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SOME PARSEVAL-GOLDSTEIN TYPE IDENTITIES WITH ILLUSTRATIVE EXAMPLES

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Abstract. In the present paper, the integral transform which will be called generalized Laplace transform is considered. Iteration of generalized Laplace transform and Fourier cosine transform and Fourier sine transform is generalized Glasser transform. Some identities involving these transforms and the Mellin transform and the generalized Stieltjes transform are given.

1. Introduction

Let Ω be the set of complex numbers which contains nonnegative real numbers and f is a holomorphic function in Ω. Temme [6] considered the following incomplete Laplace integral

$$F_\lambda(z,\alpha) = \frac{1}{\Gamma(\lambda)} \int_\alpha^\infty t^{\lambda-1} e^{-zt} f(t)dt \tag{1.1}$$

and was interested in the asymptotic behaviour of (1.1) for the large values of z which is uniformly valid for $\lambda, \alpha \geq 0$. In this paper, we consider the following Laplace type integral so-called the generalized Laplace transform

$$\mathcal{L}_\lambda\{f(x);y\} = \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} \exp(-xy) f(x)dx. \tag{1.2}$$

The purpose of this paper is to deal with the identities involving (1.2) and some well-known integral transforms. Our main goal is to generalize some identities containing classical Laplace transform and establish some Parseval-Goldstein type identities to demonstrate how these identities and relations lead a practical calculation for evaluating integrals. Of course, identities containing the images and originals of more than one operational relation are equally possible.

The well-known Laplace transfrom is defined by

$$F(x) := \mathcal{L}\{f(t);x\} = \int_0^\infty \exp(-xt) f(t)dt \tag{1.3}$$

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and the Stieltjes transform is defined by

$$\mathcal{S}\{f(x);y\} = \int_0^\infty \frac{f(x)}{x+y}dx. \tag{1.4}$$

The following result for Laplace transform is due to Goldstein [3, p. 106 (8)]:

Definition 1.1. (Goldstein Identity)
If $F(x) := \mathcal{L}\big\{f(t);x\big\}$ and $G(x) := \mathcal{L}\big\{g(t);x\big\}$, then

$$\int_0^\infty f(x)\frac{G(x)}{x}dx = \int_0^\infty \frac{F(x)}{x}g(x)dx. \tag{1.5}$$

This identity is also called *exchange identity* in the literature.

The following Parseval-Goldstein type theorem has been given in [8] which involves the Laplace transform and the Stieltjes transform:

Theorem 1.1. *(Parseval-Goldstein type identity)*

$$\int_0^\infty \mathcal{L}\big\{f(u);x\big\}\mathcal{L}\big\{g(y);x\big\}dx = \int_0^\infty g(y)\mathcal{S}\big\{f(u);y\big\}dy. \tag{1.6}$$

There are various Parseval-Goldstein type identities and a number of analogous identities in the literature on some integral transforms (for instance [4, 5, 8, 9, 10, 11]). Some of these results are applied to generalized functions by Adawi and Alawneh [1].

Now we recall the well-known integral transforms which we will use throughout the paper: Fourier cosine transform, Fourier sine transform, Mellin transform, Widder potential transform, generalized Glasser transform [7] and generalized Stieltjes transform are defined as follows, respectively.

$$\mathcal{F}_c\big\{f(x);y\big\} = \int_0^\infty \cos(xy)f(x)dx, \tag{1.7}$$

$$\mathcal{F}_s\big\{f(x);y\big\} = \int_0^\infty \sin(xy)f(x)dx, \tag{1.8}$$

$$\mathcal{M}\big\{f(x);\lambda\big\} = \int_0^\infty x^{\lambda-1}f(x)dx, \tag{1.9}$$

$$\mathcal{P}\big\{f(x);y\big\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx, \tag{1.10}$$

$$\mathcal{G}_\nu\big\{f(x);y\big\} = \int_0^\infty \frac{f(x)}{(x^2+y^2)^{\nu+\frac{1}{2}}}dx \tag{1.11}$$

and

$$\mathcal{S}_\lambda\{f(x);y\} = \int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx. \tag{1.12}$$

2. The Main Theorems

In this section, we give some useful iterations identities involving the generalized Laplace transform, Fourier sine transform, Fourier cosine transform and generalized Glasser transform.

