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LONG-RUN BEHAVIOR OF MULTIVARIATE MEANS

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Abstract. In this paper, we considered a sequence of real numbers to which we associate the sequence of averages with respect to a given weighted arithmetic multivariate mean and to Bajraktarević mean. Then we studied the long-run behavior of these averages' sequences. A special focus was given to bounded sequences with finite number of accumulation points. In this case we proved, our main result, that is, although the initial sequence is not convergent, the associated sequence of averages converges to the average of its accumulation points, in both cases, the weighted arithmetic and the Bajraktarević multivariate means.

1. Introduction and basic notions

The theory of means is as old as Pythagoras, the Greek Philosopher and Mathematician. He has introduced the arithmetic, geometric and harmonic means, at about 520BC. Two hundred years later, Pappus of Alexandria introduced many other means and he proved the well known inequalities that connect the arithmetic, geometric and harmonic means. See [6] for an extensive historical point.

More recently, many researchers have investigated new families and types of means, see [2, 5] for an exhaustive survey. It goes without saying that mean theory is in the middle of many others and has real interactions as well as applications therein, like the theories of inequalities, functional equations, and last but not least, probability & statistics. For more details about all these interactions, we give some references [1, 3, 10, 11, 12].

In the current paper, we are interested in multivariate means and more precisely to weighted quasi-arithmetic and to Bajraktarević multivariate means. Our paper is organized as follows: In the current section, after a brief historical point we introduce some notations and definitions. In Section 2 we state some technical lemmas that will be needed throughout the next section, we have made this choice to make the reader comfortable while exploring the proofs of main results. Section 3 is devoted to our main focus, that is the study of the long-run behavior of the sequence of averages with respect to a given multivariate mean, of a given initial sequence of real numbers which we suppose, not-convergent but with a finite number of accumulation points, see Theorem 3.2, Corollary 3.2 and Theorem 3.3.

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Now, let us state some definitions and notations that will be used throughout the paper. I will always denote a nonempty open real interval. For a fixed integer $n \geq 2$, an n-variate mean $\mathcal{M}_n$ is a function of n-variables that satisfies, for any $(x_1, \cdots, x_n) \in I^n$,

$$\min(x_1, \cdots, x_n) \leq \mathcal{M}_n(x_1, \cdots, x_n) \leq \max(x_1, \cdots, x_n).$$

It is said to be symmetric if

$$\mathcal{M}_n(x_1, \cdots, x_n) = \mathcal{M}_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

for every permutation σ of the set $\{1, 2, ..., n\}$.
We say that $\mathcal{M}_n$ is 1-homogeneous if for all permissible $t \in \mathbb{R}$

$$\mathcal{M}_n(tx_1, \cdots, tx_n) = t\mathcal{M}_n(x_1, \cdots, x_n).$$

A multivariate mean $\mathcal{M}$ is simply the given of a sequence of n-variate mean, i.e

$$\mathcal{M} = \{\mathcal{M}_n\}_{n\geq 2},$$

where for each $n \geq 2$, $\mathcal{M}_n$ is a given n-variate mean.
Our fundamental and generic example of multivariate mean is the well known arithmetic mean $\mathcal{A} = \{\mathcal{A}_n\}_{n\geq 2}$, defined for $n \geq 2$ and $x_1, \cdots, x_n > 0$, by

$$\mathcal{A}_n(x_1, \cdots, x_n) = \frac{x_1 + \cdots + x_n}{n}. \tag{1.1}$$

Actually, the arithmetic mean is the most known and the most used multivariate mean, this is because of its simplicity, its algebraic compatibility, and its statistical interpretation. In the sequel, we shall introduce three main ways to generalize the arithmetic multivariate mean. For this, let Φ be a continuous and strictly monotonic function $\Phi : I \longrightarrow \mathbb{R}$, and for $n \geq 2$ and a given family of real numbers $(x_1, \cdots, x_n) \in I^n$, we consider the following definitions:

(i) The quasi-arithmetic n-variate mean

$$\mathcal{A}_n^{\Phi}(x_1, \cdots, x_n) = \Phi^{-1}\left(\frac{\Phi(x_1) + \cdots + \Phi(x_n)}{n}\right). \tag{1.2}$$

(ii) The weighted arithmetic n-variate mean

$$\mathcal{A}_n^{(\Phi,\gamma)}(x_1, \cdots, x_n) = \Phi^{-1}\left(\frac{\gamma_1\Phi(x_1) + \cdots + \gamma_n\Phi(x_n)}{\sum_{i=1}^n \gamma_i}\right), \tag{1.3}$$

where $\gamma = \{\gamma_1, \gamma_2, \cdots\}$ is a sequence of strictly positive numbers.

