Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

FAST DISCRETE SOLVERS FOR NONLINEAR HAMMERSTIEN EQUATIONS

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Abstract. In this paper, discrete versions of the modified projection-type methods are studied for solving the Hammerstein integral equations with a smooth kernel. The approximating operator is either the orthogonal projection or an interpolatory projection at Gauss points onto a space of piecewise polynomials of degree $\leq r-1$. The orders of convergence that one obtains for the proposed methods shows that the used numerical quadrature for approximating the integrals preserves the orders of convergence of continuous methods. Numerical results are presented to validate the theoretical results.

1. Introduction

In this paper, we consider nonlinear Fredholm–Hammerstein integral equations of the form

$$x - \mathcal{K}x = f, \tag{1.1}$$

where $\mathcal{K}$ is the integral operator defined on $\mathcal{X} = \mathcal{L}^\infty[0,1]$ by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\psi(t,x(t))dt, \quad s \in [0,1], \quad x \in \mathcal{X}, \tag{1.2}$$

f and ψ are known functions, with $\psi(t,u)$ nonlinear in u and x is the function to be found. Hammerstein integral equations arise in many applications in science and technology such as vehicular traffic and chemical reaction including chemical kinetics, biology, heat conduction, theory of superfluidity, boundary layer and heat transfer resulting in mathematical models described by nonlinear integral equations [1, 4, 5, 19]. Since usually these equations can not be solved explicitly, it is required to obtain approximate solutions. The Galerkin, collocation and their discretized versions are the most common used projection methods for finding numerical solutions of the integral equation of type (1.1). Superconvergence results of various projection and iterated projection methods for solving nonlinear Fredholm integral equations can be found in [10, 11, 12]. Kulkarni and Nidhin [15] discuss a modified projection method to solve (1.1) with a continuous kernel, as well as a more general type of kernel in [9, 14]. Some authors proposed discrete methods to solve nonlinear integral equations using orthogonal

2010 *Mathematics Subject Classification.* 41A10, 45G10, 47H30, 65R20.
Key words and phrases. Hammerstein equation, Numerical quadrature, Hyperinterpolation projection, Interpolatory projection, Gauss points, Superconvergence.

and interpolatory projection operators (see [7, 8, 20]). There have been many methods to improve the accuracy of numerical solutions of classical methods. Furthermore, the authors in [13] introduced a discrete version of collocation and iterated collocation methods to obtain reliable superconvergence results. Moreover, an extrapolation of a discrete version of Kantorovich and degenerate kernel methods for integral equations of the second kind was presented in [3]. A new collocation method was disccused by Kumar and Sloan [16], while its superconvergence feature were studied in [18]. A superconvergent version of this method (so called modified projection-type method) was discussed in [2]. The Kumar and Sloan method is known in literature as the collocation-type method since it was defined using an interpolatory operator. This paper will specifically refer to this method as a discrete modified Galerkin-type method when using an hyperinterpolation projection and as a projection-type method when not specifying the type of projection.

The purpose of this paper is to describe the discrete modified projection-type method for solving Hammerstein equations (1.1) by using piecewise polynomial interpolation technique. Note that the continuous method is proposed in [2] for Hammerstein integral equations. When we replace the integrals by a numerical quadrature formula, we show that for a sufficiently smooth kernel, the orders of convergence of the discrete modified projection-type method and its iterated version are, respectively, $3r$ and $4r$, where r denotes the order of piecewise polynomials employed in the approximation. Accordingly, the iterated discrete modified projection-type method performs better than the modified projection-type method.

The paper is divided into four sections. In Section 2, we set up the notations, discuss the numerical methods and some relevant results are recalled. In Section 3 we give the convergence orders of the proposed method and their iterated versions. Section 4 is devoted to a presentation of numerical examples, which illustrate the theoretical estimates obtained in Section 3.

2. Preliminaries and notations

Let x_0 be a unique solution of (1.1), a and b be real numbers such that

$$\left[\min_{s\in[0,1]} x_0(s), \max_{s\in[0,1]} x_0(s)\right] \subset [a,b].$$

For $\delta_0 > 0$, let

$$\mathcal{B}(x,\delta_0) = \{x_0 \in \mathcal{X} : \|x - x_0\|_\infty < \delta_0\}.$$

Define $\Omega = [0,1] \times [a,b]$ and assume that otherwise, the following conditions on κ, f and ψ:

(i) $f \in [0,1]$ and $\psi \in \mathcal{C}(\Omega)$.

(ii) $M \equiv \sup_{s\in[0,1]} \int_0^1 |\kappa(s,t)|dt < \infty$.

(iii) The function $\psi(t,u)$ and $\frac{\partial\psi}{\partial u}(t,u)$ are Lipschitz continuous in $u \in [a,b]$, then for any $u_1, u_2 \in [a,b]$ there exists $\delta_1, \delta_2 > 0$, for which

$$|\psi(t,u_1) - \psi(t,u_2)| \le \delta_1|u_1 - u_2| \quad \text{and} \quad \left|\frac{\partial\psi}{\partial u}(t,u_1) - \frac{\partial\psi}{\partial u}(t,u_2)\right| \le \delta_2|u_1 - u_2|.$$

If the condition (iii) holds, the operator $\mathcal{K}$ is Fréchet differentiable and $\mathcal{K}'(x_0)$ is $M\delta_2$-Lipschitz . The Fréchet derivative at $x_0 \in \mathcal{C}[0,1]$ is defined by

$$(\mathcal{K}'(x_0)g)(s) = \int_0^1 \kappa(s,t)\frac{\partial \psi}{\partial u}(t, x_0(t))g(t)dt, \quad s, g \in \mathcal{C}[0,1].$$

This method involves replacing $\mathcal{K}$ by the Nyström operator $\mathcal{K}_m^N$, based on the quadrature formula in order to preserve the orders of convergence. For a positive integer n, let

$$(\Delta_n): \quad 0 = s_0 < s_1 < s_2 < \ldots < s_{n-1} < s_n = 1 \tag{2.1}$$

be the uniform partition of $[0,1]$, with nodes $\{s_i = \frac{i}{n},\ i = 0, \ldots, n\}$ and meshlength $h = \frac{1}{n}$. For $u, v \in \mathcal{C}[0,1]$, the inner product is defined by

$$\langle u, v\rangle = \int_0^1 u(t)v(t)dt \text{ and norm is } \quad \|u\|_{\mathcal{L}^2} = \left(\int_0^1 u(t)^2 dt\right)^{\frac{1}{2}}.$$

For a fixed $r \geq 1$, we denote by Π_r the space of polynomials of degree $\leq r-1$. Let

$$\mathbb{X}_n = \left\{y : [0,1] \longrightarrow \mathbb{R} : y|_{[s_{i-1}, s_i]} \in \Pi_r,\ 1 \leq i \leq n\right\}$$

be the set of functions that are polynomials of degree $\leq r-1$, on each subinterval $[s_{i-1}, s_i]$.

