MORREY-TYPE BANACH SPACES, MAXIMAL OPERATOR AND FOURIER MULTIPLIERS

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Abstract. Let \((F, \| \cdot \|_F)\) be a Banach space of complex-valued measurable functions on \(\mathbb{R}^{n+1}\). In this paper, we consider the Morrey-type Banach space \(M_F(p, \lambda)\) as well as its weak type \(M_F^*(1, \lambda)\). We develop the theory of maximal operator and Fourier multipliers on these spaces.

1. Introduction

Let \(\mathbb{R}^n\) be the \(n\)-dimensional Euclidean space of points \(x = (x_1, ..., x_n)\) with norm \(\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}\). For \(x \in \mathbb{R}^n\) and \(r > 0\), let \(B(x, r)\) be the open ball in \(\mathbb{R}^n\) centered at \(x\) with radius \(r\).

For \(f \in L_1^{loc}(\mathbb{R}^n)\), the Hardy-Littlewood maximal operator \(M\) is defined by

\[
M(f)(x) = \sup_{t>0} |B(x, t)|^{-1} \int_{B(x, t)} |f(y)| \, dy,
\]

where \(|B(x, t)|\) is the Lebesgue measure of the ball \(B(x, t)\).

For a domain \(\Omega \subset \mathbb{R}^n\) denote by \(WL_p(\Omega)\), the weak \(L_p\) space of locally integrable functions \(f\) on \(\Omega\) with the finite quasi-norm

\[
\|f\|_{WL_p(\Omega)} = \sup_{t>0} t \left| \{ x \in \Omega \mid |f(x)| > t \} \right|^{1/p}.
\]

Let \(0 < p \leq \infty\) and \(\lambda \in \mathbb{R}\). Denote by \(M(p, \lambda)\) the space of all functions \(f \in L_p^{loc}(\mathbb{R}^n)\) satisfying

\[
\|f\|_{M(p, \lambda)} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x, r))} < \infty.
\]

Denote by \(M^*(p, \lambda)\) the space of all functions \(f \in L_p^{loc}(\mathbb{R}^n)\) satisfying

\[
\|f\|_{M^*(p, \lambda)} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x, r))} < \infty.
\]

If \(0 < p \leq \infty\), then \(M(p, 0) = L_p(\mathbb{R}^n)\) and \(M(p, n) = L_\infty(\mathbb{R}^n)\) isometrically.

The space \(M(p, \lambda)\), called the Morrey space, is first introduced by C. Morrey in 1938 in [9]. It plays an important role in the study of partial differential equations, especially the local behaviour of the solutions of elliptic partial differential equations.

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equations. We refer the reader to the papers [1, 5, 10, 11] and the references therein. The following result is from [5], which is a model to our study.

**Theorem 1.1** (see [5]). Suppose $1 \leq p < \infty$ and $0 < \lambda < n$.

1. If $p > 1$, then the maximal operator $M$ is bounded on $\mathcal{M}(p, \lambda)$.
2. The maximal operator $M$ is bounded from $\mathcal{M}(1, \lambda)$ to $\mathcal{M}(1, \lambda)$.

Let $(\mathcal{F}, \| \cdot \|_{\mathcal{F}})$ be a Banach space of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$. In this paper, we consider the Morrey-type Banach space $\mathcal{M}_\mathcal{F}(p, \lambda)$ as well as the weak type $\mathcal{M}_\mathcal{F}^*(1, \lambda)$. We aim at developing the theory of maximal operator and Fourier multipliers acting on these spaces.

There are rich literatures about various Morrey-type spaces and operators acting on them. We refer the readers to [2, 3, 4, 7, 8]. It should be noted that our Morrey-type Banach space $\mathcal{M}_\mathcal{F}(p, \lambda)$ includes all the Morrey-type spaces studied in above papers. In particular, the Morrey-type space $\mathcal{M}_{\rho\theta, \lambda}$ introduced by D.R. Adams in [2] and used heavily by G. Lu in [8] for studying the embedding theorems for vector fields of Hörmander type is a special case of our Morrey-type Banach spaces.

