating from some intervals of the line of trivial solutions and contained in classes of functions having nodal properties of half-eigenfunctions (and their derivatives) of a half-linear eigenvalue problems obtained from these nonlinear problems by equating the nonlinear terms to zero.

## 1. Introduction

We consider the following nonlinear eigenvalue problem

$$\begin{gathered}\ell y \equiv (p\,(x)\,y'')'' - (q(x)y')' + r(x)y = \lambda\tau(x)\,y\, + \alpha(x)y^+(x) + \\ \beta(x)y^-(x) + f(x,y,y',y'',y''',\lambda) + g(x,y,y',y'',y''',\lambda),\ x \in (0,l),\end{gathered} \tag{1.1}$$

$$y(0) = y'(0) = y(l) = y'(l) = 0, \tag{1.2}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, $p \in C^2\left([0,l];(0,+\infty)\right)\,[0,l]$, $q \in C^1\left([0,l];\right.$ $\left.[0,+\infty)\right)$, $\tau \in C\left([0,l];(0,+\infty)\right)$ and $r, \alpha,\, \beta \in C\left([0,l];\mathbb{R}\right)$, $\alpha \not\equiv -\beta$, and $y^+ =$ $\max\{y,0\}$, $y^- = (-y)^+$. Moreover, $f, g \in C\left([0,l]\times\mathbb{R}^5);\mathbb{R}\right)$ and satisfy the following conditions: there exist small positive constant $\varkappa$ and positive constant $M$ such that

$$\begin{gathered}\left|\frac{f(x,y,s,v,w,\lambda)}{y}\right| \leq M,\ x \in [0,l],\, (y,s,v,w) \in \mathbb{R}^4,\, |y|+|s|+|v|+|w| \leq \varkappa, \\ y \neq 0,\ \lambda \in \mathbb{R};\end{gathered} \tag{1.3}$$

$$g(x,y,s,v,w,\lambda) = o(|y|+|s|+|v|+|w|) \text{ as } |y|+|s|+|v|+|w| \to 0, \tag{1.4}$$

uniformly in $(x,\lambda) \in [0,l\,] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

By $(b.c.)$ we denote the set of differentiable functions on $[0,l]$ satisfying the boundary conditions (1.2).

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Let $E$ be the Banach space $C^3[0,l] \cap (b.c.)$ with the norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where $||u||_\infty = \max_{x\in[0,\,l]} |u(x)|$

The global bifurcation of nonlinear eigenvalue problem obtained from (1.1), (1.2) by setting $\alpha \equiv 0$, $\beta \equiv 0$, $r \equiv 0$, was considered in [1]. In [1], to preserve the nodal properties along continua of solutions of this problem bifurcating from zero, the author constructed classes $S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+\,,\,-\}$, of functions of the Banach space $E$. Note that functions from these classes have the nodal properties of eigenfunctions of the linear problem (1.1), (1.2) with $\alpha \equiv 0$, $\beta \equiv 0$, $r \equiv 0$, and their derivatives.

Half-linear eigenvalue problem

$$\begin{cases} \ell y = \lambda \tau(x) y + \alpha(x) y^+ + \beta(x) y^-, \; x \in (0,l), \\ y \in (b.c.), \end{cases} \tag{1.5}$$

which obtained from (1.1), (1.2) by setting $f \equiv 0$ and $g \equiv 0$ in a more general formulations (namely, for self-adjoint differential operators of $2m$th ($m \geq 1$) order under different boundary conditions) was considered in [12]. It follows from [12, Theorem 3.3] that problem (1.5) has two unbounded sequences of simple half-eigenvalues

$$\lambda_1^+ < \lambda_2^+ < \ldots < \lambda_k^+ < \ldots$$

and

$$\lambda_1^- < \lambda_2^- < \ldots < \lambda_k^- < \ldots .$$

For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the half-eigenfunction $y_k^\nu$ corresponding to the half-eigenvalue $\lambda_k^\nu$ is contained in $S_k^\nu$. Furthermore, aside from solutions on the collection of the half-lines $\{(\lambda_k^\nu, t y_k^\nu) : t > 0\}$ and trivial ones, problem (1.1), (1.2) with $f \equiv 0$, $g \equiv 0$, has no other solutions.

The oscillatory properties of eigenfunctions of linear and half-linear eigenvalue problems play an essential role in the study of local and global bifurcation of solutions of nonlinear eigenvalue problems. Note that these properties of eigenfunctions of the half-linear Sturm-Liouville problems (both with a spectral parameter and also without a spectral parameter in the boundary conditions) are studied by various methods in [2, 5, 6, 11, 12].