Let $m = pn$ for some $p \in \mathbb{N}^*$ and let Δ_m be the uniform partition of $[0,1]$ giving by (2.1) with meshlength $\widetilde{h} = \frac{1}{m}$, such that $p\widetilde{h} = h$. To introduce the discrete methods, we consider a quadrature formula defined by

$$\int_0^1 f(t)dt \simeq \sum_{j=1}^{\rho} w_j f(\sigma_j), \tag{2.2}$$

with $\sum_{j=1}^{\rho} w_j = 1$. For $1 \leq i \leq m$ and $1 \leq j \leq \rho$, let $\zeta_{ij} = (i-1+\sigma_j)\widetilde{h}$, then (2.2) gives rise to the composite quadrature formula

$$\int_0^1 f(t)dt \simeq \widetilde{h}\sum_{i=1}^{m}\sum_{j=1}^{\rho} w_j f(\zeta_{ij}). \tag{2.3}$$

Suppose that the quadrature formula (2.2) is exact for all polynomials of degree $\leq d-1$ where $d \geq 2r$. Then, for $f \in \mathcal{C}^d[0,1]$ (See Golberg and Chen [7])

$$\left|\int_0^1 f(t)dt - \widetilde{h}\sum_{i=1}^{m}\sum_{j=1}^{\rho} w_j f(\zeta_{ij})\right| \leq c_1 \|f^{(d)}\|_\infty \widetilde{h}^d, \tag{2.4}$$

where c_1 is a constant independent of m.
Using the above numerical integration method, we define the discrete inner product as

$$\langle f, g\rangle_m = \tilde{h}\sum_{i=1}^{m}\sum_{j=1}^{\rho} \omega_j f(\zeta_{ij})g(\zeta_{ij}), \qquad f, g \in \mathcal{C}[0,1].$$

The Nyström approximation of the integral operator $\mathcal{K}$ is defined as

$$(\mathcal{K}_m^N x)(s) = \widetilde{h} \sum_{i=1}^{m} \sum_{j=1}^{\rho} \omega_j \kappa(s, \zeta_{ij}) \psi(\zeta_{ij}, x(\zeta_{ij})), \quad s \in [0,1]. \tag{2.5}$$

Since $w_j > 0$ and $1 = \sum_{i=1}^{\rho} \omega_i$. For $p = 0, 1, \ldots, r$, we have

$$\begin{aligned} \|[\mathcal{K}_m^{N'}(x_0)g]^{(p)}\|_\infty &= \max_{s\in[0,1]} \left| \widetilde{h} \sum_{i=1}^{m} \sum_{j=1}^{\rho} w_j \frac{\partial^p \kappa}{\partial s^p}(s, \zeta_{ij}) \frac{\partial \psi}{\partial u}(\zeta_{ij}, x_0(\zeta_{ij})) g(\zeta_{ij}) \right|, \\ &\leq \max_{s\in[0,1]} \widetilde{h} \sum_{i=1}^{m} \sum_{j=1}^{\rho} w_j \left| \frac{\partial^p \kappa}{\partial s^p}(s, \zeta_{ij}) \right| \left| \frac{\partial \psi}{\partial u}(\zeta_{ij}, x_0(\zeta_{ij})) \right| |g(\zeta_{ij})|, \\ &\leq \Psi_1 \|\kappa\|_{p,\infty} \|g\|_\infty, \end{aligned} \tag{2.6}$$

where

$$\|\kappa\|_{r,\infty} = \max_{s,t\in[0,1]} \left| \frac{\partial^r \kappa}{\partial s^r}(s,t) \right|, \quad \Psi_1 = \max_{t\in[0,1]} \left| \frac{\partial \psi}{\partial u}(t, x_0(t)) \right|.$$

Hence, we deduce that $\|\mathcal{K}_m^{N'}(x_0)g\|_\infty \leq \Psi_1 \|\kappa\|_{0,\infty} \|g\|_\infty$. This implies,

$$\|\mathcal{K}_m^{N'}(x_0)\|_\infty \leq \Psi_1 \|\kappa\|_{0,\infty}. \tag{2.7}$$

Then $\mathcal{K}_m^{N'}(x_0)$ is a compact operator on $\mathcal{C}[0,1]$. Assume that $\kappa \in \mathcal{C}^d[0,1]^2$ and that $x_0 \in \mathcal{C}^d[0,1]$. Then

$$\|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty \leq c_1 \|\kappa\|_{d,\infty} \widetilde{h}^d. \tag{2.8}$$

The Fréchet derivative of $\mathcal{K}_m^N$ is given by

$$(\mathcal{K}_m^{N'}(x_0)g)(s) = \widetilde{h} \sum_{i=1}^{m} \sum_{j=1}^{\rho} \omega_j \kappa(s, x_0(\zeta_{ij})) \frac{\partial \psi}{\partial u}(\zeta_{ij}, x_0(\zeta_{ij})) g(\zeta_{ij}), \quad s \in [0,1].$$

If $\frac{\partial \kappa}{\partial u} \in \mathcal{C}^d[0,1]^2$ and $g \in \mathcal{C}^d[0,1]$, then from (2.4)

$$\|\mathcal{K}'(x_0)g - \mathcal{K}_m^{N'}(x_0)g\|_\infty \leq c_1 \|\kappa\|_{d,\infty} \|g\|_\infty \widetilde{h}^d, \tag{2.9}$$

where c_1 is a constant independent of m.
For the rest of paper, we set

$$\kappa_s(t) \equiv \kappa(s,t), \quad \psi_p = \frac{\partial^p \psi}{\partial u^p}(t, x_0(t)) \quad s, t \in [0,1].$$

Now,we define two types of projections from $\mathcal{C}[0,1]$ to $\mathbb{X}_n$.

• **Discrete orthogonal projection operator:** Discrete orthogonal projection operator namely Hyperinterpolation operator $\mathcal{Q}_n^G x : \mathcal{L}^2[0,1] \to \mathbb{X}_n$ is defined by

$$(\mathcal{Q}_n^G x)(s) = \sum_{i=1}^{nr} \langle x, \varphi_i \rangle_m \varphi_i(s), \tag{2.10}$$

where $\{\varphi_1, \varphi_2, \ldots, \varphi_{nr}\}$ is an orthonormal basis for $\mathbb{X}_n$.

• **Interpolatory projection:** Let $\{\tau_1, \ldots, \tau_r\}$ be the set of r Gauss points in $[0, 1]$. For $x \in \mathcal{C}[0, 1]$, let $\mathcal{Q}_n^C x : \mathcal{C}[0, 1] \to \mathbb{X}_n$ be the interpolatory operator defined by

$$\begin{aligned} (\mathcal{Q}_n^C x)(s) &= \sum_{i=1}^{nr} x(t_i)\ell_i(s), \qquad s \in [0, 1], \\ (\mathcal{Q}_n^C x)(t_i) &= x(t_i), \qquad 1 \leq i \leq nr, \end{aligned} \tag{2.11}$$

where the collocation points are

$$\{t_i : \quad i = 1, 2, ..., nr\} = \{t_{ij} = (i - 1 + \tau_j)h, \quad 1 \leq i \leq n, \quad 1 \leq j \leq r\}, \tag{2.12}$$

and $\{\ell_i : \quad i = 1, 2, ..., nr\}$ is the Lagrange basis of $\mathbb{X}_n$.

For notational convenience from now on we write $\mathcal{Q}_n \equiv \mathcal{Q}_n^G$ or $\mathcal{Q}_n^C$. In both cases, $\mathcal{Q}_n$ converge to identity operator pointwise and for $x \in \mathcal{C}[0, 1]$, (see [6], page 328, Corollary 7.6):

$$\|(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq c_2 \|x^{(r)}\|_\infty h^r, \tag{2.13}$$

on the other hand from Allouch et al.[2], we get

$$\|(\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(x_0) - \mathcal{K}_n(x_0)]\|_\infty \leq c_2 h^{3r}, \tag{2.14}$$

where c_2 is a constant independent of n. Moreover, the projection $\mathcal{Q}_n$ is uniformly bounded with respect to n, i.e.

$$q = \sup_n \|\mathcal{Q}_n\|_{\mathcal{X} \to \mathcal{X}} < \infty. \tag{2.15}$$

Let p be a positive integer. For $x \in \mathcal{C}^p[0, 1]$, we set

$$\|x\|_{p,\infty} = \sum_{i=0}^{p} \|x^{(i)}\|_\infty.$$

If $\mathcal{Q}_n^C$ is the interpolatory projection at r Gauss points, then for $x \in \mathcal{C}^r[0, 1]$ and $y \in \mathcal{C}^{2r}[0, 1]$, (See de-Boor-Swartz [17])

$$\left| \int_0^1 x(t)(\mathcal{I} - \mathcal{Q}_n^C)y(t)dt \right| \leq c_3 \|x\|_{r,\infty} \|y\|_{2r,\infty} h^{2r}, \tag{2.16}$$

where c_3 is a constant independent of n.