### 2. Definitions and basic properties of Morrey-type spaces

Suppose $0 < p \leq \infty$, $\lambda \in \mathbb{R}$ and $f \in L_p^{loc}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $r > 0$, define

$$E_{p, \lambda}(f)(x, r) = r^{-\frac{\lambda}{p}} \| f \|_{L_p(B(x, r))} \quad \text{and} \quad E_{p, \lambda}^*(f)(x, r) = r^{-\frac{\lambda}{p}} \| f \|_{W_{L_p}(B(x, r))}.$$ 

These two quantities can be viewed as functions of $(x, r) \in \mathbb{R}^{n+1}_+.$

**Definition 2.1.** Let $0 < p \leq \infty$, $\lambda \in \mathbb{R}$ and $(\mathcal{F}, \| \cdot \|_{\mathcal{F}})$ be a Banach space of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$. Denote by $\mathcal{M}_\mathcal{F}(p, \lambda)$ the Morrey-type Banach space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ satisfying

$$\| f \|_{\mathcal{M}_\mathcal{F}(p, \lambda)} = \| E_{p, \lambda}(f) \|_{\mathcal{F}}.$$ 

Denote by $\mathcal{M}_\mathcal{F}^*(p, \lambda)$ the weak Morrey-type Banach space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ satisfying

$$\| f \|_{\mathcal{M}_\mathcal{F}^*(p, \lambda)} = \| E_{p, \lambda}^*(f) \|_{\mathcal{F}}.$$ 

It is clear that for the Banach space $L_\infty(\mathbb{R}^{n+1}_+)$, $\| \cdot \|_{\mathcal{M}_{L_\infty}(p, \lambda)} = \| \cdot \|_{\mathcal{M}(p, \lambda)}$.

For $0 < \theta, p \leq \infty$, let $L(\theta, p)$ be the space of functions $h$ on $\mathbb{R}^{n+1}_+$ satisfying

$$\| h \|_{L(\theta, p)} = \left\| \left\{ \int_0^\infty |h(x, s)|^\frac{p}{\theta} \frac{ds}{s} \right\}^{\frac{1}{p}} \right\|_{L_p(\mathbb{R}^n)} < \infty.$$ 

Clearly, $L(\theta, p)$ is a Banach space and $L(\infty, \infty) = L_\infty(\mathbb{R}^{n+1}_+)$. In fact

$$\mathcal{M}_{L(\theta, \infty)}(p, \lambda) = \mathcal{M}_{p\theta, \lambda},$$

(the space introduced by D.R. Adams in [2]). For $0 < p \leq \infty$ and $1 \leq \theta \leq \infty$, denote by $\Lambda_n(p, \theta)$ the interval $(0, n - \frac{p}{\theta})$ if $1 \leq \theta < \infty$; the interval $[0, n]$ if $\theta = \infty$. It is easy to see that for $0 < p \leq \infty$ and $1 \leq \theta \leq \infty$, $\mathcal{M}_{p\theta, \lambda}$ is trivial if $\lambda \notin \Lambda_n(p, \theta)$. Therefore the interesting case of $\lambda$ for the space $\mathcal{M}_{p\theta, \lambda}$ is $\lambda \in \Lambda_n(p, \theta)$. 
Lemma 2.1. Let $1 \leq p \leq q \leq \infty$ and $\mathcal{F}$ be a Banach space of measurable functions on $\mathbb{R}_+^{n+1}$. If $f \in \mathcal{M}_\mathcal{F}(q, \lambda)$, then $f \in \mathcal{M}_\mathcal{F}(p, (\lambda-n)\frac{q}{q}+n)$ and

$$
\|f\|_{\mathcal{M}_\mathcal{F}(p, (\lambda-n)\frac{q}{q}+n)} \leq v_n^{\frac{1}{p}-\frac{1}{q}} \|f\|_{\mathcal{M}_\mathcal{F}(q, \lambda)}. 
$$

(2.1)

Here $v_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Proof. For $x \in \mathbb{R}^n$ and $r > 0$, applying Hölder inequality, we have

$$
\|f\|_{L_p(B(x,r))} \leq \|f\|_{L_q(B(x,r))} \frac{1}{p} \frac{1}{q} = v_n^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(B(x,r))} r^{\frac{n}{q}-\frac{n}{p}}.
$$

This implies

$$
E_{p, (\lambda-n)\frac{q}{q}+n}(f)(x,r) \leq v_n^{\frac{1}{p}-\frac{1}{q}} E_{q, \lambda}(f)(x,r),
$$

and hence (2.1) follows. \(\square\)

Assume $(\mathcal{F}_i, \|\cdot\|_{\mathcal{F}_i})$, $i = 1, 2$, and $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ are Banach spaces of complex-valued measurable functions on $\mathbb{R}_+^{n+1}$. The following two properties are considered also.

(I) $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$, i.e., $fg \in \mathcal{F}$ if $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$.