Half-linearizable problem (1.1), (1.2) with $f \equiv 0$ is studied in detail in [3], where it is proved that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the trivial solution $(\lambda_k^\nu, 0)$ is a unique bifurcation point of this problem which respect to the set $\mathbb{R} \times S_k^\nu$. Moreover, it is also shown there that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the continua bifurcating from the point $(\lambda_k^\nu, 0)$ is contained in $\mathbb{R} \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$. This result was also obtained in [8] only in a particular case. The global bifurcation from infinity for problem (1.1), (1.2) with $f \equiv 0$ was considered in [9] where it is proved that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the point $(\lambda_k^\nu, \infty)$ is a unique asymptotically bifurcation point of this problem which respect to the set $\mathbb{R} \times S_k^\nu$. Note that from $(\lambda_k^\nu, \infty)$ emanates a global continuum which is contained in $\mathbb{R} \times S_k^\nu$ in some neighborhood of this point. For the half-linearizable Sturm-Liouville problems, similar results obtained earlier in [11, 12].

The purpose of this paper is to study the structure of bifurcation points with respect to the line of trivial solutions and the global bifurcation of solutions from zero to the nonlinear problem (1.1), (1.2).

The rest of this paper is organized as follows. Section 2 proves that the set of bifurcation points of problem (1.1), (1.2) with respect to the sets $\mathbb{R}\times S_k^+$ and $\mathbb{R}\times S_k^-$ are non-empty and are contained in the some intervals of the line of trivial solutions. Section 3 shows that for each $k\in\mathbb{N}$ and each $\nu\in\{+\,,\,-\}$ the union of the connected components of the solutions set bifurcating from these intervals with respect to the set $\mathbb{R}\times S_k^\nu$ is contained in $\mathbb{R}\times S_k^\nu$ and is unbounded in $\mathbb{R}\times E$.

## 2. The structure of bifurcation points of problem (1.1), (1.2) with respect to the sets $\mathbb{R}\times S_k^\nu$, $k\in\mathbb{N}$, $\nu\in\{+\,,\,-\}$

In this section, we study the structure of bifurcation points of problem (1.1), (1.2) with respect to the sets $\mathbb{R}\times S_k^\nu$, $k\in\mathbb{N}$, $\nu\in\{+\,,\,-\}$.

We call a pair $(\lambda, y)\in\mathbb{R}\times C^4[0,l]$ a solution to the problem (1.1), (1.2) if it satisfies (1.1), (1.2) (similarly for other tasks below).

By [4, Theorems 5.4 and 5.5 ] the eigenvalues of the linear problem (1.1), (1.2) with $\alpha\equiv 0$, $\beta\equiv 0$, $r\equiv 0$, are positive. Therefore, there is a Green's function of the differential operator generated by the differential expression $(p\,(x)\,y'')''-(q(x)y')'$ and the boundary conditions (1.2). Then the problem (1.1), (1.2) reduces to a nonlinear operator equation with integral operators acting in $\mathbb{R}\times E$ (see, for example, [1]). Thus, we can study the structure of the solutions set of problem (1.1), (1.2) in $\mathbb{R}\times E$.

To study the local and global bifurcations of solutions from zero to problem (1.1), (1.2), by following [1, 5, 11, 12], we introduce the approximate problem

$$\begin{cases} \ell y=\lambda\tau(x)\,y\,+\alpha(x)y^+ +\beta(x)y^- + \\ f(x,|y|^\varepsilon,y',y'',y''',\lambda)+g(x,y,y',y'',y''',\lambda),\ x\in(0,l), \\ y\in(b.c.), \end{cases} \tag{2.1}$$

where $\varepsilon\in(0,1]$.

**Remark 2.1.** By condition (1.2) we get

$$f(x,|y|^\varepsilon y,s,v,w,\lambda)=o(|y|+|s|+|v|+|w|) \text{ as } |y|+|s|+|v|+|w|\to 0,$$

uniformly for $x\in[0,l]$ and $\lambda\in\mathbb{R}$. Therefore, for each $k\in\mathbb{N}$ and each $\nu\in\{+\,,\,-\}$ there exists an unbounded continuum $C_{k,\varepsilon}^\nu$ of solutions of problem (2.1), which contains $(\lambda_k^\nu,0)$, is contained in $(\mathbb{R}\times S_k^\nu)\cup\{(\lambda_k^\nu,0)\}$ and is unbounded in $\mathbb{R}\times E$.