Lemma 2.1. *(Kulkarni-Rakshit* [13] *) Let $\mathcal{Q}_n^C : \mathcal{C}[0, 1] \to \mathbb{X}_n$ be the interpolatory projection defined by (2.11). If $\frac{\partial \kappa}{\partial u} \in \mathcal{C}^r(\Omega)$ and $g \in \mathcal{C}^{2r}[0, 1]$, then*

$$\|\mathcal{K}_m^{N'}(x_0)(\mathcal{I} - \mathcal{Q}_n^C)g\|_\infty \leq c_3 \|\kappa\|_{r,\infty} \|g\|_{2r,\infty} h^{2r}.$$

Let $z(t) = \psi(t, x(t))$, and consider the following approximate operator defined in [16] by

$$(\mathcal{K}_n x)(s) = \int_0^1 \kappa(s, t)\mathcal{Q}_n z(.)dt, \qquad s \in [0, 1]. \tag{2.17}$$

Recall that the modified projection-type method introduced in [2] consist of approximating $\mathcal{K}$ by

$$\mathcal{Q}_n\mathcal{K} + \mathcal{K}_n - \mathcal{Q}_n\mathcal{K}_n. \tag{2.18}$$

In this framework, we propose to approximate $\mathcal{K}$ by the following discrete finite rank operator

$$\mathcal{K}_n^S = \mathcal{Q}_n\mathcal{K}_m^N + \overline{\mathcal{K}}_n - \mathcal{Q}_n\overline{\mathcal{K}}_n, \tag{2.19}$$

where $\overline{\mathcal{K}}_n$ is the discrete nonlinear operator given by

$$(\overline{\mathcal{K}}_n x)(s) = \langle \kappa_s, \mathcal{Q}_n z\rangle_m = \widetilde{h}\sum_{i=1}^{m}\sum_{j=1}^{\rho}\omega_j\kappa_s(\zeta_{ij})(\mathcal{Q}_n z)(\zeta_{ij}), \quad s \in [0,1].$$

Then the discrete modified projection-type method for equation (1.1) is seeking an approximate solution x_n to x_0 such that

$$x_n - \mathcal{K}_n^S x_n = f, \tag{2.20}$$

while the discrete iterated solution is defined by

$$\widetilde{x}_n = \mathcal{K}_m^N x_n + f. \tag{2.21}$$

In the next section we consider the reduction of (2.20) to a system of nonlinear equations, and we give some details on the numerical implementation.

Implementation note. Let $\mathcal{Q}_n^G$ be the hyperinterpolation operator defined by (2.10) and $\kappa_j(s) = \langle \kappa(s,.), \varphi_j\rangle$, in order to give more information about the implementation of x_n, it is easy to show from (2.19) and (2.20), that x_n has the following form

$$x_n = f + \sum_{p=1}^{nr} a_p\varphi_p + \sum_{q=1}^{nr} b_q\kappa_q, \tag{2.22}$$

where the coefficients $\{a_i, b_i \;\; i = 0,1,...,n\}$ are obtained by substituting x_n from equation (2.22) into equation (2.20) then, we successively have

$$\begin{aligned}
(\mathcal{Q}_n^G\mathcal{K}_m^N)x_n &= \sum_{i=1}^{nr}\langle \mathcal{K}_m^N, \varphi_i\rangle_m\varphi_i = \widetilde{h}\sum_{i=1}^{nr}\left\{\sum_{j=1}^{m}\sum_{k=1}^{\rho}\omega_k z(\zeta_{jk})\langle\kappa(.,\zeta_{jk}),\varphi_i\rangle_m\right\}\varphi_i,\\
\overline{\mathcal{K}}_n x_n &= \sum_{i=1}^{nr}\langle z, \varphi_i\rangle_m\kappa_i,\\
(\mathcal{Q}_n^G\overline{\mathcal{K}}_n)x_n &= \sum_{i=1}^{nr}\langle\overline{\mathcal{K}}_n, \varphi_i\rangle_m\varphi_i = \sum_{i=1}^{nr}\left\{\sum_{l=1}^{nr}\langle\langle z,\varphi_l\rangle_m\kappa_l, \varphi_i\rangle_m\right\}\varphi_i,
\end{aligned}$$

where

$$z(t) = \psi\left(t, f(t) + \sum_{p=1}^{nr} a_p\varphi_p(t) + \sum_{q=1}^{nr} b_q\kappa_q(t)\right).$$

Therefore we can identify the coefficients of φ_i and κ_j respectively, and we obtain the nonlinear system of size $2n_r$,

$$\left\{\begin{aligned}
a_i &= \sum_{i=1}^{nr}\left\{\widetilde{h}\sum_{j=1}^{m}\sum_{k=1}^{\rho}\omega_k z(\zeta_{jk})\langle\kappa(.,\zeta_{jk}),\varphi_i\rangle_m - \sum_{l=1}^{nr}\langle\langle z,\varphi_l\rangle_m\kappa_l,\varphi_i\rangle_m\right\},\\
b_i &= \langle z, \varphi_i\rangle_m.
\end{aligned}\right.$$

For the interpolatory projection given by (2.11), we apply $\mathcal{Q}_n^C$ and $(\mathcal{I}-\mathcal{Q}_n^C)$ to equation (2.20), to obtain

$$\mathcal{Q}_n^C x_n - \mathcal{Q}_n^C \mathcal{K}_m^N = \mathcal{Q}_n^C f, \tag{2.23}$$

$$(\mathcal{I}-\mathcal{Q}_n^C)x_n - (\mathcal{I}-\mathcal{Q}_n^C)\overline{\mathcal{K}}_n x_n = (\mathcal{I}-\mathcal{Q}_n^C)f. \tag{2.24}$$

By writing

$$\mathcal{K}_m^N x_n = \mathcal{K}_m^N (I-\mathcal{Q}_n^C)x_n + \mathcal{K}_m^N \mathcal{Q}_n^C x_n, \tag{2.25}$$

and replacing $(\mathcal{I}-\mathcal{Q}_n^C)x_n$ by its expression from equation (2.24), $\mathcal{K}_m^N x_n$ becomes

$$\mathcal{K}_m^N x_n = \mathcal{K}_m^N\big((\mathcal{I}-\mathcal{Q}_n^C)\overline{\mathcal{K}}_n x_n + \mathcal{Q}_n^C x_n + (\mathcal{I}-\mathcal{Q}_n^C)f\big). \tag{2.26}$$

Now, by replacing $\mathcal{K}_m^N x_n$ in equation (2.23), we obtain

$$\mathcal{Q}_n^C x_n - \mathcal{Q}_n^C \mathcal{K}_m^N\big((\mathcal{I}-\mathcal{Q}_n^C)\overline{\mathcal{K}}_n x_n + \mathcal{Q}_n^C x_n + (\mathcal{I}-\mathcal{Q}_n^C)f\big) = \mathcal{Q}_n^C f, \tag{2.27}$$

and then for $i = 1, ..., nr$, we have

$$x_n(t_i) - \mathcal{K}_m^N\big((\mathcal{I}-\mathcal{Q}_n^C)\overline{\mathcal{K}}_n x_n + \mathcal{Q}_n^C x_n + (\mathcal{I}-\mathcal{Q}_n^C)f\big)(t_i) = f(t_i). \tag{2.28}$$