(iii) The n-variate Bajraktarević mean

$$\mathcal{A}_n^{(\Phi,p)}(x_1. \cdots, x_n) = \Phi^{-1}\left(\frac{p(x_1)\Phi(x_1) + \cdots + p(x_n)\Phi(x_n)}{\sum_{i=1}^n p(x_i)}\right), \tag{1.4}$$

where $p : I \to (0, +\infty)$ is a given strictly positive function called weight function.

In [7] and [8], one can find a good analytical characterization of quasi-arithmetic multivariate means. It is clear that quasi-arithmetic mean (1.2) can be seen as a particular case of weighted mean (1.3) and of Bajraktarević's mean (1.4), however the weighted multivariate mean doesn't fit into Bajraktarević's formulation.
Next we illustrate some examples of interest, in particular we recall that the class

of quasi-arithmetic means and more generally the Bajraktarević's class contains many known multivariate means.

Examples 1. (i) We get back the classical arithmetic mean by letting $\Phi(x) = x$ in (1.2)

$$\mathcal{A}_n^{\Phi}(x_1, \cdots, x_n) = \mathcal{A}_n(x_1, \cdots, x_n) = \frac{\sum_{i=1}^n x_i}{n}.$$

(ii) By choosing $\Phi(x) = \ln(x)$ we get back the Geometric multivariate mean

$$\mathcal{A}_n^{\Phi}(x_1, \cdots, x_n) = \mathcal{G}_n(x_1, \cdots, x_n) = \sqrt[n]{\prod_{i=1}^n x_i}.$$

(iii) If we choose $\Phi(x) = 1/x$ we get the Harmonic multivariate mean

$$\mathcal{A}_n^{\Phi}(x_1, \cdots, x_n) = \mathcal{H}_n(x_1, \cdots, x_n) = \left(\frac{\sum_{i=1}^n x_i^{-1}}{n}\right)^{-1}.$$

(iv) Let $r, s \in \mathbb{R}$ and for $x > 0$ set

$$\Phi(x) = \begin{cases} x^{\max(r,s)-\min(r,s)} & \text{if } r \neq s \\ \ln(x) & \text{if } r = s \end{cases}$$

and $p(x) = x^{\min(r,s)}$. If we apply the Bajraktarević formulation (1.4) we get the sophisticated Gini mean

$$\mathcal{A}^{(\Phi,p)}(x_1, \cdots, x_n) = G_n^{(r,s)}(x_1, \cdots, x_n) = \begin{cases} \left(\frac{\sum_{i=1}^n x_i^r}{\sum_{i=1}^n x_i^s}\right)^{\frac{1}{r-s}} & \text{if } r \neq s \\ \left(\prod_{i=1}^n x_i^{x_i^s}\right)^{\frac{1}{\sum_{i=1}^n x_i^s}} & \text{if } r = s \end{cases} \tag{1.5}$$

2. Some needed lemmas

In order to state our main results, we need more notations and some technical lemmas. Let us denote by $\mathbb{N}$ the set of all positive integers, and for a given infinite subset $N \subset \mathbb{N}$, we denote by x_N the sub-sequence of $\{x_n\}_{n\in\mathbb{N}}$ indexed by elements of N and we recall that there should be a strictly increasing function $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ such that its range $R(\varphi) = N$. Let $n \in \mathbb{N}$ be fixed and set $N_n = \{k \in N : k \leq n\}$. The notation φ^+ refers to the sub-inverse of φ defined as follows

$$\varphi^+(n) = \sup_{k\in N=R(\varphi)} \left\{\varphi^{-1}(k) : k \leq n\right\}.$$

Then we get

$$|N_n| = \varphi^+(n),$$

were $|N_n|$ denotes the cardinal of N_n. Our first needed lemma states as follows.