Let

$$N_\alpha=\max_{x\in[0,\,l]}|\alpha(x)|,\ N_\beta=\max_{x\in[0,\,l]}|\beta(x)|,$$

and

$$I_k^+=\left[\lambda_k^+-\frac{N_\alpha+N_\beta+M}{\tau_0},\lambda_k^+ +\frac{N_\alpha+N_\beta+M}{\tau_0}\right],$$

$$I_k^-=\left[\lambda_k^- -\frac{N_\alpha+N_\beta+M}{\tau_0},\lambda_k^- +\frac{N_\alpha+N_\beta+M}{\tau_0}\right],$$

where $\tau_0=\min\limits_{x\in[0,l]}\tau(x)$.

In what follows, let $\delta>0$ be a sufficiently small fixed number.

**Lemma 2.1.** *For each $k\in\mathbb{N}$ and each $\nu\in\{+\,,\,-\}$ there exists a sufficiently small $\varrho_k^\nu>0$ such that for any $\varepsilon\in(0,1)$ problem* (2.1) *has no nontrivial solution*

$(\lambda, y)$ *that satisfying the following conditions:*

$$\operatorname{dist}\{\lambda, I_k^\nu\} > \delta,\ y \in S_k^\nu \ \textit{and}\ ||y||_3 < \varrho_k^\nu.$$

*Proof.* We will prove the theorem for some fixed (arbitrary) $k_0 \in \mathbb{N}$ and $\nu = +$ (the case $\nu = -$ is considered similarly).

Assume the contrary, i.e. there exist $\varepsilon \in (0, 1)$ and sufficiently large $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ problem (2.1) for $\varepsilon = \varepsilon_0$ has nontrivial solution $(\lambda_{n,0}, y_{n,0})$ which satisfy the following conditions:

$$\operatorname{dist}\{\lambda_{n,0}, I_{k_0}^+\} > \delta,\ y_{n,0} \in S_{k_0}^+ \ \text{and}\ ||y_{n,0}|||_3 < \frac{1}{n} < \delta. \tag{2.2}$$

Let $\psi_{n,0}(x)$ be the function defined by

$$\psi_{n,0}(x) = \begin{cases} \frac{f(x, |y_{n,0}|^{\varepsilon_0} y_{n,0}, y'_{n,0}, y''_{n,0}, y'''_{n,0}, \lambda_{n,0})}{y_{n,0}(x)} & \text{if } y_{n,0}(x) \neq 0, \\ 0 & \text{if } y_{n,0}(x) = 0. \end{cases} \tag{2.3}$$

By (2.3), it follows from (2.1) that for each $n \geq n_0$ the pair $(\lambda_{n,0}, y_{n,0})$ solves the half-linearizable eigenvalue problem

$$\begin{cases} \ell y = \lambda \tau(x)\, y + \alpha(x) y^+(x) + \beta(x) y^-(x) + \psi_{n,0}(x) y + \\ g(x, y, y', y'', y''', \lambda),\ x \in (0, l), \\ y \in (b.c.). \end{cases} \tag{2.4}$$

By [12, Theorem 3.3] the half-linear eigenvalue problem

$$\begin{cases} \ell y = \lambda \tau(x) y + \alpha(x) y^+ + \beta(x) y^- + \psi_{n,0}(x) y,\ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{2.5}$$

obtained from (2.5) by setting $g \equiv 0$ has two unbounded sequences of simple half-eigenvalues

$$\lambda_{1,n,0}^+ < \lambda_{2,n,0}^+ < \ \ldots\ < \lambda_{k,n,0}^+ < \ \ldots$$

and

$$\lambda_{1,n,0}^- < \lambda_{2,n,0}^- < \ \ldots\ < \lambda_{k,n,0}^- < \ \ldots\ .$$

Moreover, for each $k \in \mathbb{N}$ the half-eigenfunctions $y_{k,n,0}^+$ and $y_{k,n,0}^-$, which correspond to the half-eigenvalues $\lambda_{k,n,0}^+$ and $\lambda_{k,n,0}^-$ are contained in $S_k^+$ and $S_k^-$ respectively. Furthermore, aside from solutions on the collection of the half-lines $\{(\lambda_{k,n,0}^+, t y_{k,n,0}^+) : t > 0\}$ and $\{(\lambda_{k,n,0}^-, t y_{k,n,0}^-) : t > 0\}$, and trivial ones, problem (2.5) has no other solutions.