Now using the expressions of the operators $\mathcal{Q}_n^C$, $\mathcal{K}_m^N$ and $\overline{\mathcal{K}}_n$, we obtain the following nonlinear system of size n_r

$$a_i - \widetilde{h}\sum_{i=1}^{m}\sum_{j=1}^{\rho}\omega_j\kappa(t_i,\zeta_{ij})\psi\Bigg(\zeta_{ij}, \sum_{i=1}^{nr}(a_i - f_i)\ell_i + \sum_{i=1}^{nr}\langle\kappa(t_i,\zeta),\psi(\zeta,a_i)\ell_i\rangle_m$$
$$-\sum_{l=1}^{nr}\Bigg\{\sum_{i=1}^{nr}\langle\kappa(t_l,\zeta),\psi(\zeta,a_i)\ell_i\rangle_m\Bigg\}\ell_l + f(t)\Bigg) = f_i,$$

where $f_i := f(t_i)$, $\zeta := \zeta_{ij}$ for any $1 \le i \le m$ and $1 \le j \le \rho$ with $\{a_i = x_n(t_i), i = 0, 1, ..., n\}$ are the unknowns. From (2.24), the approximate solution is given by

$$\begin{aligned} x_n &= \mathcal{Q}_n^C x_n + (\mathcal{I}-\mathcal{Q}_n^C)\overline{\mathcal{K}}_n x_n + (\mathcal{I}-\mathcal{Q}_n^C)f, \\ &= f + \sum_{i=1}^{nr}(a_i - f_i)\ell_i + \sum_{i=1}^{nr}\langle\kappa(.,\zeta),\psi(\zeta,a_i)\ell_i\rangle_m \\ &\qquad - \sum_{l=1}^{nr}\Bigg\{\sum_{i=1}^{nr}\langle\kappa(t_l,\zeta),\psi(\zeta,a_i)\ell_i\rangle_m\Bigg\}\ell_l. \end{aligned} \tag{2.29}$$

3. Convergence rate

In this section, we analyse the existence and uniqueness of the approximate solutions of (1.1) and we discuss the superconvergence results. The following theorem can be proved by using Theorem 2 given in [21] .

Theorem 3.1. *Let $x_0 \in \mathcal{C}[0,1]$ be a unique solution of (1.1). Assume that* 1 *is not an eigenvalue of $\mathcal{K}'(x_0)$. Then there exists a real number $\delta_0 > 0$ such that the approximate equation (2.20) has a unique solution x_n in $\mathcal{B}(x_0,\delta_0)$ for a sufficiently large n. Moreover, there exists a constant $0 < p < 1$, independent of n such that*

$$\frac{\alpha_n}{1+p} \le \|x_0 - x_n\|_\infty \le \frac{\alpha_n}{1-p}, \tag{3.1}$$

where $\alpha_n = \big\|(\mathcal{I}-\mathcal{K}_n^{S'}(x_0))^{-1}(\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0))\big\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 3.1. *Assume that* 1 *is not an eigenvalue of* $\mathcal{K}'(x_0)$. *Then, for* n *large enough,* $(\mathcal{I}-\mathcal{K}_n^{S'}(x_0))^{-1}$ *exists and it is a bounded linear operator, then, there exists a constant* $A>0$ *such that*

$$\|(\mathcal{I}-\mathcal{K}_n^{S'}(x_0))^{-1}\|_\infty \leq A. \tag{3.2}$$

Proof. Since the operator $\mathcal{Q}_n$ converge pointwise to the identity operator and $\mathcal{K}'(x_0), \overline{\mathcal{K}'_n}(x_0)$ are compact, it follows that

$$max\{\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(x_0)\|_\infty, \|(\mathcal{I}-\mathcal{Q}_n)\overline{\mathcal{K}'_n}(x_0)\}\|_\infty\} \to 0, \qquad n\to\infty.$$

Choose $m \geq n$, then from (2.8), we obtain $\mathcal{K}_m^N(x_0) \to \mathcal{K}(x_0)$ pointwise in $\mathcal{C}[0,1]$ as $n\to\infty$. We get

$$\mathcal{K}'(x_0)-\mathcal{K}_n^{S'}(x_0) = \mathcal{Q}_n(\mathcal{K}'(x_0)-\mathcal{K}_m^{N'}(x_0)) + (\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}'(x_0)-\overline{\mathcal{K}'_n}(x_0)).$$

Thus, since the projection $\mathcal{Q}_n$ is uniformly bounded, we obtain

$$\begin{aligned}&\|\mathcal{K}'(x_0)-\mathcal{K}_n^{S'}(x_0)\|_\infty \leq q\|\mathcal{K}'(x_0)-\mathcal{K}_m^{N'}(x_0))\|_\infty +\\ &\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(x_0)\|_\infty + \|(\mathcal{I}-\mathcal{Q}_n)\overline{\mathcal{K}'_n}(x_0)\}\|_\infty \to 0, \quad \text{as} \quad n\to\infty.\end{aligned}$$

Hence by Lemma 2.6 in [4], the operators $(\mathcal{I}-\mathcal{K}_n^{S'}(x_0))^{-1}$ exists and are uniformly bounded, for some sufficiently large n. This completes the proof. ☐

Lemma 3.2. *Let* x_0 *be the unique solution of (1.1). In case of the hyperinterpolation projection, we assume that* $\kappa \in \mathcal{C}^r[0,1]^2$, $\psi \in \mathcal{C}^r(\Omega)$ *and* $f \in \mathcal{C}^r[0,1]$, *while, in the case of the interpolatory projection, we assume that* $\kappa \in \mathcal{C}^{2r}[0,1]^2$, $\psi \in \mathcal{C}^{2r}(\Omega)$ *and* $f \in \mathcal{C}^{2r}[0,1]$. *Then*

$$\|(\mathcal{I}-\mathcal{Q}_n)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\|_\infty = \mathcal{O}\left(\max\left\{h^{3r}, \widetilde{h}^d\right\}\right). \tag{3.3}$$

Proof. We write

$$\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0) = \mathcal{K}(x_0)-\mathcal{K}_n(x_0)+\mathcal{K}_n(x_0)-\overline{\mathcal{K}}_n(x_0) \tag{3.4}$$

Now, applying $(\mathcal{I}-\mathcal{Q}_n)$ to both sides of equations (3.4), we have that

$$\begin{aligned}\|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0))\|_\infty &\leq \|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}(x_0)-\mathcal{K}_n(x_0))\|_\infty \\ &+ \|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}_n(x_0)-\overline{\mathcal{K}}_n(x_0)\|_\infty.\end{aligned} \tag{3.5}$$

Let $\mathcal{Q}_n^C$ be the interpolatory operator defined by (2.11). By the formula

$$[\mathcal{K}_n(x_0)-\overline{\mathcal{K}}_n(x_0)](s) = \sum_{i=1}^{nr} z(t_i)\left[\int_0^1 \kappa_s(t)\ell_i(t)dt - \langle \kappa_s, \ell_i\rangle_m\right],$$

it holds that

$$[\mathcal{K}_n(x_0)-\overline{\mathcal{K}}_n(x_0)]^{(r)}(s) = \sum_{i=1}^{nr} z(t_i)\left[\int_0^1 q_s(t)\ell_i(t)dt - \widetilde{h}\sum_{j=1}^{m}\sum_{k=1}^{\rho} w_k q_s(\zeta_{jk})\ell_i(\zeta_{jk})\right],$$

where $q_s(t) = \dfrac{\partial^r \kappa}{\partial s^r}(s,t)$. Since ℓ_i is a polynomial of degree $r-1$ on each subinterval, then $q_s\ell_i \in \mathcal{C}^d[s_{i-1}, s_i]$. Hence by (2.4) we have for any $s\in[0,1]$,

$$\left|[\mathcal{K}_n(x_0)-\overline{\mathcal{K}}_n(x_0)]^{(r)}(s)\right| \leq nr\|z\|_\infty \|(q_s\ell_i)^{(r)}\|_\infty \widetilde{h}^d,$$