(II) If $\mathcal{F}_1 \mathcal{F}_2 \subseteq \mathcal{F}$, then the Hölder inequality holds, i.e.,

$$
\|fg\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}_1} \|g\|_{\mathcal{F}_2}
$$

holds for any $f \in \mathcal{F}_1$ and any $g \in \mathcal{F}_2$.

Lemma 2.2 (the Hölder inequality). Suppose $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ are Banach spaces of functions on $\mathbb{R}_+^{n+1}$ satisfying (I) and (II). Let $0 < p, p_1, p_2 \leq \infty$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}^n$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$. If $f \in \mathcal{M}_{\mathcal{F}_1}(p_1, \lambda_1)$ and $g \in \mathcal{M}_{\mathcal{F}_2}(p_2, \lambda_2)$, then $fg \in \mathcal{M}_{\mathcal{F}}(p, \lambda)$ and

$$
\|fg\|_{\mathcal{M}_\mathcal{F}(p, \lambda)} \leq \|f\|_{\mathcal{M}_{\mathcal{F}_1}(p_1, \lambda_1)} \|g\|_{\mathcal{M}_{\mathcal{F}_2}(p_2, \lambda_2)}.
$$

Proof. For $x \in \mathbb{R}^n$ and $r > 0$, applying Hölder inequality, we have

$$
\|fg\|_{L_p(B(x,r))} \leq \|f\|_{L_{p_1}(B(x,r))} \|g\|_{L_{p_2}(B(x,r))}.
$$

Therefore,

$$
E_{p, \lambda}(fg)(x,r) = r^{-\frac{1}{\lambda}} \|fg\|_{L_p(B(x,r))} \leq r^{-\frac{1}{\lambda_1}} r^{-\frac{1}{\lambda_2}} \|f\|_{L_{p_1}(B(x,r))} \|g\|_{L_{p_2}(B(x,r))} = E_{p_1, \lambda_1}(f)(x,r) E_{p_2, \lambda_2}(g)(x,r).
$$

Because of the assumption (II), the above estimate implies the desired result. \(\square\)

3. Maximal operator on $\mathcal{M}_\mathcal{F}(p, \lambda)$

For a positive integer $k \geq 1$ and a measurable function $h$ defined on $\mathbb{R}_+^{n+1}$, denote by $V_k$, the compression of $h$ by a factor of $2^k$ on the $(n+1)^{th}$ variable, i.e.,

$$
V_k(h)(x,y) = h(x, 2^k y), \quad \text{for all } (x, y) \in \mathbb{R}_+^{n+1}.
$$
Clearly $V_k$ is a linear operator. A Banach space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$ is said to be $\{V_k\}_{k \geq 1}$ admissible if there exists a $C > 0$ such that

$$\|V_k(h)\|_{\mathcal{F}} \leq C \|h\|_{\mathcal{F}}, \quad \text{for all } h \in \mathcal{F} \text{ and } k \geq 1.$$ 

It is easy to check that the space $L(\theta,p)$ is $\{V_k\}_{k \geq 1}$ admissible. In fact, it is $\{V_k\}_{k \geq 1}$ invariant, i.e., $\|V_k(h)\|_{L(\theta,p)} = \|h\|_{L(\theta,p)}$ for all $h \in L(\theta,p)$ and $k \geq 1$.

Inspired by the results in [3] about boundedness of the maximal operator in the local Morrey-type spaces, we establish the following theorem, which clearly implies Theorem 1.1.

**Theorem 3.1.** Let $1 \leq p < \infty$, $\lambda < n$, and $\mathcal{F}$ be a $\{V_k\}_{k \geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$. Then

1. For $1 < p < \infty$, the maximal operator $M$ is bounded on $\mathcal{M}_F(p,\lambda)$, i.e., there is a constant $C > 0$ such that
   $$\|M(f)\|_{\mathcal{M}_F(p,\lambda)} \leq C \|f\|_{\mathcal{M}_F(p,\lambda)}, \quad \text{for all } f \in \mathcal{M}_F(p,\lambda).$$

2. The maximal operator $M$ is bounded from $\mathcal{M}_F(1,\lambda)$ to $\mathcal{M}_F^*(1,\lambda)$, i.e., there is a constant $C > 0$ such that
   $$\|M(f)\|_{\mathcal{M}_F^*(1,\lambda)} \leq C \|f\|_{\mathcal{M}_F(1,\lambda)}, \quad \text{for all } f \in \mathcal{M}_F(1,\lambda).$$

Instead of proofing Theorem 3.1, we prove the following generalized result for the Morrey-type Banach spaces of vector-valued functions.