By [3, Lemma 3.3], $(\lambda_{k_0,n,0}^+, 0)$ is an unique bifurcation point of problem (2.5) with respect to the set $\mathbb{R} \times S_{k_0}^+$ . Moreover, according to [3, Theorem 3.1], from the point $(\lambda_{k_0,n,0}^+, 0)$ bifurcates an unbounded continuum $D_{k_0,n,0}^+$ contained in $\mathbb{R} \times S_{k_0}^+$. Then for every sufficiently large $n > n_0$ there exists a sufficiently small $\delta_n > 0$ such that

$$\delta_n < \delta \ \text{and}\ |\lambda_{n,0} - \lambda_{k_0,n,0}^+| < \delta_n. \tag{2.6}$$

Note that for any function $y \in E$ the following relations hold

$$y^+(x) + y^-(x) = |y(x)| \ \text{and}\ y^+(x) - y^-(x) = y(x),\ x \in [0, l],$$

which implies that

$$y^+(x) = \frac{|y(x)| + y(x)}{2} \text{ and } y^-(x) = \frac{|y(x)| - y(x)}{2},\ x \in [0, l].$$

Then $\lambda_{k_0}^+$ and $\lambda_{k_0,n,0}^+$ are $k_0$th eigenvalues of the eigenvalue problems

$$\begin{cases} \ell y = \lambda \tau(x)\, y(x) + \frac{1}{2}(\alpha(x) + \beta(x))(\operatorname{sgn} y_{k_0}^+(x))y(x) + \\ \frac{1}{2}(\alpha(x) - \beta(x))y(x),\ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{2.7}$$

and

$$\begin{cases} (\ell y)(x) = \lambda \tau(x)\, y(x) + \frac{1}{2}(\alpha(x) + \beta(x))(\operatorname{sgn} y_{k_0,n,0}^+(x))y(x) + \\ \frac{1}{2}(\alpha(x) - \beta(x))y(x) + \psi_{n,\,0}(x)y(x),\ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{2.8}$$

respectively.

According to the maximum-minimum property of eigenvalues [7, Ch. 6, § 1, p. 405-406] the eigenvalues $\lambda_{k_0}^+$ and $\lambda_{k_0,n,0}^+$ of problems (2.7) and (2.8) are defined by

$$\lambda_{k_0}^+ = \max_{V^{(k_0-1)}} \min_{y \in (b.c.)} \left\{ R_{\alpha,\,\beta,\,k_0}^+[y] \,:\, \int_0^1 \tau(x)y(x)\phi(x)dx = 0,\ \phi \in V^{(k_0-1)} \right\}, \tag{2.9}$$

and

$$\lambda_{k_0,n,0}^+ = \max_{V^{(k_0-1)}} \min_{y \in (b.c.)} \left\{ R_{\alpha,\,\beta,\,k_0,\,n,\,0}^+[y] \,:\, \int_0^1 \tau(x)y(x)\phi(x)dx = 0,\ \phi \in V^{(k_0-1)} \right\}, \tag{2.10}$$

respectively, where

$$R_{\alpha,\,\beta,\,k_0}^+[y] = \frac{\int_0^1 \{p(x)y''^2(x) + q(x)y'^2(x) + r(x)y^2(x)\}dx}{\int_0^1 \tau(x)y^2(x)dx} + \frac{\frac{1}{2}\int_0^1 (\alpha(x) + \beta(x)(\operatorname{sgn} y_{k_0}^+(x))y^2(x)dx + \frac{1}{2}\int_0^1 (\alpha(x) - \beta(x)y^2(x)\,dx}{\int_0^1 \tau(x)y^2(x)dx}, \tag{2.11}$$

$$R^{+}_{\alpha,\,\beta,\,k_0,n,0}[y] = \frac{\int\limits_0^1 \{p(x)y''^2(x)+q(x)y'^2(x)+r(x)y^2(x)\}dx}{\int\limits_0^1 \tau(x)y^2(x)dx} + \frac{\frac{1}{2}\int\limits_0^1 (\alpha(x)+\beta(x)(\operatorname{sgn} y^{+}_{k_0,\,n,\,0}(x))y^2(x)dx+\frac{1}{2}\int\limits_0^1 (\alpha(x)-\beta(x)y^2(x)\,dx}{\int\limits_0^1 \tau(x)y^2(x)dx} - \frac{\int\limits_0^1 \psi_{n,0}(x)y^2(x)dx}{\int\limits_0^1 \tau(x)y^2(x)dx}, \tag{2.12}$$

and $V^{(k-1)}$ is an arbitrary set of linearly independent functions $\phi_s \in (b.c.)$, $1 \le s \le k-1$.