Lemma 2.1. *Let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing function and suppose that it satisfies $\lim_{n\to+\infty} \frac{\varphi(n)}{n} = l$. Then we have*

$$l \in [1, +\infty] \quad \text{and} \quad \lim_{n\to+\infty} \frac{\varphi^+(n)}{n} = \frac{1}{l}.$$

Proof. Since φ is strictly increasing we get by induction $\varphi(n) \geq n$, this implies immediately that $l \in [1, +\infty]$, and by using the definition of $\varphi^+(n)$ we see that $\lim\limits_{n \to +\infty} \varphi^+(n) = +\infty$ and that $\varphi\left(\varphi^+(n)\right) \leq n < \varphi\left(\varphi^+(n) + 1\right)$. So, by dividing by $\varphi^+(n)$ we finish the proof. □

The second needed lemma reads as follows.

Lemma 2.2. *Let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing function such that*

$$\limsup_{n \to +\infty} \frac{\varphi(n)}{n} = l \in [1, +\infty].$$

Then there exist a subsequence $\{\psi(n)\}_{n \in \mathbb{N}}$ of $\{\varphi(n)\}_{n \in \mathbb{N}}$ and $l^* \in [1, +\infty]$ such that

$$\lim_{n \to +\infty} \frac{\psi(n)}{n} = l^*.$$

Proof. Since $\varphi : \mathbb{N} \to \mathbb{N}$ is strictly increasing then for all $p \geq 1$ we have

$$\varphi(n + p) \geq \varphi(n) + p.$$

Let ψ be the subsequence of φ defined as follows

$$\begin{cases} \psi(1) & = & \varphi(1) \\ \psi(n+1) & = & \varphi\Big(1 + \Sigma_{i=1}^{n} \psi(i)\Big) \end{cases},$$

we have

$$\psi(n+1) \geq \varphi\Big(1 + \Sigma_{i=1}^{n-1} \psi(i)\Big) + \psi(n) = 2\psi(n) \geq \frac{n+1}{n} \psi(n).$$

Then the sequence $\{\frac{\psi(n)}{n}\}_{n \in \mathbb{N}}$ is increasing, and so $\lim\limits_{n \to +\infty} \frac{\psi(n)}{n} = l^*$ for some $l^* \in [1, +\infty[$, which finishes the proof. □

Lemma 2.3. *Let $\{x_n\}_{n \in \mathbb{N}}$ be a bounded sequence of elements of I and let x_∞ be one of its accumulation points. Then there exists a sub-sequence x_N that converges to x_∞ such that*

$$\frac{|N_n|}{n} \xrightarrow[n \to +\infty]{} \theta,$$

for some $\theta \in [0, 1]$.

Proof. Let x_M be a sub-sequence that converges to x_∞ and let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be an increasing function such that $M = R(\varphi)$.

Let $\limsup\limits_{n \to +\infty} \frac{\varphi(n)}{n} = l \in [1, +\infty]$ then due to Lemma 2.2, there exists $\{\psi(n)\}$, a subsequence of $\{\varphi(n)\}$, such that $\lim\limits_{n \to +\infty} \frac{\psi(n)}{n} = l^* \in [1, +\infty]$. Denote by $N = R(\psi)$ and by $N_n = \{k \in N : k \leq n\}$, then by using Lemma 2.1 we obtain

$$\frac{|N_n|}{n} = \frac{\psi^+(n)}{n} \xrightarrow[n \to +\infty]{} \frac{1}{l^*} := \theta \in [0, 1].$$

We choose x_N as the desired subsequence of $\{x_n\}_{n \in \mathbb{N}}$, which finishes the proof. □