Let $0 < q \leq \infty$. If $f = \{f_j\}_{j=-\infty}^{\infty}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, we write $f \in \mathcal{M}_F(p,\lambda)(l_q)$, if

$$\|f\|_{\mathcal{M}_F(p,\lambda)(l_q)} = \left\|\|f\|_{l_q}\right\|_{\mathcal{M}_F(p,\lambda)}.$$ 

Denote $M(f) = \{M(f_j)\}_{j=-\infty}^{\infty}$, if $f = \{f_j\}_{j=-\infty}^{\infty}$.

**Theorem 3.2.** Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $\lambda < n$, and $\mathcal{F}$ be a $\{V_k\}_{k \geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$. Then

1. For $1 < p < \infty$, the maximal operator $M$ is bounded on $\mathcal{M}_F(p,\lambda)(l_q)$, i.e., there is a constant $C > 0$ such that
   $$\|M(f)\|_{\mathcal{M}_F(p,\lambda)(l_q)} \leq C \|f\|_{\mathcal{M}_F(p,\lambda)(l_q)}$$
   holds for all $f \in \mathcal{M}_F(p,\lambda)(l_q)$.

2. The maximal operator $M$ is bounded from $\mathcal{M}_F(1,\lambda)(l_q)$ to $\mathcal{M}_F^*(1,\lambda)(l_q)$, i.e., there is a constant $C > 0$ such that
   $$\|M(f)\|_{\mathcal{M}_F^*(1,\lambda)(l_q)} \leq C \|f\|_{\mathcal{M}_F(1,\lambda)(l_q)}$$
   holds for all $f \in \mathcal{M}_F(1,\lambda)(l_q)$.

**Remark 3.1.** From the proof below, we will see that Theorem 3.2 remains true if the condition of “$\mathcal{F}$ is a $\{V_k\}_{k \geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$” is replaced by following ($1 < p < \infty$ for case (1) and $p = 1$ for case (2), respectively)

$$\sum_{k=0}^{\infty} (2^k)^{-n/p+\lambda/p} \|V_k\|_{\mathcal{F}} < \infty.$$
Remark 3.2. In Theorem 3.2, let $F = L(\theta, t)$ (which is clearly $\{V_k\}_{k \geq 1}$ admissible), especially for $\theta = t = \infty$ or $t = \infty$, we obtain results about the boundedness of the maximal operator on the classical Morrey spaces $M(p, \lambda)(l_q)$ (see [12]), or on the space $M_{p,\theta,\lambda}(l_q)$ for $1 \leq p, \theta \leq \infty$ and $\lambda \in \Lambda_n(p, \theta)$.

Proof of Theorem 3.2. Let $1 \leq p < \infty$ and $f = \{f_j\}_{j=-\infty}^{\infty}$. For fixed $u \in \mathbb{R}^n$ and $r > 0$ denote

$$T^0_{u,r}(f) = \chi_{B(u,2r)}f = \{\chi_{B(u,2r)}f_j\}_{j=-\infty}^{\infty},$$

$$T^k_{u,r}(f) = \chi_{B(u,2^{k+1}r) \setminus B(u,2^kr)}f = \{\chi_{B(u,2^{k+1}r) \setminus B(u,2^kr)}f_j\}_{j=-\infty}^{\infty}, \quad k = 1, 2, \ldots$$

Clearly $f = \sum_{k=0}^{\infty} T^k_{u,r}(f)$ and

$$\|M(f)(x)\|_{l_q} \leq \sum_{k=0}^{\infty} \|M(T^k_{u,r}(f))(x)\|_{l_q}, \quad \forall x \in \mathbb{R}^n.$$  

This implies that

$$E_{p,\lambda}(\|M(f)\|_{l_q})(u, r) \leq \sum_{k=0}^{\infty} E_{p,\lambda}(\|M(T^k_{u,r}(f))\|_{l_q})(u, r), \quad (3.1)$$

$$E^*_{p,\lambda}(\|M(f)\|_{l_q})(u, r) \leq \sum_{k=0}^{\infty} E^*_{p,\lambda}(\|M(T^k_{u,r}(f))\|_{l_q})(u, r). \quad (3.2)$$