It follows from (2.11) and (2.12) that

$$R^{+}_{\alpha,\,\beta,\,k_0,n,0}[y] = R^{+}_{\alpha,\,\beta,\,k_0}[y] + \frac{\frac{1}{2}\int\limits_0^1 (\alpha(x)+\beta(x)\left\{\operatorname{sgn}\, y^{+}_{k_0,\,n,\,0}(x) - \operatorname{sgn}\, y^{+}_{k_0}(x)\right\}y^2(x)dx}{\int\limits_0^1 \tau(x)y^2(x)dx} - \frac{\int\limits_0^1 \psi_{n,0}(x)y^2(x)dx}{\int\limits_0^1 \tau(x)y^2(x)dx}. \tag{2.13}$$

By condition (1.4) it follows from (2.3) that

$$|\psi_{n,0}(x)| \le M|y^{+}_{n,0}(x)|^{\varepsilon_0} < M,\ \ x \in [0,l]. \tag{2.14}$$

Then, in view of (2.14), from (2.13) we get

$$R^{+}_{\alpha,\,\beta,\,k_0}[y] - \tfrac{N_\alpha+N_\beta+M}{\tau_0} \le R^{+}_{\alpha,\,\beta,\,k_0,\,n,\,0}[y] \le R^{+}_{\alpha,\,\beta,\,k_0}[y] + \tfrac{N_\alpha+N_\beta+M}{\tau_0}, \tag{2.15}$$

which, with regard to (2.9)-(2.10), we obtain

$$\lambda^{+}_{k_0} - \frac{N_\alpha+N_\beta+M}{\tau_0} \le \lambda^{+}_{k_0,n,0} \le \lambda^{+}_{k_0} + \frac{N_\alpha+N_\beta+M}{\tau_0},$$

i.e.

$$|\lambda^{+}_{k_0,n,0} - \lambda^{+}_{k_0}| \le \frac{N_\alpha+N_\beta+M}{\tau_0}. \tag{2.16}$$

By inequalities (2.6) and (2.16) we have

$$|\lambda_{n,0} - \lambda^{+}_{k_0}| \le |\lambda_{n,0} - \lambda^{+}_{k_0,\,n,\,0}| + |\lambda^{+}_{k_0,\,n,\,0} - \lambda^{+}_{k_0}| < \tfrac{N_\alpha+N_\beta+M}{\tau_0} + \delta. \tag{2.17}$$

It follows from (2.17) that

$$\operatorname{dist}\{\lambda_{n,0}, I^{+}_{k_0}\} < \delta,$$

which contradicts the first relation in (2.2). The proof of this lemma is complete.

**Lemma 2.2.** *For each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the set of bifurcation points of problem* (1.1)-(1.2) *with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty. Moreover, if $(\lambda, 0)$ is a bifurcation point of this problem with respect to the set $\mathbb{R} \times S_k^\nu$, then $\lambda \in I_k^\nu$.*

*Proof.* By Lemma 2.1 and [3, Theorem 3.1] for each $k \in \mathbb{N}$, each $\nu \in \{+, -\}$, any $\varepsilon \in (0, 1)$ and any positive $\varrho < \varrho_k^\nu$ there exists a solution $(\lambda_{k,\nu,\varepsilon,\varrho}, y_{k,\nu,\varepsilon,\varrho})$ of problem (2.1) such that

$$\mathrm{dist}\,\{\lambda_{k,\,\nu,\,\varepsilon,\,\varrho}, I_k^\nu\} \leq \delta,\ y_{k,\,\nu,\,\varepsilon,\,\varrho} \in S_k^\nu \text{ and } ||y_{k,\,\nu,\,\varepsilon,\,\varrho}||_3 = \varrho.$$

Let $\{\varepsilon_n\}_{n=1}^\infty$, $\varepsilon_n \in (0, 1)$, be a sequence that converges to 0. Then, by (2.1), we get

$$\begin{cases} \ell y_{k,\nu,\varepsilon_n,\varrho} = \lambda_{k,\nu,\varepsilon_n,\varrho}\, \tau(x)\, y_{k,\nu,\varepsilon_n,\varrho} + \alpha(x) y^+_{k,\nu,\varepsilon_n,\varrho} + \beta(x) y^-_{k,\nu,\varepsilon_n,\varrho} + \\ f(x, |y_{k,\nu,\varepsilon,\varrho}|^{\varepsilon_n} y_{k,\nu,\varepsilon,\varrho}, y'_{k,\nu,\varepsilon,\varrho}, y''_{k,\nu,\varepsilon,\varrho}, y'''_{k,\nu,\varepsilon,\varrho}, \lambda_{k,\nu,\varepsilon_n,\varrho}) + \\ g(x, y_{\varepsilon_{k,\nu,\varepsilon,\varrho}}, y'_{k,\nu,\varepsilon,\varrho}, y''_{k,\nu,\varepsilon,\varrho}, y'''_{k,\nu,\varepsilon,\varrho}, \lambda_{k,\nu,\varepsilon_n,\varrho}),\ x \in (0, l), \\ y_{k,\nu,\varepsilon_n,\varrho} \in (b.c.), \end{cases} \tag{2.18}$$

