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ON DISCRETENESS OF THE SPECTRUM OF A HIGH ORDER DIFFERENTIAL OPERATOR IN MULTIDIMENSIONAL CASE

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Abstract. In the paper, the self-adjointness of a high order differential operator in multidimensional case and the discreteness of its spectrum are proved. Furthermore, the expansion in terms of eigenfunctions of an arbitrary function from the space $L_2(\mathbb{R}_n)$ is obtained.

1. Introduction

Consider in $\mathbb{R}_n$ the differential expression ℓ_{V} given by the formula

$$\ell_{\mathrm{V}} = \sum_{|\alpha|=1}^{m} a_\alpha D^\alpha + \mathrm{V}(x),$$

where m is a natural number, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}_n$, $\mathrm{V}(x)$ is a real measurable function, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, i.e. its components α_j are integer non-negative numbers, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}},$$

a_α is a real number if $|\alpha|$ is even, is pure imaginary if $|\alpha|$ is odd.

Assume that the following conditions are fulfilled:

a) $\mathrm{V}(x) \in L_{2,loc}(\mathbb{R}_n)$;

b) $\lim\limits_{|\mathrm{x}| \to +\infty} \mathrm{V}(x) = +\infty$, where $|x| = \sqrt{\sum_{k=1}^{n} x_k^2}$;

c) for any $p = (p_1, p_2, ..., p_n) \in \mathbb{R}_n$, $G(p) = \sum_{|\alpha|=1}^{m} (-1)^{|\alpha|} a_\alpha p^\alpha \geq 0$, where $p^\alpha = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$;

d) $\lim\limits_{|\mathrm{p}| \to +\infty} \mathrm{G}(p) = +\infty$, where $|p| = \sqrt{\sum_{k=1}^{n} p_k^2}$.

Introduce the operator $\tilde{\mathrm{H}}_{\mathrm{V}} : L_2(\mathbb{R}_n) \to L_2(\mathbb{R}_n)$ by the formula $\tilde{\mathrm{H}}_{\mathrm{V}} u = \ell_{\mathrm{V}} u$ with domain of definition $D\left(\tilde{\mathrm{H}}_{\mathrm{V}}\right) = C_0^\infty(\mathbb{R}_n)$ ($C_0^\infty(\mathbb{R}_n)$ is the totality of all finite, infinitely differentiable in $\mathbb{R}_n$ functions). If $\mathrm{V}(x) = 0$, then denote $\tilde{\mathrm{H}}_{\mathrm{V}}$ by

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$\tilde{\mathrm{H}}_0$. Denote the closure of the operators $\tilde{\mathrm{H}}_{\mathrm{V}}$ and $\tilde{\mathrm{H}}_0$, by H_{V} and H_0, respectively. Obviously, H_0 is a self-adjoint absolutely continuous operator with domain of definition $W_2^m(\mathbb{R}_n)$ (Sobolev's space of m order) and its spectrum fills the positive semi-axis, i.e.

$$\sigma(\mathrm{H}_0) = \sigma_{ac}(\mathrm{H}_0) = [0, +\infty).$$

The goal of the paper is to prove the self-adjointness of the operator H_{V} and discreteness of its spectrum under conditions a)-d) and to obtain expansion in eigenfunctions of an arbitrary function from space $L_2(\mathbb{R}_n)$.

At present, there are many papers on investigation of the character of the spectrum of differential operators of any order (see, e.g. [1,3-5,10-12,14,16,18,19]). Both mathematicians and physicists pay rather great attention to studying spectral properties of differential operator. This is connected with applications of such problems, for example, in quantum mechanics and acoustics (see, e.g. [7,17]). The problem on control of a discrete spectrum of differential operators (see, e.g. [2]) is of special interest. A discrete spectrum consists of isolated eigenvalues of finite algebraic multiplicity, and describes important characteristics of physical and chemical objects (squares of frequencies of eigen vibrations of mechanical systems, energy levels of quantum objects, etc). Investigation of many problems of mathematical physics and quantum mechanics is connected with expansions in series in terms of eigenfunctions of differential operators (see, e.g., [6,20]).

2. Self-adjointness

Theorem 2.1. *In conditions a)-d) the operator* H_{V} *is self-adjoint in* $L_2(\mathbb{R}_n)$.

Proof. M be a positive number. Then from condition b) it follows that there exists a positive number R_M that for $|x| > R_M$ there will be $\mathrm{V}(x) \geq M$. Introduce the functions $\mathrm{V}_1(x)$ and $\mathrm{V}_2(x)$ assuming

$$\mathrm{V}_1(x) = \begin{cases} \mathrm{V}(x), & if \quad |x| \leq R_M, \\ 0, & if \quad |x| > R_M, \end{cases}$$

and

$$\mathrm{V}_2(x) = \begin{cases} 0, & if \quad |x| \leq R_M, \\ \mathrm{V}(x), & if \quad |x| > R_M. \end{cases}$$