To prove Theorem 3.2 (1), we estimate $E_{p,\lambda}(\|M(T^0_{u,r}(f))\|_{l_q})(u, r)$ first. Recall the well known Fefferman-Stein maximal inequality (see [6])

$$\left\|\|M(f)\|_{l_q}\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|f\right\|_{l_q}\left\|_{L^p(\mathbb{R}^n)},$$

where $C > 0$ is independent of the vector-valued function $f$. We have

$$\left\|\|M(T^0_{u,r}(f))\|_{l_q}\right\|_{L^p(B(2r))} \leq \left\|\|M(T^0_{u,r}(f))\|_{l_q}\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\|T^0_{u,r}(f)\|_{l_q}\right\|_{L^p(\mathbb{R}^n)} = C \left\|f\right\|_{L^p(B(2r))},$$

where $C > 0$ is independent of $u \in \mathbb{R}^n$, $r > 0$ and the vector-valued function $f$. This yields

$$E_{p,\lambda}(\|M(T^0_{u,r}(f))\|_{l_q})(u, r) \leq CV_1(E_{p,\lambda}(\|f\|_{l_q}))(u, r).$$

It remains to estimate $E_{p,\lambda}(\|M(T^k_{u,r}(f))\|_{l_q})(u, r)$ for $k \geq 1$. Let $x \in B(u, r)$ and let

$$t_x = \inf\{t : B(x,t) \setminus B(u,2^{k+1}r) \neq \emptyset\}.$$  

It is easy to see that

$$t_x \asymp 2^{kr} \quad \text{for all} \quad x \in B(u, r).$$
Therefore for a complex-valued function \( g \in L^\infty_p(\mathbb{R}^n) \) and \( x \in B(u, r) \), we have
\[
M(T^k_{u,r}(g))(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |T^k_{u,r}(g)(y)| \, dy \\
\leq |B(x, t_x)|^{-1} \int_{\mathbb{R}^n} |T^k_{u,r}(g)(y)| \, dy \\
\asymp C(2^k r)^{-n} \int_{\mathbb{R}^n} |T^k_{u,r}(g)(y)| \, dy.
\]

Therefore by the Minkowski inequality, we have
\[
\left\| M(T^k_{u,r}(f))(x) \right\|_{L^q} \leq C(2^k r)^{-n} \left\{ \int_{\mathbb{R}^n} |T^k_{u,r}(f_j)(y)| \, dy \right\}_j^{\infty} \\
\leq C \left( 2^k r \right)^{-n} \int_{\mathbb{R}^n} \left\| T^k_{u,r}(f)(y) \right\|_{L^q} \, dy \\
= C \left( 2^k r \right)^{-n} \int_{B(u,2^k+1)r \setminus B(u,2^k r)} \left\| f(y) \right\|_{L^q} \, dy
\]

(Hölder) \( \leq C(2^k r)^{-n/p} \left\| f \right\|_{L^p(B(u,2^k+1r))} \).

This yields
\[
E_{p,\lambda}\left( \left\| M(T^k_{u,r}(f)) \right\|_{L^q} \right)(u, r) \leq C(2^k)^{-n/p+\lambda/p} V_{k+1}(E_{p,\lambda}(\left\| f \right\|_{L^q}))(u, r).
\]

Combining the estimates above and using estimate (3.1), we obtain
\[
E_{p,\lambda}\left( \left\| M(f) \right\|_{L^q} \right)(u, r) \leq \sum_{k=0}^{\infty} E_{p,\lambda}\left( \left\| M(T^k_{u,r}(f)) \right\|_{L^q} \right)(u, r) \\
\leq C \sum_{k=0}^{\infty} (2^k)^{-n/p+\lambda/p} V_{k+1}(E_{p,\lambda}(\left\| f \right\|_{L^q}))(u, r).
\]

Applying \( \left\| \cdot \right\|_{\mathcal{F}} \) to both sides of the above estimate and using the fact that \( \mathcal{F} \) is a \( \{V_k\}_{k \geq 1} \) admissible Banach space of functions on \( \mathbb{R}^{n+1}_+ \) and the series \( \sum_{k=0}^{\infty} (2^k)^{-\frac{n}{p}+\frac{\lambda}{p}} \) converges when \( \lambda < n \), we can conclude the desired result.