By virtue of the conditions imposed on the functions $p$, $q$, $r$, $\tau$, $\alpha$, $\beta$, $f$ and $g$ from (2.18) we obtain

$$|y^{(4)}_{k,\nu,\varepsilon_n,\varrho}(x)| \leq C_k^\nu, \tag{2.19}$$

where

$$C_k^\nu = p_0^{-2} \varrho\{p_2 + 2p_1 + q_0 + q_1 + (\lambda_k^\nu + M_{\alpha,\beta})\tau_1 + N_\alpha + N_\alpha + M + 1\},$$

$p_s = \max\limits_{x\in[0,1]} |p^{(s)}(x)|$, $s = 0, 1, 2$, $q_s = \max\limits_{x\in[0,1]} |q^{(s)}(x)|$, $s = 0, 1$, $r_0 = \max\limits_{x\in[0,1]} |r(x)|$, $\tau_1 = \max\limits_{x\in[0,1]} |\tau(x)|$. Therefore, by the Arzelà-Ascoli theorem from the sequence $\{(\lambda_{k,\nu,\varepsilon_n,\varrho}, y_{k,\nu,\varepsilon_n,\varrho})\}_{n=1}^\infty$ we can extract a subsequence $\{(\lambda_{k,\nu,\varepsilon_{n_m},\varrho}, y_{k,\nu,\varepsilon_{n_m},\varrho})\}_{m=1}^\infty$ which converges to $(\lambda_{k,\nu,\varrho}, y_{k,\nu,\varrho})$ in $\mathbb{R} \times E$. Then, replacing $(\lambda_{k,\nu,\varepsilon_n,\varrho}, y_{k,\nu,\varepsilon_n,\varrho})$ by $(\lambda_{k,\nu,\varepsilon_{n_m},\varrho},\, y_{k,\nu,\varepsilon_{n_m},\varrho})$ in relations (2.18) and passing to the limit in these relations as $m \to \infty$, we fobtain that $(\lambda_{k,\nu,\varrho}, y_{k,\nu,\varrho})$ is a solution to problem (1.1)-(1.2) such that

$$\mathrm{dist}\,\{\lambda_{k,\,\nu,\,\varrho}, I_k^\nu\} \leq \delta,\ y_{k,\,\nu,\,\varrho} \in \overline{S_k^\nu} = S_k^\nu \cup \partial S_k^\nu \text{ and } ||y_{k,\,\nu,\,\varrho}||_3 = \varrho.$$

Since $||y_{k,\,\nu,\,\varrho}||_3 = \varrho$ it follows from [1, Lemma 1.1] that $y_{k,\,\nu,\,\varrho} \in S_k^\nu$.

Now let $\{\varrho_n\}_{n=1}^\infty$, $\varrho_n \in (0, \varrho)$, be a sequence that converges to 0. It follows from above that for any $n \in \mathbb{N}$ there exists a solution $(\lambda_{k,\nu,\varrho_n}, y_{k,\nu,\varrho_n})$ of problem (1.1)-(1.2) such that

$$\mathrm{dist}\,\{\lambda_{k,\,\nu,\,\varrho_n}, I_k^\nu\} \leq \delta,\ y_{k,\,\nu,\,\varrho_n} \in S_k^\nu, \text{ and } ||y_{k,\,\nu,\,\varrho_n}||_3 = \varrho_n.$$

By the condition $\mathrm{dist}\,\{\lambda_{k,\,\nu,\,\varrho_n}, I_k^\nu\} \leq \delta$ from the sequence $\{(\lambda_{k,\nu,\varrho_n}, y_{k,\nu,,\varrho_n})\}_{n=1}^\infty$ we can extract a subsequence $\{(\lambda_{k,\nu,,\varrho_{n_m}}, y_{k,\nu,\varrho_{n_m}})\}_{m=1}^\infty$ which converges to $(\lambda_{k,\nu}, 0)$ in $\mathbb{R} \times E$, i.e. $(\lambda_{k,\nu}, 0)$ is a bifurcation point of problem (1.1)-(1.2) with respect to the set $\mathbb{R} \times S_k^\nu$.