Lemma 2.4. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of elements of I, x_∞ be one of its accumulation points and x_N be the sub-sequence given in Lemma 2.3. And let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a given sequence of weights that converges to $l \in (0,+\infty)$. Then there exists $\theta \in [0,1]$ such that*

$$\lim_{n\to+\infty} \frac{\gamma^*_{N_n}}{\gamma^*_n} = \theta, \tag{2.1}$$

where

$$\gamma^*_n = \sum_{k=1}^{n} \gamma_k \text{ and } \gamma^*_{N_n} = \sum_{k\in N_n} \gamma_k.$$

Proof. Write

$$\frac{\gamma^*_{N_n}}{\gamma^*_n} = \frac{\gamma^*_{N_n}}{|N_n|} \times \frac{|N_n|}{n} \times \frac{n}{\gamma^*_n}.$$

The $\lim\limits_{n\to+\infty} \dfrac{|N_n|}{n} = \theta$, for some $\theta \in [0,1]$ due to Lemma 2.3. The limits of the two other ratio exist by using Soltz-Cézaro Theorem, and worth l and $\dfrac{1}{l}$ respectively, which finishes the proof. □

3. Long run average behavior

Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of elements of I and $\mathcal{M} = \{\mathcal{M}_n\}_{n\geq 2}$ a multivariate mean defined on I, we associate a sequence of averages

$$\{\mathcal{M}_n(x_1, \cdots, x_n)\}_{n\geq 2}, \tag{3.1}$$

such sequence of averages is called $\mathcal{M}$-averages.
Our first result, see Corollary 3.1, shows that if the multivariate mean considered is either the weighted-arithmetic or the Bajraktarević mean, then the associated sequence of $\mathcal{M}$-averages behaves asymptotically the same as the initial sequence $\{x_n\}_{n\in\mathbb{N}}$. Actually it is a direct application of Soltz-Cézaro Theorem, which we recall here.

Theorem 3.1. [4]
If $\{v_n\}_{n\in\mathbb{N}}$ is a positive, strictly increasing and unbounded sequence and $\{u_n\}_{n\in\mathbb{N}}$ is an arbitrary real sequence, then

$$\liminf_{n\to+\infty} \frac{u_{n+1}-u_n}{v_{n+1}-v_n} \leq \liminf_{n\to+\infty} \frac{u_n}{v_n} \leq \limsup_{n\to+\infty} \frac{u_n}{v_n} \leq \limsup_{n\to+\infty} \frac{u_{n+1}-u_n}{v_{n+1}-v_n}$$

Corollary 3.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of I and Φ a strictly monotonic function defined on I.*

(i) If $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of weights that satisfies moreover $\sum_{n\geq 1} \gamma_n = +\infty$, then

$$\liminf_{n\to+\infty} x_n \leq \liminf_{n\to+\infty} \mathcal{A}^{(\Phi,\gamma)}_n(x_1, \cdots, x_n) \leq \limsup_{n\to+\infty} \mathcal{A}^{(\Phi,\gamma)}_n(x_1, \cdots, x_n) \leq \limsup_{n\to+\infty} x_n. \tag{3.2}$$

(ii) If $p : I \longrightarrow (0,+\infty)$ is a weight function that is bounded below by $c > 0$, then

$$\liminf_{n\to+\infty} x_n \le \liminf_{n\to+\infty} \mathcal{A}_n^{(\Phi,p)}(x_1,\cdots,x_n) \le \limsup_{n\to+\infty} \mathcal{A}_n^{(\Phi,p)}(x_1,\cdots,x_n) \le \limsup_{n\to+\infty} x_n. \tag{3.3}$$

(iii) If moreover the initial sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then both $\mathcal{M}$-average sequences are convergent and

$$\lim_{n\to+\infty} \mathcal{A}_n^{(\Phi,\gamma)}(x_1,\cdots,x_n) = \lim_{n\to+\infty} \mathcal{A}_n^{(\Phi,p)}(x_1,\cdots,x_n) = \lim_{n\to+\infty} x_n. \tag{3.4}$$

Proof. By using the property that for any sequence of real numbers $\{u_n\}_{n\in\mathbb{N}}$, $\limsup(-u_n) = -\liminf(u_n)$ we only need to prove the right hand side of inequalities (3.2) and (3.3). Also, since $\mathcal{A}_n^{(-\Phi,\gamma)} = \mathcal{A}_n^{(\Phi,\gamma)}$ and $\mathcal{A}_n^{(-\Phi,p)} = \mathcal{A}_n^{(\Phi,p)}$, we can suppose that Φ is increasing without loss of generality. Then the proof becomes straightforward from Theorem 3.1 with