The weak case can be proved similarly by using the weak type Fefferman-Stein maximal inequality [6]
\[
\left\| \left\| M(f) \right\|_{L^q} \right\|_{W L^1(\mathbb{R}^n)} \leq C \left\| \left\| f \right\|_{L^q} \right\|_{L^1(\mathbb{R}^n)},
\]
where \( C > 0 \) is independent of the vector-valued function \( f \). Indeed, we have
\[
\left\| \left\| M(T^0_{u,r}(f)) \right\|_{L^q} \right\|_{W L^1(B(u,r))} \leq \left\| \left\| M(T^0_{u,r}(f)) \right\|_{L^q} \right\|_{W L^1(\mathbb{R}^n)} \\
\leq C \left\| T^0_{u,r}(f) \right\|_{L^1(\mathbb{R}^n)} \\
= C \left\| f \right\|_{L^q(B(u,2r))},
\]
where $C > 0$ is independent of $u \in \mathbb{R}^n$, $r > 0$ and the vector-valued function $f$. This implies
\[ E_{1,\lambda}^*(\|M(T_{u,r}^0(f))\|_{l_q})(u,r) \leq CV_1(E_{1,\lambda}(\|f\|_{l_q}))(u,r). \]

On the other hand, by (3.3) we have
\[ E_{1,\lambda}^*(\|M(T_{u,r}^k(f))\|_{l_q})(u,r) \leq C(2^k)^{-n+\lambda}V_{k+1}(E_{1,\lambda}(\|f\|_{l_q}))(u,r). \]

Hence by (3.2), we obtain
\[ E_{1,\lambda}^*(\|M(f)\|_{l_q})(u,r) \leq \sum_{k=0}^{\infty} E_{1,\lambda}^*(\|M(T_{u,r}^k(f))\|_{l_q})(u,r) \leq C \sum_{k=0}^{\infty} (2^k)^{-n+\lambda}V_{k+1}(E_{1,\lambda}(\|f\|_{l_q}))(u,r). \]

Applying $\|\cdot\|_F$ to both sides of the above estimate and using the fact that $F$ is a $\{V_k\}_{k \geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}$ and the series $\sum_{k=0}^{\infty} (2^k)^{-n+\lambda}$ converges when $\lambda < n$, we can conclude the desired result. □

4. Fourier multipliers on $M_\mathcal{F}(p, \lambda)$

In this section, we establish an application of our theorems in previous section for Fourier multipliers on $M_\mathcal{F}(p, \lambda)$. Our approach has its root in [13].

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing infinitely differential complex-valued functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all complex-valued tempered distributions on $\mathbb{R}^n$. Let
\[ (F\phi)(\xi) = (2\pi)^{n/2} \int_{\mathbb{R}^n} \phi(x)e^{-ix\cdot\xi}dx \]
and let $F^{-1}$ denote the Fourier transform and its inverse on $\mathcal{S}'(\mathbb{R}^n)$, respectively.

If $s \in \mathbb{R}$, we write
\[ H^s_2(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^s_2} = \left\| (1 + |x|^2 )^{s/2} Ff(x) \right\|_{L^2} < \infty \right\}. \]

If $\Omega$ is a compact set of $\mathbb{R}^n$, we write
\[ L_{p,\Omega} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \text{supp } Ff \subset \Omega, \|f\|_{L^p} < \infty \right\}. \]

For $p \geq 1$ and $0 < s < p$, it is proved in [13] (page 22) that if $g \in L_{p,B(0,1)}$, then
\[ \frac{|g(x - z)|}{1 + |z|^{n/s}} \leq C \{M(\{|g|^s\}(x))\}^{1/s}, \quad \text{for all } x, z \in \mathbb{R}^n, \quad (4.1) \]
where the constant $C > 0$ is independent of $x, z$, and $g$.

For $d, s > 0$, consider the following maximal operator $N_{d,s}$ defined by
\[ N_{d,s}(g)(x) = \sup_{y \in \mathbb{R}^n} \frac{|g(x - y)|}{1 + (d|y|)^{n/s}}. \]

We have the following result.
Theorem 4.1. Suppose $\mathcal{F}$ is a $\{V_k\}_{k \geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}$. Let $1 \leq p < \infty$, $\lambda < n$, $\Omega$ be a compact set, $d$ be the radius of $\Omega$, $0 < s < p$ and $f \in L_{p,\Omega}$. Then there is a constant $C > 0$ such that

1. For $1 < p < \infty$
   \[
   \|N_{d,s}(f)\|_{\mathcal{M}_s^{(p,\lambda)}} \leq C \|f\|_{\mathcal{M}_s^{(p,\lambda)}}
   \]
   holds for all $f \in \mathcal{M}_s^{(p,\lambda)}$.