Lemma 2.1. *The identities given in the following equations are valid provided that the integrals involved are absolutely convergent:*

$$\mathcal{L}_\lambda\big\{\mathcal{F}_s\{f(x);y\};t\big\} = \mathcal{G}_{\frac{\lambda-1}{2}}\left\{f(x)\sin\left[\lambda\arctan\left(\frac{x}{t}\right)\right];t\right\} \tag{2.1}$$

and

$$\mathcal{F}_s\big\{y^{\lambda-1}\mathcal{L}_\lambda\{f(x);y\};t\big\} = \mathcal{G}_{\frac{\lambda-1}{2}}\left\{x^{\lambda-1}f(x)\sin\left[\lambda\arctan\left(\frac{t}{x}\right)\right];t\right\}. \tag{2.2}$$

Proof. We begin by proving (2.1). By the definitions (1.2) and (1.8) of the $\mathcal{L}_\lambda$-transform and $\mathcal{F}_s$-transform, we get

$$\mathcal{L}_\lambda\big\{\mathcal{F}_s\{f(x);y\};t\big\} = \frac{1}{\Gamma(\lambda)}\int_0^\infty y^{\lambda-1}e^{-ty}\left(\int_0^\infty \sin(xy)f(x)dx\right)dy.$$

If we change the order of integration, which is permissible by the absolute convergence of the involved integrals, we find that

$$\mathcal{L}_\lambda\big\{\mathcal{F}_s\{f(x);y\};t\big\} = \frac{1}{\Gamma(\lambda)}\int_0^\infty f(x)\left(\int_0^\infty \sin(xy)y^{\lambda-1}e^{-ty}dy\right)dx. \tag{2.3}$$

Using the formula [2, p. 152, Eq. 4. 7 (15)] on the right-hand side of (2.3) and the definition (1.11) of generalized Glasser integral transform, we obtain

$$\begin{aligned}\mathcal{L}_\lambda\big\{\mathcal{F}_s\{f(x);y\};t\big\} =& \frac{1}{\Gamma(\lambda)}\int_0^\infty f(x)\left[\Gamma(\lambda)\left(t^2+x^2\right)^{-\frac{\lambda}{2}}\sin\left[\lambda\arctan\left(\frac{x}{t}\right)\right]\right]dx\\ =&\int_0^\infty \frac{f(x)}{(t^2+x^2)^{\frac{\lambda}{2}}}\sin\left[\lambda\arctan\left(\frac{x}{t}\right)\right]dx\\ =&\mathcal{G}_{\frac{\lambda-1}{2}}\left\{f(x)\sin\left[\lambda\arctan\left(\frac{x}{t}\right)\right];t\right\}\end{aligned}$$

where $\mathrm{Re}(\lambda) > -1$ and $\mathrm{Re}(t) > 0$. For the proof of (2.2), we use definitions (1.2) and (1.8) of $\mathcal{L}_\lambda$-transform and $\mathcal{F}_s$-transform and we get,

$$\mathcal{F}_s\big\{y^{\lambda-1}\mathcal{L}_\lambda\{f(x);y\};t\big\} = \frac{1}{\Gamma(\lambda)}\int_0^\infty y^{\lambda-1}\sin(ty)\left(\int_0^\infty x^{\lambda-1}e^{-yx}f(x)dx\right)dy.$$