It is easy to see that $\mathrm{V}(x) = \mathrm{V}_1(x) + \mathrm{V}_2(x)$. From condition a) it follows that the finite function $\mathrm{V}_1(x)$ belongs to the space $L_2(\mathbb{R}_n)$. To prove the theorem it is enough to prove the self-adjointness of the operator $\mathrm{H}_{\mathrm{V}_2}$ that is generated by the differential expression

$$\ell_{\mathrm{V}_2} = \sum_{|\alpha|=1}^{m} a_\alpha D^\alpha + \mathrm{V}_2(x).$$

For that, as in the paper [15] (see also [12]), establish that the equation

$$\mathrm{H}_{\mathrm{V}_2} u(x) + u(x) = 0 \tag{2.1}$$

may not have non-trivial solutions in $L_2(\mathbb{R}_n)$. Let the function $u(x) \in L_2(\mathbb{R}_n)$ be the solution of equation (2.1). Then this function in the sense of distributions satisfies the equation

$$\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u(x) + \mathrm{V}_2(x)\, u(x) + u(x) = 0. \tag{2.2}$$

Using condition c) and Kato's inequality [8] (see also [9]), we get

$$\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha |u(x)| \le Re\left(sgn\overline{u(x)} \sum_{|\alpha|=1}^{m} a_\alpha D^\alpha u(x) \right), \tag{2.3}$$

where $\overline{u(x)}$ is complex conjugated to $u(x)$,

$$sgn\overline{u(x)} = \begin{cases} \frac{\overline{u(x)}}{|u(x)|}, & if \;\; u(x) \ne 0, \\ 0, & if \;\; u(x) = 0. \end{cases}$$

Remark 2.1. From condition c) it follows that if $|\alpha|$ is an odd number, then $a_\alpha = 0$, and from condition d) it follows that even one among the numbers a_α is non-zero. Thus, the operator H_0 is a non-negative self-adjoint operator. The operator in the Kato inequality is non-positive. Therefore, in our case, the Kato inequality is of the form (2.3).

It follows from (2.2) and (2.3) that in the sense of distributions, the following inequality is valid

$$-\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha |u(x)| \ge (\mathrm{V}_2(x) + 1)\, |u(x)|. \tag{2.4}$$

Let (see [21, p. 16])

$$\omega_\varepsilon(x) = \begin{cases} c_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - x^2}}, & if \;\; |x| < \varepsilon, \\ 0, & if \;\; |x| \ge \varepsilon, \end{cases}$$

is an averaging kernel, where ε and c_ε are positive numbers, c_ε is chosen so that

$$\int_{\mathbb{R}_n} \omega_\varepsilon(x)\, dx = 1.$$

Determine the convolution $\omega_\varepsilon(x)$ and $|u(x)|$:

$$\varphi_\varepsilon(x) = \omega_\varepsilon(x) * |u(x)| = \int_{\mathbb{R}_n} \omega_\varepsilon(x-y)\, |u(y)|\, dy.$$

From (2.4) it follows that

$$-\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha \varphi_\varepsilon(x) \ge 0 \tag{2.5}$$

in the sense of distributions. Since the function $\varphi_\varepsilon(x)$ is infinitely differentiable, inequality (2.5) is true and pointwise. Thus, for any positive number ε we have

$$\left(\varphi_\varepsilon(x), \quad -\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha \varphi_\varepsilon(x)\right) \geq 0. \tag{2.6}$$

On the other hand, from conditions c) and d) we have

$$\left(\varphi_\varepsilon(x), \quad -\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha \varphi_\varepsilon(x)\right) \leq 0. \tag{2.7}$$

From inequalities (2.6) and (2.7) we have

$$\left(\varphi_\varepsilon(x), \quad -\sum_{|\alpha|=1}^{m} a_\alpha D^\alpha \varphi_\varepsilon(x)\right) = 0. \tag{2.8}$$

Since $\varphi_\varepsilon(x) \in L_2(\mathbb{R}_n)$, and the operator H_0 has no eigenfunctions, from (2.8) we get $\varphi_\varepsilon(x) = 0$. Since

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}_n} \omega_\varepsilon(x-y)\,|u(y)|\,dy = |u(x)|,$$

we have $|u(x)| = 0$, and consequently, $u(x) = 0$. The theorem is proved. □

3. The spectrum of operator H_V

There exist different definitions of the discrete spectrum of an operator. Their equivalence is formulated in [1] in the form of a proposal.

We need the following two lemmas.

Lemma 3.1 (see [16, p. 269, Theorem XIII.64])**.** *Let A be a self-adjoint operator. The following two statements are equivalent:*

(i) the operator $(A-\mu E)^{-1}$ is compact for some $\mu \in \rho(A)$;

(ii) the operator $(A-\lambda E)^{-1}$ is compact for all $\lambda \in \rho(A)$.

(Here E is a unit operator, $\rho(A)$ is a resolvent set of the operator A).

Remark 3.1. In book [16], the lower boundedness of the operator A is required. But Lemma 1 is a special case of Theorem XIII.64 from [16], and its statement follows from the resolvent formula

$$(A-\lambda E)^{-1} = (A-\mu E)^{-1} + (\mu-\lambda)(A-\mu E)^{-1}(A-\lambda E)^{-1},$$

therefore, here this requirement is not necessary.