2. For $p = 1$
   \[
   \|N_{d,s}(f)\|_{\mathcal{M}_s^{+(1,\lambda)}} \leq C \|f\|_{\mathcal{M}_s^{+(1,\lambda)}}
   \]
   holds for all $f \in \mathcal{M}_s^{+(1,\lambda)}$.

Instead of proofing Theorem 4.1, we prove the following generalized result for vector-valued functions. Assume $0 < p, q < \infty$ and $\lambda \in \mathbb{R}^n$. If $\Omega = \{\Omega_j\}_{j=-\infty}^{\infty}$ is a sequence of compact sets on $\mathbb{R}^n$, we denote $f = \{f_j\}_{j=-\infty}^{\infty} \in \mathcal{M}_s^{(p,\lambda)}(l_q)$, if $f_j \in L_{p,\Omega_j}$ for $j \in \mathbb{Z}$ and $f$ is in $\mathcal{M}_s^{(p,\lambda)}(l_q)$.

Theorem 4.2. Suppose $\mathcal{F}$ is a $\{V_k\}$ admissible Banach space of functions on $\mathbb{R}^{n+1}$, $1 \leq p, q < \infty$, $\lambda < n$, $\Omega = \{\Omega_j\}_{j=-\infty}^{\infty}$ is a sequence of compact sets, and $d_j$ is the radius of $\Omega_j$ for $j \in \mathbb{Z}$. If $f = \{f_j\}_{j=-\infty}^{\infty} \in \mathcal{M}_s^{(p,\lambda)}(l_q)$ and $0 < s < \min\{p, q\}$, then exists a constant $C$ such that

1. For $1 < p < \infty$
   \[
   \|\{N_{d_j,s}(f_j)\}\|_{\mathcal{M}_s^{(p,\lambda)}(l_q)} \leq C \|f\|_{\mathcal{M}_s^{(p,\lambda)}(l_q)}
   \]
   holds for all $f \in \mathcal{M}_s^{(p,\lambda)}(l_q)$.

2. For $p = 1$
   \[
   \|\{N_{d_j,s}(f_j)\}\|_{\mathcal{M}_s^{+(1,\lambda)}(l_q)} \leq C \|f\|_{\mathcal{M}_s^{+(1,\lambda)}(l_q)}
   \]
   holds for all $f \in \mathcal{M}_s^{+(1,\lambda)}(l_q)$.

Proof. Let $1 \leq p < \infty$ and $f = \{f_j\}_{j=-\infty}^{\infty} \in \mathcal{M}_s^{(p,\lambda)}(l_q)$, and $y_j$ be the center of $\Omega_j$. Define $h_j(x) = e^{-ix \cdot y_j} f_j(x)$ for $j \in \mathbb{Z}$. We have clearly $|h_j(x)| = |f_j(x)|$, $F(h_j)(x) = F(f_j)(x)(x + y_j)$ and therefore $\text{supp } F(h_j) \subset \Omega_j - y_j$. Thus, without lost of generality, we assume that $0 \in \Omega_j$, and $\Omega_j = B(0, d_j)$ for all $j \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, let $g_j(x) = f_j(d_j^{-1}x)$. Then $F(g_j)(x) = d_j^n F(f_j)(d_j x)$ and $\text{supp } F(g_j) \subset B(0, 1)$.

From (4.1), we obtain

\[
N_{d_j,s}(f_j)(x) \leq C [M(|f_j|^s)(x)]^{\frac{1}{s}} \quad \text{for all } x, z \in \mathbb{R}^n,
\]

(4.2)

where the constant $C > 0$ is independent of $x$, $j$, and $f_j$.

The above estimate implies

\[
\|\{N_{d_j,s}(f_j)\}\|_{l_q} \leq C \|\{[M(|f_j|^s)(x)]^{\frac{1}{s}}\}\|_{l_q} = C \|\{M(|f_j|^s)(x)\}\|_{\frac{1}{s} l_q}^{\frac{1}{s}},
\]

and therefore

\[
E_{p,\lambda}(\|\{N_{d_j,s}(f_j)\}\|_{l_q})(u, r) \leq C \left( E_{p/s,\lambda}(\|\{M(|f_j|^s)\}\|_{\frac{1}{s} l_q})(u, r) \right)^{\frac{1}{s}}.
\]

(4.3)
We conclude together with (4.3), we conclude the following theorem.