Changing the order of the integration and using the formula [2, p. 152, Eq. 4. 7 (15)] for $\mathrm{Re}(\lambda) > -1$ and $\mathrm{Re}(t) > 0$, we find

$$\mathcal{F}_s\big\{y^{\lambda-1}\mathcal{L}_\lambda\{f(x);y\};t\big\} = \int_0^\infty \frac{x^{\lambda-1}f(x)}{(t^2+x^2)^{\frac{\lambda}{2}}}\sin\left[\lambda\arctan\left(\frac{t}{x}\right)\right]dx.$$

On the right-hand side of above equality, using the definition (1.11) of generalized Glasser transform, we arrive at

$$\mathcal{F}_s\big\{y^{\lambda-1}\mathcal{L}_\lambda\{f(x);y\};t\big\} = \mathcal{G}_{\frac{\lambda-1}{2}}\left\{x^{\lambda-1}f(x)\sin\left[\lambda\arctan\left(\frac{t}{x}\right)\right];t\right\}.$$

□

Lemma 2.2. *The identities given in the following equations are valid provided that the integrals involved are absolutely convergent:*

$$\mathcal{L}_\lambda\Big\{\mathcal{F}_c\big\{f(x);y\big\};t\Big\}=\mathcal{G}_{\frac{\lambda-1}{2}}\Big\{f(x)\cos\Big[\lambda\arctan\Big(\frac{x}{t}\Big)\Big];t\Big\}, \tag{2.4}$$

and

$$\mathcal{F}_c\Big\{y^{\lambda-1}\mathcal{L}_\lambda\big\{f(x);y\big\};t\Big\}=\mathcal{G}_{\frac{\lambda-1}{2}}\Big\{x^{\lambda-1}f(x)\cos\Big[\lambda\arctan\Big(\frac{t}{x}\Big)\Big];t\Big\}. \tag{2.5}$$

Proof. The proof for Lemma 2.2 is similar to proof of previous Lemma. □

Corollary 2.1. *If* $F_c(t)=\mathcal{F}_c\left\{y^{\lambda-1}\mathcal{L}_\lambda\{h(x);y\};t\right\}$ *and* $F_s(t)=\mathcal{F}_s\left\{y^{\lambda-1}\mathcal{L}_\lambda\left\{h(x);y\right\};t\right\}$ *then,*

$$\mathcal{L}_\lambda\left\{\mathcal{F}_s\big\{x^{\lambda-1}h(x);y\big\};t\right\}=\sin\frac{\lambda\pi}{2}F_c(t)-\cos\frac{\lambda\pi}{2}F_s(t) \tag{2.6}$$

and

$$\mathcal{L}_\lambda\left\{\mathcal{F}_c\big\{x^{\lambda-1}h(x);y\big\};t\right\}=\cos\frac{\lambda\pi}{2}F_c(t)+\sin\frac{\lambda\pi}{2}F_s(t). \tag{2.7}$$

Proof. If we set $f(x)=x^{\lambda-1}h(x)$ in (2.1) and (2.4) we get (2.6) and (2.7). □

Theorem 2.1. *Provided that the conditions given in Lemma 2.1 and Lemma 2.2, the following Parseval-Goldstein type identities hold:*

$$\int_0^\infty t^{\lambda-1}\mathcal{L}_\lambda\big\{f(x);t\big\}\mathcal{F}_c\big\{g(y);t\big\}dt = \mathcal{M}\Big\{f(x)\mathcal{G}_{\frac{\lambda-1}{2}}\Big\{g(y)\cos\Big[\lambda\arctan\Big(\frac{y}{x}\Big)\Big];x\Big\};\lambda\Big\}, \tag{2.8}$$

$$\int_0^\infty t^{\lambda-1}\mathcal{L}_\lambda\big\{f(x);t\big\}\mathcal{F}_c\big\{g(y);t\big\}dt = \int_0^\infty g(y)\mathcal{G}_{\frac{\lambda-1}{2}}\Big\{x^{\lambda-1}f(x)\cos\Big[\lambda\arctan\Big(\frac{y}{x}\Big)\Big];y\Big\}dy, \tag{2.9}$$