Since $m = np$, it follows that

$$\|[\mathcal{K}_n(x_0) - \overline{\mathcal{K}}_n(x_0)]^{(r)}\|_\infty \leq \binom{r}{p} \|z\|_\infty \|(q_s \ell_i)^{(r)}\|_\infty \widetilde{h}^{d-1}, \tag{3.6}$$

which means that

$$\begin{aligned}\|(\mathcal{I} - \mathcal{Q}_n^G)[\mathcal{K}_n(x_0) - \overline{\mathcal{K}}_n(x_0)]\|_\infty &\leq c_2 h^r \|[\mathcal{K}_n(x_0) - \overline{\mathcal{K}}_n(x_0)]^{(r)}\|_\infty, \\ &\leq c_2 r \|z\|_\infty \|(q_s \ell_i)^{(r)}\|_\infty \widetilde{h}^d. \end{aligned} \tag{3.7}$$

A similar estimate can be obtained in the case of the hyperinterpolation projection. Thus, (3.3) follows from (2.14), (3.5) and (3.7). □

Now we are ready to state and prove the main theorem of this subsection.

Theorem 3.2. *Let x_0, x_n be the solution of (1.1) and (2.20) respectively, and asume that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. In case of the projection, we assume that $\kappa \in \mathcal{C}^r[0,1]^2$, $\psi \in \mathcal{C}^r(\Omega)$ and $f \in \mathcal{C}^r[0,1]$, while, in the case of the interpolatory projection, we assume that $\kappa \in \mathcal{C}^{2r}[0,1]^2$, $\psi \in \mathcal{C}^{2r}(\Omega)$ and $f \in \mathcal{C}^{2r}[0,1]$. Then*

$$\|x_0 - x_n\|_\infty = \mathcal{O}\left(\max\left\{h^{3r}, \widetilde{h}^d\right\}\right). \tag{3.8}$$

Proof. We see from Theorem 3.1 that to estimate $\|x_0 - x_n\|_\infty$ we need to estimate $\|\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)\|_\infty$. By writing

$$\|\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)\|_\infty = \|\mathcal{Q}_n[\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)] + (\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(x_0) - \overline{\mathcal{K}}_n(x_0)]\|_\infty.$$

From Lemma 3.1, we have

$$\|x_0 - x_n\|_\infty \leq Aq\|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty + A\|(\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(x_0) - \overline{\mathcal{K}}_n(x_0)]\|_\infty. \tag{3.9}$$

By combining (2.8) and (3.3), the estimate (3.8) follows. □

Now, we prove the following crucial lemma.

Lemma 3.3. *Let $\kappa \in \mathcal{C}^d[0,1]^2$ and $\frac{\partial \psi}{\partial u} \in \mathcal{C}^r(\Omega)$. Then $\mathcal{K}_m^{N'}(x_0)$ is Lipschitz continuous in $\mathcal{B}(x, \delta_0)$, that is, there exists a constant $\delta_2 > 0$ independent of n such that*

$$\|[(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g]^{(p)}\|_\infty \leq \delta_2 \|\kappa\|_{p,\infty} \|x_0 - x\|_\infty \|g\|_\infty, \quad x \in \mathcal{B}(x_0, \delta_0). \tag{3.10}$$

Proof. Using Lipschitz continuity of $\frac{\partial \psi}{\partial u}(t,u)$ and the estimate (2.6), we have for $p = 0, 1, \ldots, r$

$$\begin{aligned}&\|(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g]^{(p)}\|_\infty = \\ &\quad \max_{s\in[0,1]} |\widetilde{h} \sum_{i=1}^m \sum_{j=1}^\rho w_j \frac{\partial^p \kappa}{\partial s^p}(s, \zeta_{ij})[\frac{\partial \psi}{\partial u}(\zeta_{ij}, x_0(\zeta_{ij})) - \frac{\partial \psi}{\partial u}(\zeta_{ij}, x(\zeta_{ij}))] g(\zeta_{ij})|, \\ &\quad \leq \|\kappa\|_{p,\infty} \widetilde{h} \sum_{i=1}^m \sum_{j=1}^\rho w_j \left| \frac{\partial \psi}{\partial u}(\zeta_{ij}, x_0(\zeta_{ij})) - \frac{\partial \psi}{\partial u}(\zeta_{ij}, x(\zeta_{ij})) \right| |g(\zeta_{ij})|, \\ &\quad \leq \delta_2 \|\kappa\|_{p,\infty} \|x_0 - x\|_\infty \|g\|_\infty.\end{aligned}$$

This completes the proof. □

The following results are needed to obtain the order of convergence of $\widetilde{x}_n$ to x_0.

Lemma 3.4. *Assume that $\kappa \in \mathcal{C}^d[0,1]^2$ and $\frac{\partial \psi}{\partial u} \in \mathcal{C}^r(\Omega)$. Then, the linear operator $\mathcal{K}'_n(x_0)$ is Lipschitz continuous in a neiberhood of x_0, that is, there exists a constant $\delta_3 > 0$ independent of n such that*

$$\|\mathcal{K}_n^{S'}(x_0) - \mathcal{K}_n^{S'}(x)\|_\infty \leq \delta_3 \|x_0 - x\|_\infty, \quad x \in \mathcal{B}(x_0, \delta_0). \tag{3.11}$$

Proof. From equation (2.19), we have

$$\mathcal{K}_n^{S'}(y) = \mathcal{Q}_n \mathcal{K}_m^{N'}(y) + (\mathcal{I} - \mathcal{Q}_n)\overline{\mathcal{K}'_n}(y), \qquad y \in \mathcal{C}[0,1]. \tag{3.12}$$

Hence, for any $g \in \mathcal{C}[0,1]$,

$$\begin{aligned}\|[\mathcal{K}_n^{S'}(x_0) - \mathcal{K}_n^{S'}(x)]g\|_\infty &= \|\mathcal{Q}_n(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty \\ &\quad + \|(\mathcal{I} - \mathcal{Q}_n)(\overline{\mathcal{K}'_n}(x_0) - \overline{\mathcal{K}'_n}(x))g\|_\infty.\end{aligned}$$

Now using the Lipschitz's continuity of $\frac{\partial \psi}{\partial u}$ and (3.10), we get

$$\begin{aligned}&\|\mathcal{Q}_n(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty \leq \|(\mathcal{Q}_n - \mathcal{I})(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty \\ &+ \|(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty, \\ &\leq c_2 h^r \|[(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g]^{(r)}\|_\infty + \|(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty, \\ &\leq c_2 h^r \delta_2 \|\kappa\|_{r,\infty} \|x_0 - x\|_\infty \|g\|_\infty + \delta_2 \|\kappa\|_{0,\infty} \|x_0 - x\|_\infty \|g\|_\infty.\end{aligned} \tag{3.13}$$

Similarly, the Cauchy-Schwarz inequality and (2.13), yields

$$\begin{aligned}&\|(\mathcal{I} - \mathcal{Q}_n)(\overline{\mathcal{K}'}_n(x_0) - \overline{\mathcal{K}'_n}(x))g\|_\infty \leq c_2 h^r \|[(\overline{\mathcal{K}'_n}(x_0) - \overline{\mathcal{K}'_n}(x))g]^{(r)}\|_\infty, \\ &\leq c_2 h^r \|\kappa\|_{r,\infty} \|\mathcal{Q}_n[\frac{\partial \psi}{\partial u}(.,x_0(.)) - \frac{\partial \psi}{\partial u}(.,x(.))]\|_{\mathcal{L}^2} \|g\|_{\mathcal{L}^2}, \\ &\leq c_2 q \delta_2 h^r \|\kappa\|_{r,\infty} \|x_0 - x\|_\infty \|g\|_\infty.\end{aligned} \tag{3.14}$$

The desired result follows from (3.13) and (3.14) with,

$$\delta_3 = [\|\kappa\|_{0,\infty} + (1+q) c_2 h^r \|\kappa\|_{r,\infty}]\delta_2.$$

This completes the proof. □

In the next theorem we give the approximation error of the iterated discrete modified projection-type method.