Since \( \frac{q}{s}, \frac{p}{s} > 1 \), by (3.4), we have

\[
E_{p/s, \lambda}(\|M(\{f_j \}^s)\|_{l_{q/2}})(u, r)
\leq C \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} V_{k+1}(E_{p/s, \lambda}(\|\{f_j \}^s\|_{l_{q/2}}))(u, r)
\]

\[
= C \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} \left( V_{k+1}(E_{p, \lambda}(\|f\|_{l_q}))(u, r) \right)^s.
\]

If \( s \geq 1 \), then

\[
\left( E_{p/s, \lambda}(\|M(\{f_j \}^s)\|_{l_{q/2}})(u, r) \right)^{\frac{1}{s}} \leq C \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} V_{k+1}(E_{p, \lambda}(\|f\|_{l_q}))(u, r).
\]

If \( 0 < s < 1 \), by Hölder inequality, we have

\[
\left( E_{p/s, \lambda}(\|M(\{f_j \}^s)\|_{l_{q/2}})(u, r) \right)^{\frac{1}{s}} \leq C \left( \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} \right)^{\frac{1}{s}} \left( \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} V_{k+1}(E_{p, \lambda}(\|f\|_{l_q}))(u, r) \right).
\]

Together with (4.3), we conclude

\[
\|\{N_{d_j, \lambda}(f_j)\}\|_{M\mathcal{F}(p, \lambda)(l_q)} \leq C \|f\|_{M\mathcal{F}(p, \lambda)(l_q)}.
\]

For the weak estimation, it is easy to see that if

\[
f = \{f_j\}_{j=-\infty}^{\infty} \in W\mathcal{M}\mathcal{F}(1, \lambda)(l_q)
\]

then

\[
E_{1, \lambda}^*\left(\|\{N_{d_j, \lambda}(f_j)\}\|_{l_q}(u, r)\right) \leq C \left( E_{1/s, \lambda}^*(\|M(\{f_j \}^s)\|_{l_{q/2}})(u, r) \right)^{\frac{1}{s}}.
\]

Since \( \frac{q}{s}, \frac{1}{s} > 1 \), we can continue the above estimation by

\[
\leq C \left( E_{1/s, \lambda}(\|M(\{f_j \}^s)\|_{l_{q/2}}))(u, r) \right)^{\frac{1}{s}}
\leq C \left( \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} \left( V_{k+1}(E_{1, \lambda}(\|f\|_{l_q}))(u, r) \right)^s \right)^{\frac{1}{s}}
\leq C \left( \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} \right)^{\frac{1}{s}} \left( \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} \right)^{\frac{1}{s} - 1} \left( \sum_{k=0}^{\infty} \left( \frac{2^{n-k}}{p-q} \right)^{-k} V_{k+1}(E_{1, \lambda}(\|f\|_{l_q}))(u, r) \right).
\]

We conclude

\[
\|\{N_{d_j, \lambda}(f_j)\}\|_{W\mathcal{M}\mathcal{F}(1, \lambda)(l_q)} \leq C \|f\|_{M\mathcal{F}(1, \lambda)(l_q)}.
\]

By Theorem 4.2 and the proof of p. 31–32 in [13], it is easy to obtain the following theorem.
Theorem 4.3. Let \( 1 \leq p < \infty, 1 \leq q \leq \infty, 1 \leq \theta < \infty, 0 < \lambda < n - \frac{p}{q} \). Let also \( \Omega = \{ \Omega_j \}_{j=-\infty}^\infty \) is a sequence of compact sets on \( \mathbb{R}^n \), \( f_j \in L^p(\Omega_j) \) for \( j \in \mathbb{Z} \), and \( d_j \) be the radius of \( \Omega_j \). If \( \nu > n/2 + n/\min\{p, q\} \), then exists a constant \( C \) such that

\[
\left\| \left\{ F^{-1} G_j F f_j \right\}_j \right\|_{M_{p\theta,\lambda}(l_q)} \leq C \sup_j \left\| G_j(d_j \cdot) \right\|_{H^\nu_2} \left\| \left\{ f_j \right\}_j \right\|_{M_{p\theta,\lambda}(l_q)}
\]

and

\[
\left( \sum_{j=\infty}^\infty \left\| F^{-1} G_j F f_j \right\|_{M_{p\theta,\lambda}}^q \right)^{1/q} \leq C \sup_j \left\| G_j(d_j \cdot) \right\|_{H^\nu_2} \left( \sum_{j=\infty}^\infty \left\| f_j \right\|_{M_{p\theta,\lambda}}^q \right)^{1/q}.
\]

for any sequence \( \{G_j\}_j \in H^\nu_2(\mathbb{R}^n) \).

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