$$\int_0^\infty t^{\lambda-1}\mathcal{L}_\lambda\big\{f(x);t\big\}\mathcal{F}_s\big\{g(y);t\big\}dt = \mathcal{M}\Big\{f(x)\mathcal{G}_{\frac{\lambda-1}{2}}\Big\{g(y)\sin\Big[\lambda\arctan\Big(\frac{y}{x}\Big)\Big];x\Big\};\lambda\Big\}, \tag{2.10}$$

$$\int_0^\infty t^{\lambda-1}\mathcal{L}_\lambda\big\{f(x);t\big\}\mathcal{F}_s\big\{g(y);t\big\}dt = \int_0^\infty g(y)\mathcal{G}_{\frac{\lambda-1}{2}}\Big\{x^{\lambda-1}f(x)\sin\Big[\lambda\arctan\Big(\frac{y}{x}\Big)\Big];y\Big\}dy. \tag{2.11}$$

Proof. We only give the proofs of (2.8) and (2.9). Because proofs of (2.10) and (2.11) are similar. To prove (2.8), we start with the definition (1.2) of generalized Laplace transform, we have

$$\int_0^\infty t^{\lambda-1}\mathcal{L}_\lambda\big\{f(x);t\big\}\mathcal{F}_c\big\{g(y);t\big\}dt = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\mathcal{F}_c\big\{g(y);t\big\}\left(\int_0^\infty e^{-tx}x^{\lambda-1}f(x)dx\right)dt.$$

Interchanging the order of integration and then making use of (2.4), we get

$$\begin{aligned}\int_0^\infty & t^{\lambda-1}\mathcal{L}_\lambda\{f(x);t\}\mathcal{F}_c\{g(y);t\}dt \\ &= \frac{1}{\Gamma(\lambda)}\int_0^\infty x^{\lambda-1}f(x)\left(\int_0^\infty t^{\lambda-1}e^{-tx}\mathcal{F}_c\{g(y);t\}dt\right)dx \\ &= \int_0^\infty x^{\lambda-1}f(x)\mathcal{L}_\lambda\{\mathcal{F}_c\{g(y);t\};x\}dx \\ &= \int_0^\infty x^{\lambda-1}f(x)\mathcal{G}_{\frac{\lambda-1}{2}}\left\{g(y)\cos\left[\lambda\arctan\left(\frac{y}{x}\right)\right];x\right\}dx.\end{aligned}$$

Hence by the definition (1.9) of Mellin transform, we obtain (2.8).

Similarly, using the definition (1.7) of Fourier cosine transform, we have

$$\begin{aligned}\int_0^\infty & t^{\lambda-1}\mathcal{L}_\lambda\{f(x);t\}\mathcal{F}_c\{g(y);t\}dt \\ &= \int_0^\infty t^{\lambda-1}\mathcal{L}_\lambda\{f(x);t\}\left(\int_0^\infty g(y)\cos(ty)dy\right)dt.\end{aligned}$$

Interchanging the order of integration and then making use of (2.5), we find

$$\begin{aligned}\int_0^\infty & t^{\lambda-1}\mathcal{L}_\lambda\{f(x);t\}\mathcal{F}_c\{g(y);t\}dt \\ &= \int_0^\infty g(y)\left(\int_0^\infty t^{\lambda-1}\cos(ty)\mathcal{L}_\lambda\{f(x);t\}dt\right)dy \\ &= \int_0^\infty g(y)\mathcal{F}_c\{t^{\lambda-1}\mathcal{L}_\lambda\{f(x);t\};y\}dy \\ &= \int_0^\infty g(y)\mathcal{G}_{\frac{\lambda-1}{2}}\left\{x^{\lambda-1}f(x)\cos\left[\lambda\arctan\left(\frac{y}{x}\right)\right];y\right\}dy.\end{aligned}$$