(i) $u_n = \gamma_1\Phi(x_1) + \cdots + \gamma_n\Phi(x_n)$ and $v_n = \sum_{i=1}^n \gamma_i$.
(ii) $u_n = p(x_1)\Phi(x_1) + \cdots + p(x_n)\Phi(x_n)$ and $v_n = \sum_{i=1}^n p(x_i)$.

Where we have used interchangeability between the two functions Φ, Φ^{-1} and both inferior and superior limits. Indeed, for any given sequence of real numbers $\{u_n\}_{n\in\mathbb{N}}$, $\Phi(\limsup_n u_n) = \Phi(\sup_n \inf_{k\ge n} u_k) = \sup_n \inf_{k\ge n} \Phi(u_k) = \limsup_n \Phi(u_n)$, where we have supposed that Φ is increasing, same for Φ^{-1}. The conditions on γ_n and on p guarantee the assumption on v_n. □

The next Example shows a case where the initial sequence doesn't converge but the averages' sequence does! This leads immediately to the main question of the current work, see Question 1 below.

Example 3.1. Consider the alternate sequence $x_n = (-1)^n$, then

$$\mathcal{A}_{2n}^{\Phi}(x_1,\cdots,x_{2n}) = \Phi^{-1}\left[\frac{\Phi(-1)+\Phi(1)+\cdots+\Phi(1)}{2n}\right] = \Phi^{-1}\left[\frac{\Phi(-1)+\Phi(1)}{2}\right]$$

and

$$\mathcal{A}_{2n+1}^{\Phi}(x_1,\cdots,x_{2n+1}) = \Phi^{-1}\left[\frac{n(\Phi(-1)+\Phi(1))}{2n+1} + \frac{\Phi(-1)}{2n+1}\right].$$

So it is easy to see that

$$\lim_{n\to+\infty} \mathcal{A}^{\Phi}(x_1,\cdots,x_n) = \Phi^{-1}\left[\frac{\Phi(-1)+\Phi(1)}{2}\right].$$

Question 1. If we suppose now that the initial sequence $\{x_n\}_{n\in\mathbb{N}}$ is only bounded but not convergent, what can we say about the averages' sequence, in both cases weighted quasi-arithmetic and Bajraktarević ?

Since our initial sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded, it has either a finite or an infinite number of accumulation points. The next theorem answers the above question when our initial sequence has a finite number of accumulation points.

Theorem 3.2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of elements of I, that has a finite number of accumulation points $\{x_\infty^{(1)}, \cdots, x_\infty^{(d)}\}$ for some integer $d \geq 2$. Given Φ a continuous and strictly monotonic function defined on I and $\{\gamma_n\}_{n\geq 1}$ a sequence of weights that satisfies the condition of Lemma 2.4. Then there exists a family $\theta_1, \cdots, \theta_d$ in $[0,1]$ that satisfies $\theta_1 + \cdots + \theta_d = 1$, such that*

$$\mathcal{A}_n^{(\Phi,\gamma)}(x_1, \cdots, x_n) \xrightarrow[n\to+\infty]{} \Phi^{-1}\left(\sum_{i=1}^{d} \theta_i \Phi(x_\infty^{(i)})\right).$$

Proof. We note that the proof is recursive so we just need to consider the case $d = 2$, also because Φ and Φ^{-1} are both continuous, it is enough to consider $\Phi = id$ the identity function. Let $x_{N^{(1)}}$ be a sub-sequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to $x_\infty^{(1)}$ and such that there exists $\theta_1 \in [0,1]$ for which

$$\frac{\gamma^*_{N_n^{(1)}}}{\gamma_n^*} \xrightarrow[n\to+\infty]{} \theta_1, \tag{3.5}$$

where existence of such sub-sequence is guaranteed by Lemma 2.4. Let $N^{(2)} \subset \mathbb{N}\backslash N^{(1)}$ be an infinite subset such that $x_{N^{(2)}}$ converges to $x_\infty^{(2)}$, again by using Lemma 2.4 there exists $\theta_2 \in [0,1]$ such that

$$\frac{\gamma^*_{N_n^{(2)}}}{\gamma_n^*} \xrightarrow[n\to+\infty]{} \theta_2. \tag{3.6}$$