Theorem 3.3. *We suppose that $\kappa \in \mathcal{C}^r[0,1]^2$ and $\frac{\partial \psi}{\partial u} \in \mathcal{C}(\Omega)$. Let $x_0 \in \mathcal{C}[0,1]$ be the unique solution of (1.1). Then, for n sufficiently large, the iterated solution $\widetilde{x}_n$ given by (2.21), satisfies*

$$\begin{aligned}\|x_0 - \widetilde{x}_n\|_\infty \leq c_4 \|x_0 - x_n\|_\infty^2 + A\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)]\|_\infty \\ + \|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty,\end{aligned} \tag{3.15}$$

where c_4 is a constant independent of n and A is such that $\|(\mathcal{I} - \mathcal{K}_n^{S'}(x_0))^{-1}\|_\infty \leq A < \infty$.

Proof. Note that from (1.1) and (2.21) we have

$$\begin{aligned}x_0 - \widetilde{x}_n &= \mathcal{K}x_0 - \mathcal{K}_m^N x_n, \\ &= \mathcal{K}_m^N x_0 - \mathcal{K}_m^N x_n - \mathcal{K}_m^N x_0 + \mathcal{K}x_0.\end{aligned} \tag{3.16}$$

Therefore, for some $0 < \theta < 1$, we get

$$\begin{aligned}\mathcal{K}_m^N x_0 - \mathcal{K}_m^N x_n &= \mathcal{K}_m^{N'}(x_0 + \theta(x_0 - x_n))(x_0 - x_n),\\ &= [\mathcal{K}_m^{N'}(x_0 + \theta(x_0 - x_n)) - \mathcal{K}_m^{N'}(x_0) + \mathcal{K}_m^{N'}(x_0)](x_0 - x_n).\end{aligned}$$

Taking the norm on both sides of the above equation and applying the Lipschitz's continuity of $\mathcal{K}_m^{N'}$ and using (3.10) , we can show that

$$\begin{aligned}\|x_0 - \widetilde{x}_n\|_\infty \le \delta_2\theta\|\kappa\|_{0,\infty}\|x_0 - x_n\|_\infty^2 + \|\mathcal{K}_m^{N'}(x_0)(x_0 - x_n)\|_\infty\\ + \|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty.\end{aligned} \tag{3.17}$$

Let

$$\begin{aligned}(\mathcal{I} - \mathcal{K}_n^{S'}(x_0))(x_0 - x_n) = \mathcal{K}(x_0) - \mathcal{K}_n^S(x_0) - \mathcal{K}_n^{S'}(x_0)(x_0 - x_n)\\ + \mathcal{K}_n^S(x_0) - \mathcal{K}_n^S(x_n).\end{aligned}$$

Applying $\mathcal{K}_m^{N'}(x_0)$ to both sides and using the mean value theorem, we obtain

$$\begin{aligned}\mathcal{K}_m^{N'}(x_0)(x_0 - x_n) &= \mathcal{K}_m^{N'}(x_0)(\mathcal{I} - \mathcal{K}_n^{S'}(x_0))^{-1}[\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)\\ &\quad - \mathcal{K}_n^{S'}(x_0)(x_0 - x_n) + \mathcal{K}_n^S(x_0) - \mathcal{K}_n^S(x_n)]\\ &= \mathcal{K}_m^{N'}(x_0)(\mathcal{I} - \mathcal{K}_n^{S'}(x_0))^{-1}[\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)] + \mathcal{K}_m^{N'}(x_0)\times\\ &\quad (\mathcal{I} - \mathcal{K}_n^{S'}(x_0))^{-1}[\mathcal{K}_n^{S'}(x_0 + \theta(x_0 - x_n)) - \mathcal{K}_n^{S'}(x_0)](x_0 - x_n),\end{aligned}$$

where $0 < \theta < 1$. Now from estimates (2.7), (3.2) and Lemma 3.4 one has

$$\begin{aligned}\|\mathcal{K}_m^{N'}(x_0)(x_0 - x_n)\|_\infty \le A\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)]\|_\infty\\ + A\theta\Psi_1\delta_3\|\kappa\|_{0,\infty}\|x_0 - x_n\|_\infty^2.\end{aligned}$$

By Combining (3.17) with the above estimate, we get

$$\begin{aligned}\|x_0 - \widetilde{x}_n\|_\infty \le & c_4\|x_0 - x_n\|_\infty^2 + A\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)]\|_\infty\\ & + \|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty,\end{aligned}$$

with $c_4 = \theta\|\kappa\|_{0,\infty}(\delta_2 + A\Psi_1\delta_3)$. This completes the proof. □

The following Lemma is needed to obtain an error estimate for the term $\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0) - \mathcal{K}_n^S(x_0)]\|_\infty$.

Lemma 3.5. *Let $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathbb{X}_n$ be the hyperinterpolation or the interpolatory projection operator defined by (2.10) and (2.11). Assume that $\kappa \in \mathcal{C}^d[0,1]^2$, $\frac{\partial\psi}{\partial u} \in \mathcal{C}^r(\Omega)$ and that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. Then,*

$$\|\mathcal{K}_m^{N'}(x_0)\mathcal{Q}_n[\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)]\|_\infty = \mathcal{O}(\widetilde{h}^d). \tag{3.18}$$

Proof. From estimates (2.7) and (2.15), we have

$$\begin{aligned}&\left|\mathcal{K}_m^{N'}(x_0)\mathcal{Q}_n[\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)](s)\right| =\\ &\left|\widetilde{h}\sum_{i=1}^{m}\sum_{j=1}^{\rho} w_j\kappa(s,\zeta_{ij})\frac{\partial\psi}{\partial u}(\zeta_{ij}, x_0(\zeta_{ij}))\mathcal{Q}_n[\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)](\zeta_{ij})\right|,\\ &\le \|\mathcal{K}_m^{N'}(x_0)\|_\infty\|\mathcal{Q}_n[\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)]\|_\infty,\\ &\le q\Psi_1\|\kappa\|_{0,\infty}\|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty.\end{aligned}$$

Thus, by using (2.8), we deduce that

$$\|\mathcal{K}_m^{N'}(x_0)\mathcal{Q}_n[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty \leq c_1 q\Psi_1\|\kappa\|_{0,\infty}\|\kappa\|_{d,\infty}\widetilde{h}^d,$$

which completes the proof. □

The result below state that the iterated discrete modified Galerkin-type solution defined by (2.21) can converges to x_0 faster than $\widetilde{x}_n$.