□

Remark 2.1. If we set $\lambda = 1$ in Theorem 2.1, then we get the following equations which were obtained before in [7, p. 539] and [4, p. 518, (1.3)].

$$\begin{aligned}\int_0^\infty \mathcal{L}\{f(x);t\}\mathcal{F}_c\{g(y);t\}dt &= \int_0^\infty xf(x)\mathcal{G}_{1/2}\Big\{g(y);x\Big\}dx, \\ \int_0^\infty \mathcal{L}\{f(x);t\}\mathcal{F}_c\{g(y);t\}dt &= \int_0^\infty g(y)\mathcal{P}\Big\{f(x);y\Big\}dy, \\ \int_0^\infty \mathcal{L}\{f(x);t\}\mathcal{F}_s\{g(y);t\}dt &= \int_0^\infty f(x)\mathcal{P}\Big\{g(y);x\Big\}dx, \\ \int_0^\infty \mathcal{L}\{f(x);t\}\mathcal{F}_s\{g(y);t\}dt &= \int_0^\infty yg(y)\mathcal{G}_{1/2}\Big\{f(x);y\Big\}dy.\end{aligned}$$

Remark 2.2. Taking into account identities (2.8) and (2.9), we obtain a new relation between generalized Glasser transform and Mellin transform,

$$\begin{aligned}\int_0^\infty & x^{\lambda-1}f(x)\mathcal{G}_{\frac{\lambda-1}{2}}\{g(y)\cos\left[\lambda\arctan\left(\frac{y}{x}\right)\right];x\}dx \\ &= \int_0^\infty g(y)\mathcal{G}_{\frac{\lambda-1}{2}}\{x^{\lambda-1}f(x)\cos\left[\lambda\arctan\left(\frac{y}{x}\right)\right];y\}dy. \qquad (2.12)\end{aligned}$$

Similarly, taking into account (2.10) and (2.11), we get

$$\int_0^\infty x^{\lambda-1} f(x) \mathcal{G}_{\frac{\lambda-1}{2}} \{g(y) \sin\left[\lambda \arctan\left(\frac{y}{x}\right)\right]; x\} dx$$
$$= \int_0^\infty g(y) \mathcal{G}_{\frac{\lambda-1}{2}} \{x^{\lambda-1} f(x) \sin\left[\lambda \arctan\left(\frac{y}{x}\right)\right]; y\} dy. \tag{2.13}$$

3. Illustrative Examples

Example 3.1. We show that

$$\int_0^\infty \frac{\cos[\lambda \arctan(\frac{x}{t})]}{(x^2+a^2)(x^2+t^2)^{\lambda/2}} dx = \frac{\pi}{2a(t+a)^\lambda}, \tag{3.1}$$

where $\mathrm{Re}(a) > 0$.

We put $f(x) = (x^2+a^2)^{-1}$ in (2.4). Using [2, p. 8, Eq. 1. 2 (11)], we find that

$$\mathcal{F}_c \{f(x); y\} = \frac{\pi e^{-ay}}{2a}. \tag{3.2}$$

Substituting (3.2) into identity (2.4), we obtain

$$\mathcal{L}_\lambda\{\mathcal{F}_c \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-yt} y^{\lambda-1} \left[\frac{\pi}{2a} e^{-ay}\right] dy$$
$$= \frac{\pi}{2a\Gamma(\lambda)} \int_0^\infty e^{-y(t+a)} y^{\lambda-1} dy. \tag{3.3}$$

Using the known result [2, p. 144, Eq. 4. 5 (3)] for the inner integral in (3.3) and the equality (2.4) for $\mathrm{Re}(t) > -\mathrm{Re}(a)$, we find that

$$\mathcal{G}_{\frac{\lambda-1}{2}} \left\{(x^2+a^2)^{-1} \cos\left[\lambda \arctan\left(\frac{x}{t}\right)\right]; t\right\} = \int_0^\infty \frac{\cos[\lambda \arctan(\frac{x}{t})]}{(x^2+a^2)(x^2+t^2)^{\lambda/2}} dx$$
$$= \frac{\pi}{2a(t+a)^\lambda}.$$