To complete the proof of the theorem, we show that $\lambda_{k,\nu} \in I_k^\nu$. Assume the contrary, i.e. let $\lambda_{k,\nu} \notin I_k^\nu$. Then $\mathrm{dist}\,\{\lambda_{k,\nu}, I_k^\nu\} > 0$, and let $\rho_{k,\nu} = \mathrm{dist}\,\{\lambda_{k,\nu}, I_k^\nu\}$. It is obvious that there exists $m_{k,\,\nu} \in \mathbb{N}$ such that for any $m \geq m_{k,\,\nu}$ we have $|\lambda_{k,\nu,,\varrho_{n_m}} - \lambda_{k,\nu}| < \frac{1}{2}\rho_{k,\nu}$, and consequently, $\mathrm{dist}\,\{\lambda_{k,\nu,,\varrho_{n_m}}, I_k^\nu\} > \frac{1}{2}\rho_{k,\nu}$.

Note that $(\lambda_{k,\nu,,\varrho_{n_m}}, y_{k,\nu,\varrho_{n_m}})$ solves the following half-linearizable problem

$$\begin{cases} \ell y = \lambda\, \tau(x)\, y\, + \alpha(x) y^+ + \beta(x) y^- + \psi_{k,\nu,m} y + \\ g(x, y, y', y'', y''', \lambda),\ x \in (0, l), \\ y \in (b.c.), \end{cases} \tag{2.20}$$

where

$$\psi_{k,\nu,m}(x) = \begin{cases} \frac{f(x, y_{k,\nu,\rho_{n_m}}, y'_{k,\nu,\rho_{n_m}}, y''_{k,\nu,\rho_{n_m}}, y'''_{k,\nu,\rho_{n_m}}, \lambda_{k,\nu,\rho_{n_m}})}{y_{k,\nu,\rho_{n_m}}(x)} & \text{if } \ y_{k,\nu,\rho_{n_m}}(x) \neq 0, \\ 0 & \text{if } \ y_{k,\nu,\rho_{n_m}}(x) = 0. \end{cases}$$

By [3, Lemma 3.3] for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the point $(\lambda^{\nu}_{k,\nu,m,\psi}, 0)$ is a unique bifurcation point of problem (2.20) with respect to the set $\mathbb{R} \times S^{\nu}_k$, where $\lambda^{\nu}_{k,\nu,m,\psi}$ is a half-eigenvalue of the half-linear eigenvalue problem (2.20) with $g \equiv 0$. Then we can choose $m \geq m_{k,\nu}$ so large that

$$|\lambda^{\nu}_{k,\nu,\rho_{n_m}} - \lambda^{\nu}_{k,\nu,m,\psi}| < \frac{1}{2}\rho^{\nu}_k$$

holds. Moreover, by following the arguments in the proof of Lemma 2.2 (see (2.16)) we can show that

$$|\lambda^{\nu}_{k,\nu,m,\psi} - \lambda^{\nu}_k| \leq \frac{N_\alpha + N_\beta + M}{\tau_0}.$$

Then from the last two relation for the above $m \geq m_{k,\nu}$ we obtain

$$|\lambda^{\nu}_{k,\nu,\rho_{n_m}} - \lambda^{\nu}_k| < \frac{N_\alpha + N_\beta + M}{\tau_0} + \frac{1}{2}\rho^{\nu}_k,$$

whence implies that $\operatorname{dist}\{\lambda_{k,\nu,,\varrho_{n_m}}, I^{\nu}_k\} < \frac{1}{2}\rho^{\nu}_k$ which contradicts the relation $\operatorname{dist}\{\lambda_{k,\nu,,\varrho_{n_m}}, I^{\nu}_k\} > \frac{1}{2}\rho^{\nu}_k$. The lemma is proved.

## 3. Global bifurcation of solutions of problem (1.1)-(1.2)

Denote by $D \subset \mathbb{R} \times E$ the closure of the set of nontrivial solutions of the nonlinear problem (1.1), (1.2).

For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ let $\tilde{D}^{\nu}_k$ be the union of all the components of the set $D$ which bifurcating from the interval $I^{\nu}_k \times \{0\}$ with respect to the set $\mathbb{R} \times S^{\nu}_k$. Using Lemma 2.2 and following the arguments in Theorem 3.1 of [12] we can show that $\tilde{D}^{\nu}_k \neq \emptyset$. Let now $D^{\nu}_k = \tilde{D}^{\nu}_k \cup (I^{\nu}_k \times \{0\})$. Should be noted that the set $\tilde{D}^{\nu}_k$ may not be connected in $\mathbb{R} \times E$, while $D^{\nu}_k$ is connected in $\mathbb{R} \times E$.

The main result of this paper is the following theorem.