Theorem 3.4. *Let $\mathcal{Q}_n^G : \mathcal{C}[0,1] \to \mathbb{X}_n$ be the hyperinterpolation projection defined by (2.10). Assume that $\kappa \in \mathcal{C}^d[0,1]^2$, $\frac{\partial\psi}{\partial u} \in \mathcal{C}^r(\Omega)$ and let $x_0 \in \mathcal{C}[0,1]$ be a unique solution of (1.1). Then, for n sufficiently large, the iterated discrete solution $\widetilde{x}_n$ given by (2.21), satisfies*

$$\|x_0-\widetilde{x}_n\|_\infty = \mathcal{O}\left(\max\left\{h^{4r},\widetilde{h}^d\right\}\right). \tag{3.19}$$

Proof. For the second term of (3.15), we can write

$$\begin{aligned}\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^S(x_0)]\|_\infty = \|&\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\|_\infty \\ &+\|\mathcal{K}_m^{N'}(x_0)\mathcal{Q}_n^G[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty.\end{aligned} \tag{3.20}$$

First, using the estimate (2.13), we get

$$\begin{aligned}&\left|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)](s)\right| = \\ &\left|\widetilde{h}\sum_{i=1}^m\sum_{j=1}^\rho w_j\kappa(s,\zeta_{ij})\frac{\partial\psi}{\partial u}(\zeta_{ij},x_0(\zeta_{ij}))(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)](\zeta_{ij})\right| \\ &=\left\langle \kappa_s\psi_1,(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\right\rangle_m \\ &=\left\langle (\mathcal{I}-\mathcal{Q}_n^G)\kappa_s\psi_1,(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\right\rangle_m \\ &\leq c_2\|(\kappa_s\psi_1)^{(r)}\|_\infty\|(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)\|_\infty h^r.\end{aligned}$$

Hence, from (3.3)

$$\|(\mathcal{I}-\mathcal{Q}_n^G)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\|_\infty = \mathcal{O}\left(\max\left\{h^{3r},\tilde{h}^d\right\}\right), \tag{3.21}$$

Then by combining (3.15), (3.18), (3.20) and (3.21), the estimate (3.19) is proved. □

The following result give the superconvergence of the iterated discrete modified collocation-type solution $\widetilde{x}_n$ to x_0.

Theorem 3.5. *Let $\mathcal{Q}_n^C : \mathcal{C}[0,1] \to \mathbb{X}_n$ be the interpolatory projection defined by (2.11). Assume that $\kappa \in \mathcal{C}^d[0,1]^2$, $\frac{\partial\psi}{\partial u} \in \mathcal{C}^r(\Omega)$ and let $x_0 \in \mathcal{C}[0,1]$ be a unique solution of (1.1). Then, for n sufficiently large, the iterated discrete solution $\widetilde{x}_n$ given by (2.21), satisfies*

$$\|x_0-\widetilde{x}_n\|_\infty = \mathcal{O}\left(\max\left\{h^{4r},\widetilde{h}^d\right\}\right). \tag{3.22}$$

Proof. By (3.15), we consider

$$\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^S(x_0)]\|_\infty = \|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathcal{Q}_n^C)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\|_\infty + \|\mathcal{K}_m^{N'}(x_0)\mathcal{Q}_n^C[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty. \tag{3.23}$$

Then, by using Lemma 2.1, we write

$$\|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathcal{Q}_n^C)[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]\|_\infty \le c_5\Psi_2\|\kappa\|_{r,\infty}\|\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)\|_{2r,\infty}h^{2r}, \tag{3.24}$$

then for $0 \le p \le 2r$, we have

$$[\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)]^{(p)}(s) = [\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(p)}(s) + [\mathcal{K}_n(x_0)-\overline{\mathcal{K}}_n(x_0)]^{(p)}(s),$$

we have

$$[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(p)}(s) = \int_0^1 q_s(t)(\mathcal{I}-\mathcal{Q}_n^C)z_0(t)dt,$$

where $z_0(t) \equiv \psi(t, x_0(t))$. By using (2.16) we obtain,

$$\begin{aligned}\|\mathcal{K}(x_0)-\mathcal{K}_n(x_0)\|_{2r,\infty} &= \sum_{p=0}^{2r}\|[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(p)}\|_\infty, \\ &\le c_3(2r+1)\|q_s\|_{r,\infty}\|\psi\|_{2r,\infty}h^{2r},\end{aligned} \tag{3.25}$$

Since $d \ge 2r$, from (3.6) and (3.25) ,we deduce that

$$\|\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)\|_{2r,\infty} = \mathcal{O}\left(h^{2r}\right).$$

Replacing $\|\mathcal{K}(x_0)-\overline{\mathcal{K}}_n(x_0)\|_{2r,\infty}$ by its expression in (3.24) and combining (3.15), (3.18) and (3.23), we obtain the desired result. □

4. Numerical results

In this section, numerical examples are given to illustrate the theory established in the previous sections. It is noted that the Newton–Raphson method was used to solve the nonlinear systems. The numerical algorithms are compiled by using `WOLFRAM MATHEMATICA`. Let $\mathbb{X}_n$ be the space of piecewise constant functions $(r = 1)$ with respect to the uniform partition of $[0, 1]$

$$0 = \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n} = 1.$$

The projection $\mathcal{Q}_n^C$ is chosen to be the interpolatory projection at the $nr = n$ midpoints

$$t_i^{(n)} = \frac{2i-1}{2n},\ i = 1, \ldots, n$$

or the restriction to $\mathcal{L}^\infty[0,1]$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ to $\mathbb{X}_n$. Let

$$\|x_0 - x_n\|_\infty = \mathcal{O}(h^\alpha) \quad \text{and} \quad \|x_0 - \widetilde{x}_n\|_\infty = \mathcal{O}(h^\beta).$$

Note that, for evaluating the required integrals we use the composite 2 points Gauss quadrature with respect to the uniform partition of $[0,1]$ with $m = 128$ intervals. The computations are done for $n = 2, 4, 8, 16$ and 32. Thus,

$$r = 1, \quad d = 4, \quad \widetilde{h} = 2^{-7} \quad \text{and} \quad 2^{-5} \leq h \leq 2^{-1}.$$

Hence

$$\widetilde{h}^d \leq h^{4r} \leq h^{3r} \quad \text{this implies} \quad 2^{-28} \leq 2^{-20} \leq 2^{-15}.$$

The orders of convergence are computed by using the following formula

$$\alpha = \frac{log\left(\frac{\|x_0 - x_n\|_\infty}{\|x_0 - x_{2n}\|_\infty}\right)}{log(2)} \quad \text{and} \quad \beta = \frac{log\left(\frac{\|x_0 - \widetilde{x}_n\|_\infty}{\|x_0 - \widetilde{x}_{2n}\|_\infty}\right)}{log(2)}.$$

From Theorems 3.8, 3.19 and 3.22, the expected orders of convergence are

$$\alpha = 3 \quad \text{and} \quad \beta = 4.$$

We present the errors of the discrete solution and discrete iterated solution in the infinity norm. In Tables 1 and 3, we present the maximum errors of the approximation solution obtained by using the discrete modified Galerkin-type method and its iterated version. The corresponding errors obtained by using the discrete modified collocation-type method and its iterated version are given in Table 2 and 4.

Example 1. We consider the following Hammerstein equation

$$x(s) - \int_0^1 e^{s-t} \cos(x(t))dt = f(s) \quad s \in [0,1],$$

where $f(s)$ is selected so that $x_0(t) = t$ is the exact solution.

Example 2. Now we consider the nonlinear integral equation defined by

$$x(s) - \int_0^1 \frac{\sin\left(\frac{\pi}{2}(t-s)\right)}{1 + x(t)} dt = \left(\frac{2}{\pi} - 1\right) \cos\left(\frac{\pi}{2}s\right) + \left(\frac{\log(4)}{\pi} + 1\right) \sin\left(\frac{\pi}{2}s\right),$$

with $s \in [0,1]$ and the exact solution is unknown. The results are given in Tables 3 and 4.

TABLE 1. Discrete modified Galerkin-type method.

	$\|x_0 - x_n\|_\infty$	α	$\|x_0 - \widetilde{x}_n\|_\infty$	β
2	2.96×10^{-3}	–	8.04×10^{-5}	–
4	3.98×10^{-4}	2.89	4.81×10^{-6}	4.06
8	5.19×10^{-5}	2.94	2.97×10^{-7}	4.01
16	6.64×10^{-6}	2.96	1.85×10^{-8}	4.00
32	8.39×10^{-7}	2.98	1.15×10^{-9}	4.00

For the sake of completeness, we illustrate in Figures 1 and 2 the errors in absolute value obtained by various methods using example 1 (Hyperinterpolation in red and interpolatory projection in blue) with $n = 4$.