Corollary 3.1. *The following iteration identity holds true for the generalized Laplace transform*

$$\mathcal{L}_\lambda \{\mathcal{L}_\lambda \{f(x); t\}; a\} = \frac{1}{\Gamma(\lambda)} \mathcal{S}_\lambda \left\{x^{\lambda-1} f(x); a\right\} \tag{3.4}$$

where $\mathrm{Re}(a) > 0$.

Proof. If we put $g(y) = \frac{1}{y^2+a^2}$ in the equality (2.8) of Theorem 2.1, then we have,

$$\int_0^\infty t^{\lambda-1} \mathcal{L}_\lambda \{f(x); t\} \mathcal{F}_c \left\{\frac{1}{y^2+a^2}; t\right\} dt$$
$$= \int_0^\infty x^{\lambda-1} f(x) \mathcal{G}_{\frac{\lambda-1}{2}} \left\{\frac{\cos\left[\lambda \arctan\left(\frac{y}{x}\right)\right]}{y^2+a^2}; x\right\} dx. \tag{3.5}$$

Using the known result [2, p. 8, Eq. 1. 2 (11)] and (3.1) in (3.5), we find

$$\frac{\pi}{2a} \int_0^\infty t^{\lambda-1} e^{-at} \mathcal{L}_\lambda \{f(x); t\} dt = \frac{\pi}{2a} \int_0^\infty \frac{x^{\lambda-1} f(x)}{(x+a)^\lambda} dx.$$

□

Remark 3.1. If we set $\lambda = 1$ in (3.4) then we get the known relation [8, p. 243, (13)]

$$\mathcal{L}\left\{\mathcal{L}\left\{f(x);t\right\};a\right\} = \mathcal{S}\left\{f(x);a\right\}.$$

Example 3.2. We show that

$$\int_0^\infty \frac{\sin[b(x^2+a^2)^{1/2}]\cos[\lambda\arctan(\frac{x}{t})]}{(x^2+a^2)^2(x^2+t^2)^{\lambda/2}}dx = \frac{b\pi}{2a(t+a)^\lambda}, \tag{3.6}$$

where $\mathrm{Re}(a) > 0$.

We put

$$f(x) = (x^2+a^2)^{-2}\sin[b(x^2+a^2)^{1/2}]$$

in (2.4). Using [2, p. 26, Eq. 1. 7 (29)], we find that

$$\mathcal{F}_c\left\{f(x);y\right\} = \frac{b\pi e^{-ay}}{2a}. \tag{3.7}$$

Substituting (3.7) into (1.2), we get

$$\mathcal{L}_\lambda\big\{\mathcal{F}_c\left\{f(x);y\right\};t\big\} = \frac{b\pi}{2a\Gamma(\lambda)}\int_0^\infty e^{-y(t+a)}y^{\lambda-1}dy. \tag{3.8}$$

Making use of the known result [2, p. 144, Eq. 4. 5 (3)] and then identity (2.4), we find

$$\begin{aligned}
&\mathcal{G}_{\frac{\lambda-1}{2}}\left\{\frac{\sin[b(x^2+a^2)^{1/2}]\cos\left[\lambda\arctan\left(\frac{x}{t}\right)\right]}{(x^2+a^2)^2};t\right\}\\
&\quad=\int_0^\infty \frac{\sin[b(x^2+a^2)^{1/2}]\cos[\lambda\arctan(\frac{x}{t})]}{(x^2+a^2)(x^2+t^2)^{\lambda/2}}dx\\
&\quad=\frac{b\pi}{2a(t+a)^\lambda}
\end{aligned}$$

where $\mathrm{Re}(t) > -\mathrm{Re}(a)$.