We have either $\mathbb{N}\backslash(N^{(1)} \cup N^{(2)})$ is finite or infinite.
Suppose first that it is finite (actually it can be considered as empty), then $|N_n^{(2)}| = n - |N_n^{(1)}|$ and so $\theta_2 = 1 - \theta_1$, let us decompose

$$\begin{aligned}
\mathcal{A}_n^{(id,\gamma)}(x_1, \cdots, x_n) &= \frac{\gamma_1 x_1 + \cdots + \gamma_n x_n}{\gamma_n^*} \\
&= \frac{1}{\gamma_n^*}\left[\sum_{N_n^{(1)}} \gamma_k x_k + \sum_{N_n^{(2)}} \gamma_k x_k\right] \\
&= \frac{\gamma^*_{N_n^{(1)}}}{\gamma_n^*}\left[\frac{1}{\gamma^*_{N_n^{(1)}}} \sum_{N_n^{(1)}} \gamma_k x_k\right] + \frac{\gamma^*_{N_n^{(2)}}}{\gamma_n^*}\left[\frac{1}{\gamma^*_{N_n^{(2)}}} \sum_{N_n^{(2)}} \gamma_k x_k\right].
\end{aligned}$$

We use Corollary 3.1 to get the convergence of the two weighted arithmetic means between square brackets, and the two ratios' limits above (3.5) and (3.6), to obtain that

$$\mathcal{A}_n^{(id,\gamma)}(x_1, \cdots, x_n) \xrightarrow[n\to+\infty]{} \theta_1 x_\infty^{(1)} + (1-\theta_1) x_\infty^{(2)}.$$

Suppose now that $|\mathbb{N}\backslash(N^{(1)} \cup N^{(2)})| = +\infty$, we consider $N^{(3)} \subset \mathbb{N}\backslash(N^{(1)} \cup N^{(2)})$ such that $|N^{(3)}| = +\infty$ and $x_{N^{(3)}} \xrightarrow[n\to+\infty]{} x_\infty^{(1)}$, for instance. So, if $|\mathbb{N}\backslash(N^{(1)} \cup N^{(2)} \cup N^{(3)})| < \infty$, we use a similar decomposition as above to obtain

$$\mathcal{A}^{(id,\gamma)}(x_1, \cdots, x_n) \xrightarrow[n\to+\infty]{} (\theta_1 + \theta_3) x_\infty^{(1)} + \theta_2 x_\infty^{(2)},$$

where

$$\theta_3 = \lim_{n\to+\infty} \frac{\gamma^*_{N_n^{(3)}}}{\gamma_n^*} = 1 - (\theta_1 + \theta_2).$$

A recursive process finishes the proof. $\square$

If we consider Theorem 3.2 for a constant sequence $\gamma_n = 1$ for all $n \in \mathbb{N}$, we derive the next Corollary for the quasi-arithmetic multivariate mean.

Corollary 3.2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of elements of I, that has a finite number of accumulation points $\{x_\infty^{(1)}, \cdots, x_\infty^{(d)}\}$ for some integer $d \geq 2$, and let Φ be a continuous and strictly monotonic function defined on I. Then there exists a family $\theta_1, \cdots, \theta_d$ in $[0,1]$, that satisfies $\theta_1 + \cdots + \theta_d = 1$, such that*

$$\mathcal{A}_n^{\Phi}(x_1, \cdots, x_n) \xrightarrow[n\to+\infty]{} \Phi^{-1}\left(\sum_{i=1}^{d} \theta_i \Phi(x_\infty^{(i)})\right).$$