**Theorem 3.1.** *For each $k \in \mathbb{N}$ the sets $D^+_k$ and $D^-_k$ are contained in $(\mathbb{R} \times S^+_k) \cup (I^+_k \times \{0\})$ and $(\mathbb{R} \times S^-_k) \cup (I^-_k \times \{0\})$, respectively, and are unbounded in $\mathbb{R} \times E$.*

*Proof.* We will prove the theorem for the set $D^+_k$, $k \in \mathbb{N}$ (for the set $D^-_k$, $k \in \mathbb{N}$, the theorem is proved similarly).

If $(\lambda, y) \in D^+_k$ and $y \in \partial S^+_k$, then by Lemma 1.1 of [1] we get $y \equiv 0$. Hence $(\lambda, y) \in D^+_k \cap (\mathbb{R} \times \{0\})$. It follows from Lemma 2.2 that $(D^+_k \cap (\mathbb{R} \times \{0\})) \subset (I^+_k \times \{0\})$. Therefore, $D^+_k \subset (\mathbb{R} \times S^+_k) \cup (I^+_k \times \{0\})$.

Now suppose that $D_k^+$ is bounded in $\mathbb{R}\times E$. Then by the estimate (2.19) it follows that $D_k^+$ is compact in $\mathbb{R}\times E$. Hence by [10, Lemma 1.2] there exists a neighborhood $Q_k^+$ of the set $D_k^+$ such that

$$\partial Q_k^+ \cap D \cap (\mathbb{R}\times S_k^+) = \emptyset. \tag{3.1}$$

By Remark 2.1 for any $\varepsilon \in (0,1)$ there exists $(\lambda_{k,\nu,\varepsilon}^+, y_{k,\nu,\varepsilon}^+) \in C_{k,\varepsilon}^+ \cap \partial Q_k^+$. In view of (2.19) the set $\{(\lambda_{k,\nu,\varepsilon}^+, y_{k,\nu,\varepsilon}^+)\}_{\varepsilon\in(0,1)}$ is precompact in $\mathbb{R}\times E$. Consequently, we can find a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ converging to 0 such that $(\lambda_{k,\nu,\varepsilon}^+, y_{k,\nu,\varepsilon_n}^+) \to (\lambda_{k,\nu,*}^+, y_{k,\nu,*}^+)$ which is a solution to problem (1.1), (1.2). Note that $y_{k,\nu,*}^+ \in \overline{S_k^\nu} = S_k^\nu \cup \partial S_k^\nu$. If $y_{k,\nu,*}^+ \in \partial S_k^\nu$, then by the above arguments we get $\lambda_{k,\nu,*}^+ \in I_k^+$, which implies that $(\lambda_{k,\nu,*}^+, 0) \notin \partial Q_k^+$, a contradiction. If $y_{k,\nu,*}^+ \in S_k^\nu$, then we have $\partial Q_k^+ \cap D \cap (\mathbb{R}\times S_k^+) \neq \emptyset$ which contradicts the condition (3.1). The proof of this theorem is complete.

**Remark 3.1.** The statements of Theorem 3.1 are also hold under the following boundary conditions:

(i) $y(0)=y'(0)=y(l)=y''(l)=0$; (ii) $y(0)=y'(0)=y'(l)=y'''(l)=0$;
(iii) $y(0)=y'(0)=y''(l)=y'''(l)=0$; (iv) $y(0)=y''(0)=y(l)=y''(l)=0$;
(v) $y(0)=y''(0)=y(l)=y'(l)=0$; (vi) $y(0)=y''(0)=y'(l)=y'''(l)=0$;
(vii) $y'(0)=y'''(0)=y(l)=y'(l)=0$; (viii) $y'(0)=y'''(0)=y(l)=y''(l)=0$;
(ix) $y''(0)=y'''(0)=y(l)=y'(l)=0$; (x) $y''(0)=y'''(0)=y(l)=y''(l)=0$.

**Remark 3.2.** If $g\equiv 0$, then the assertions of Theorem 3.1 hold.

**Remark 3.3.** Let $g\equiv 0$ and condition (1.3) is satisfied for all $(x,y,s,v,w,\lambda)\in[0,l]\times\mathbb{R}^5$. If $(\lambda,y)\in\mathbb{R}\times E$ is a solution of problem (1.1), (1.2) such that $y\in S_k^\nu$, $k\in\mathbb{N}$, $\nu\in\{+,-\}$, then the proof of Lemmas 2.1, 2.2 implies that $\lambda\in I_k^\nu$. Therefore, it follows from Theorem 3.1 and Remark 3.2 that

$$D_k^\nu \subset (I_k^\nu\times S_k^\nu)\cup(I_k^\nu\times\{0\}).$$

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