Table 2. Discrete modified collocation-type method.

	$\|x_0 - x_n\|_\infty$	α	$\|x_0 - \widetilde{x}_n\|_\infty$	β
2	2.70×10^{-4}	–	7.43×10^{-6}	–
4	4.83×10^{-5}	2.48	6.80×10^{-7}	3.45
8	6.65×10^{-6}	2.86	8.58×10^{-8}	3.89
16	8.60×10^{-7}	2.95	2.91×10^{-9}	3.97
32	1.08×10^{-7}	2.98	1.83×10^{-10}	3.99

Table 3. Discrete modified Galerkin-type method.

	$\|x_0 - x_n\|_\infty$	α	$\|x_0 - \widetilde{x}_n\|_\infty$	β
2	4.13×10^{-3}	–	3.81×10^{-5}	–
4	6.24×10^{-4}	2.72	2.44×10^{-6}	3.96
8	8.28×10^{-5}	2.91	1.51×10^{-7}	4.01
16	1.05×10^{-5}	2.96	9.41×10^{-9}	4.00
32	1.33×10^{-7}	2.98	5.87×10^{-10}	4.00

Table 4. Discrete modified collocation-type method.

	$\|x_0 - x_n\|_\infty$	α	$\|x_0 - \widetilde{x}_n\|_\infty$	β
2	6.68×10^{-3}	–	4.73×10^{-4}	–
4	8.99×10^{-4}	2.89	3.07×10^{-5}	3.94
8	1.16×10^{-4}	2.95	1.93×10^{-6}	3.98
16	1.47×10^{-5}	2.97	1.21×10^{-7}	3.99
32	1.85×10^{-6}	2.99	7.60×10^{-9}	4.00

The results illustrated in the above tables show that the computed orders of convergence confirm the theoretical ones. From Tables 1 and 2, it can be seen that to obtain the error $\|x_0 - x_{32}\|_\infty$ of order 10^{-7}, a system of size 32 is needed to be solved in the discrete modified collocation-type method, while as in the discrete modified Galerkin-type method we need to solve a system of size 64 to obtain an accuracy of comparable order. Also when computing $\widetilde{x}_8$ by a discrete modified Galerkin-type method, which is obtained by solving a system of size 16, we get an error of the order of 10^{-7}. As a result, the discrete modified collocation-type method has benefits theoretically and computationally over the discrete modified Galerkin-type method, which require solving an extremely large nonlinear system that is computationally very expensive. It should be mentioned that the iterated discrete modified projection-type method converges faster than the discrete modified projection-type method. There are similar observations to be made in Tables 3 and 4.

5. Conclusion

The main purpose of this article is to investigate a discrete versions of the modified projection-type method for approximating the solutions of Hammesrtein integral equations. Theoretically, the error bound and the convergence rate of the presented method are obtained. Finally, we have presented some numerical

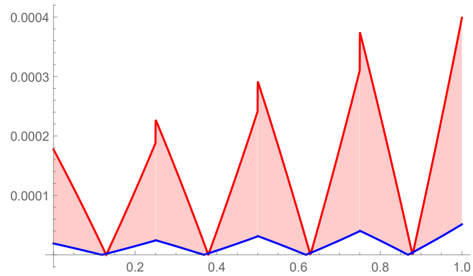

FIGURE 1. Discrete modified projection-type method.

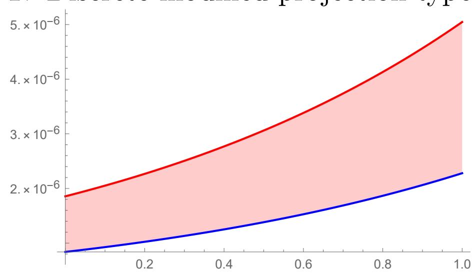

FIGURE 2. Iterated discrete modified projection-type method.

examples to show the validity of the method and confirm the theoretical error estimates. The results in this paper can be extended to weakly singular kernels Hammerstein integral equations. This study can be a topic for another paper.

References

[1] R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive solutions of differential, difference and integral equations,* Kluwer Academic Publishers, Dordrecht, 1999.

[2] C. Allouch, D. Sbibih, M. Tahrichi, Superconvergent product integration method for Hammerstein integral equations, *J. Int. Eq. Appl.* **31** (2019), 1–28.

[3] C. Allouch, M. Arrai, H. Bouda, Richardson extrapolation of Kantorovich and degenerate kernel methods for Fredholm integral equations of the second kind, *Khayyam J. Math.* **8** (2022) 204–218.

[4] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind,* Cambridge University Press, Cambridge, 1997.

[5] T.A. Burton, *Volterra integral and differential equations,* Academic Press, New York, 1983.

[6] F. Chatelin, *Spectral approximation of linear operators,* Academic Press, New York, 1983.

[7] M. Golberg, C. Chen, *Discrete Projection Methods for Integral Equations,* Computational Mechanics Publications, 1997.

[8] M. Golberg, Improved convergence rates for some discrete Galerkin methods, *J. Int. Eqn.Appl.* **8** (1996), 307–335.

[9] L. Grammont, R. P. Kulkarni, T.J. Nidhin, Modified projection method for Urysohn integral equations with non-smooth kernels, *J. Comp. Appl. Math.* **294** (2016), 309–322.

[10] H. Kaneko, R. Noren and P.A. Padilla, Superconvergence of the iterated collocation methods for Hammerstein equations, *J. Comp. Appl. Math.* **80** (1997) 335–349.

[11] H. Kaneko, Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations, *SIAM J. Numer. Anal.* **33** (1996) 1048–1064.

[12] K. Kant, M. Mandal and G. Nelakanti, Jacobi spectral Galerkin methods for a class of nonlinear weakly singular volterra integral equations, *Adv. Appl. Math. Mech.* **13** (2021) 1227–1260.

[13] R.P. Kulkarni, G. Rakshit, Discrete modified projection method for Urysohn integral equations with smooth kernels, *Appl Num. Math.* **126** (2018), 180–198.

[14] R. P. Kulkarni, A. Rane, Asymptotic expansions for approximate solutions of Hammerstein integral equations with Green's function type kernels, *Math. Model. Anal.* **19** (2014) 127–143.

[15] R.P. Kulkarni, T J. Nidhin, Asymptotic error analysis of projection and modified projection methods for nonlinear integral equations, *J. Integr. Equations Appl.* **27** (2015) 67–101.

[16] S. Kumar, I.H. Sloan, A new collocation-type method for Hammerstein equations, *Math.Comp.* **178** (1987) 585–593.

[17] C. de Boor, B. Swartz, Collocation at Gaussian points, *SIAM J. Numer. Anal.* **10** (1973) 582–606.

[18] S. Kumar, Superconvergence of a collocation-type method for Hammerstein equations, *IMA J. Numer. Anal.* **7** (1987) 313–325.

[19] D. O'Regan and M. Meehan, *Existence theory for nonlinear integral and integro-differential equations,* Kluwer Academic Publishers, Dordrecht, 1998.

[20] I. H. Sloan, Polynomial interpolation and hyperinterpolation over general regions, *J. Approx.Theory.* **83** (1995), 238–254.

[21] G. M. Vainikko, Galerkin's perturbation method and the general theory of approximate methods for non-linear equations, *USSR Comput. Math. Math. Phys.* **7** (1967) 1–41.

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Received: November 30, 2022; Revised: March 10, 2023; Accepted: May 16, 2023