Theorem 3.3. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of elements of I, that has a finite number of accumulation points $\{x_\infty^{(1)}, \cdots, x_\infty^{(d)}\}$ for some integer $d \geq 2$. Given Φ a continuous and strictly monotonic function defined on I and let $p : I \longrightarrow (0, +\infty)$ a continuous weight function that is bounded below by $c > 0$. Then there exists a family $\alpha_1, \cdots, \alpha_d$ in $[0,1]$ that satisfies $\alpha_1 + \cdots + \alpha_d = 1$, such that*

$$\mathcal{A}_n^{(\Phi,p)}(x_1, \cdots, x_n) \xrightarrow[n\to+\infty]{} \Phi^{-1}\left(\sum_{i=1}^{d} \alpha_i \Phi(x_\infty^{(i)})\right).$$

Proof. The proof is similar to the proof of Theorem 3.2, and so we proceed recursively which means that it is sufficient to consider the case $d = 2$, also because Φ and Φ^{-1} are both continuous, it is enough to consider $\Phi = id$ the identity function. Let $x_{N^{(1)}}$ be a sub-sequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to $x_\infty^{(1)}$ and $N^{(2)} \subset \mathbb{N}\backslash N^{(1)}$ be an infinite subset such that $x_{N^{(2)}}$ converges to $x_\infty^{(2)}$. We have either $\mathbb{N}\backslash(N^{(1)} \cup N^{(2)})$ is finite or infinite.
Suppose first that it is finite (actually it can be considered as empty), then $|N_n^{(2)}| = n - |N_n^{(1)}|$, let us decompose

$$\begin{aligned}
\mathcal{A}_n^{(id,p)}(x_1, \cdots, x_n) &= \frac{p(x_1)x_1 + \cdots + p(x_n)x_n}{p_n^*} \\
&= \frac{1}{p_n^*}\left[\sum_{N_n^{(1)}} p(x_k)x_k + \sum_{N_n^{(2)}} p(x_k)x_k\right] \\
&= \frac{p^*_{N_n^{(1)}}}{p_n^*}\left[\frac{1}{p^*_{N_n^{(1)}}} \sum_{N_n^{(1)}} p(x_k)x_k\right] + \frac{p^*_{N_n^{(2)}}}{p_n^*}\left[\frac{1}{p^*_{N_n^{(2)}}} \sum_{N_n^{(2)}} p(x_k)x_k\right],
\end{aligned}$$

where $p_n^* = \displaystyle\sum_{k=1}^{n} p(x_k)$ and for $i = 1, 2$, $p^*_{N_n^{(i)}} = \displaystyle\sum_{N_n^{(i)}} p(x_k)$.

The two quantities between square brackets converges respectively to $x_\infty^{(1)}$ and $x_\infty^{(2)}$ due to Corollary 3.1.
For $i = 1, 2$

$$\frac{p^*_{N_n^{(i)}}}{p_n^*} = \frac{p^*_{N_n^{(i)}}}{|N_n^{(i)}|} \times \frac{|N_n^{(i)}|}{n} \times \frac{n}{p_n^*}$$

and then by using Corollary 3.2, Lemma 2.3 and the continuity of p, we obtain

$$\lim_{n\to+\infty} \frac{p^*_{N_n^{(i)}}}{p^*_n} = p(x^{(i)}_\infty) \times \theta_i \times \frac{1}{\theta_1 p(x^{(1)}_\infty) + \theta_2 p(x^{(2)}_\infty)} = \alpha_1 \in [0,1].$$

Finally we get

$$\mathcal{A}_n^{(id,\gamma)}(x_1, \cdots, x_n) \xrightarrow[n\to+\infty]{} \alpha_1 x^{(1)}_\infty + (1-\alpha_1) x^{(2)}_\infty.$$

We finish the proof recursively as we did for Theorem 3.2 . □

Remarks 1. It is worth noting that, despite the different formulations of the multivariate weighted mean and the Bajraktarević one, Theorems 3.2 and 3.3 shows that in average and in the long run they behave exactly the same as the multivariate quasi-arithmetic mean, see Corollary 3.2.
This could be interpreted in the scope of huge masses' behavior, more precisely it inspires us that if a huge mass of individuals has a finite number of options, in the long run and with respect to all types of averages, we will end up with behavior that is the natural average of all options.

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