Example 3.3. We have

$$\int_0^\infty \frac{\left[\left(a^2+x^2\right)^{1/2}+a\right]^{1/2}\cos\left[\lambda\arctan(\frac{x}{t})\right]}{(a^2+x^2)^{1/2}(x^2+t^2)^{\lambda/2}}dx = \frac{\sqrt{\pi}\Gamma(\lambda-1/2)}{\sqrt{2}\Gamma(\lambda)(a+t)^{\lambda-1/2}}, \tag{3.9}$$

where $\mathrm{Re}(a) > 0$.

We put

$$f(x) = (a^2+x^2)^{-1/2}\left[(a^2+x^2)^{1/2}+a\right]^{1/2}$$

in (2.4). Using [2, p. 10, Eq. 1. 2 (25)], we find that

$$\mathcal{F}_c\{f(x);y\} = \left(\frac{\pi}{2y}\right)^{1/2}e^{-ay}. \tag{3.10}$$

Substituting (3.10) into identity (2.4), we obtain

$$\mathcal{L}_\lambda\big\{\mathcal{F}_c\left\{f(x);y\right\};t\big\} = \frac{\sqrt{\pi}}{\sqrt{2}\Gamma(\lambda)}\int_0^\infty e^{-(a+t)y}y^{\lambda-3/2}dy.$$

Using the known result [2, p. 137, Eq. 4. 3 (1)] for $\mathrm{Re}(\lambda) > \frac{1}{2}$ and from (2.4), we get

$$\int_0^\infty \frac{\left[\left(a^2+x^2\right)^{1/2}+a\right]^{1/2}\cos\left[\lambda\arctan(\frac{x}{t})\right]}{(a^2+x^2)^{1/2}(x^2+t^2)^{\lambda/2}}dx = \frac{\sqrt{\pi}\Gamma(\lambda-1/2)}{\sqrt{2}\Gamma(\lambda)}\frac{1}{(a+t)^{\lambda-1/2}}.$$

Example 3.4. We show

$$\int_0^\infty \frac{x\sin[\lambda\arctan(\frac{x}{t})]}{(x^2+a^2)(x^2+t^2)^{\lambda/2}}dx = \frac{\pi}{2(t+a)^\lambda}, \tag{3.11}$$

where $\mathrm{Re}(a) > 0$.

We put $f(x) = x(x^2+a^2)^{-1}$ in (2.1). Using the formula [2, p. 65, Eq. 2. 2 (15)], we find that

$$\mathcal{F}_s\left\{f(x);y\right\} = \frac{\pi e^{-ay}}{2}. \tag{3.12}$$

Substituting (3.12) into identity (2.1), we obtain

$$\mathcal{L}_\lambda\big\{\mathcal{F}_s\left\{f(x);y\right\};t\big\} = \frac{\pi}{2\Gamma(\lambda)}\int_0^\infty e^{-y(t+a)}y^{\lambda-1}dy. \tag{3.13}$$

Using the known result [2, p. 144, Eq. 4. 5 (3)] for right-hand side of (3.13) and then using (2.1), we get

$$\begin{aligned}\mathcal{G}_{\frac{\lambda-1}{2}}\left\{\frac{x\sin\left[\lambda\arctan\left(\frac{x}{t}\right)\right]}{(x^2+a^2)};t\right\} &= \int_0^\infty \frac{\sin[\lambda\arctan(\frac{x}{t})]}{(x^2+a^2)(x^2+t^2)^{\lambda/2}}dx \\ &= \frac{\pi}{2(t+a)^\lambda}.\end{aligned}$$

4. Concluding Remarks

In this article, we aimed to obtain some new identities that will enable the calculation of some generalized integrals. Using these identities, some new Parseval-Goldstein type relations are obtained for given integral transforms and many other well-known. As applications of the identities and theorems, some illustrative examples are also given. More generalized integral solutions can be obtained using these identities.

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