Denote by $Q(\mathrm{H}_\mathrm{V})$ the domain of definition of quadratic form, generated by the linear operator H_V. Obviously, $D(\mathrm{H}_\mathrm{V}) \subset Q(\mathrm{H}_\mathrm{V})$.

Lemma 3.2. *Let conditions a)-d) be fulfilled. Then for any positive number b, the set*

$$F^b_{\mathrm{H}_\mathrm{V}} = \{\psi(x) \in Q(\mathrm{H}_\mathrm{V}) : \|\psi\| \leq 1,\ (\mathrm{H}_\mathrm{V}\psi,\ \psi) \leq b\}$$

is compact.

Proof. Going over to the Fourier transform

$$\hat{\psi}(p) = \int_{\mathbb{R}_n} \psi(x) e^{i(p,x)} dx$$

in the inequality

$$(\mathrm{H}_0\psi,\ \psi) \leq b$$

and using the Plancherel theorem, we get

$$\int_{\mathbb{R}_n} G(p) \left|\hat{\psi}(p)\right|^2 dp \leq \tilde{b}, \tag{3.1}$$

where $\tilde{b} = \frac{b}{(2\pi)^n}$.

Using inequality (3.1) and $(\mathrm{V}(x)\psi,\ \psi) \leq b$, we get "$x-p$"-representation for the set

$$F^b_{\mathrm{H_V}} = \left\{\psi(x) \in Q(\mathrm{H_V}) : \|\psi\| \leq 1,\ \int_{\mathbb{R}_n} \mathrm{V}(x) |\psi(x)|^2 dx \leq b,\right.$$
$$\left.\int_{\mathbb{R}_n} G(p) \left|\hat{\psi}(p)\right|^2 dx \leq \tilde{b}\right\}.$$

Applying the Rellich compactness criterion (see [16, p. 271, Theorem XIII.65]), we complete the proof of the lemma. □

Theorem 3.1. *In conditions a)-d), the spectrum of the operator* $\mathrm{H_V}$ *is purely discrete.*

Proof. By the theorem condition, the operator $\mathrm{H_V}$ is lower semi-bounded. Denote its lower bound by m, i.e.

$$m = \inf_{u \in D(\mathrm{H_V})} \frac{(\mathrm{H_V}u,\ u)}{(u, u)} > -\infty.$$

Then for $\mu > -m$ the operator $\mathrm{H}_{\mathrm{V},\mu} = \mathrm{H_V} + \mu E$ is non-negative. From Lemma 1, to prove the theorem, it is enough to prove that the operator $(\mathrm{H}_{\mathrm{V},\mu} + E)^{-1} \in \sigma_\infty$. Let $\varphi \in D(\mathrm{H}_{\mathrm{V},\mu}) = D(\mathrm{H_V})$. Then from the equality

$$\|(\mathrm{H}_{\mathrm{V},\mu} + E)\varphi\|^2 = \|\mathrm{H}_{\mathrm{V},\mu}\varphi\|^2 + \|\varphi\|^2 + 2(\mathrm{H}_{\mathrm{V},\mu}\varphi,\ \varphi)$$

it follows that

$$\|\mathrm{H}_{\mathrm{V},\mu}\varphi\| \leq \|(\mathrm{H}_{\mathrm{V},\mu} + E)\varphi\|$$

and

$$\|\varphi\| \leq \|(\mathrm{H}_{\mathrm{V},\mu} + E)\varphi\|.$$

From these inequalities we get

$$\left\|\mathrm{H}_{\mathrm{V},\mu}(\mathrm{H}_{\mathrm{V},\mu} + E)^{-1}\right\| \leq 1 \tag{3.2}$$

and

$$\left\| \left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1} \right\| \leq 1. \tag{3.3}$$

Now prove that the operator $\left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1}$ takes the unit ball to the compact set, i.e. the set

$$L = \left\{ \psi\left(x\right) = \left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1} \varphi : \left\| \varphi \right\| \leq 1 \right\} \subset L_2\left(\mathbb{R}_n\right)$$

is a compact set. From inequalities (3.2) and (3.3) it follows

$$\left\| \mathrm{H}_{\mathrm{V},\mu} \psi \right\| = \left\| \mathrm{H}_{\mathrm{V},\mu} \left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1} \varphi \right\| \leq \left\| \mathrm{H}_{\mathrm{V},\mu} \left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1} \right\| \leq 1 \tag{3.4}$$

and

$$\left\| \psi \right\| = \left\| \left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1} \varphi \right\| \leq \left\| \left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1} \right\| \leq 1. \tag{3.5}$$

From inequalities (3.4) and (3.5) it follows

$$L \subset \left\{ \psi \in D\left(\mathrm{H}_{\mathrm{V},\mu}\right) : \left\| \psi \right\| \leq 1, \ \ \left\| \mathrm{H}_{\mathrm{V},\mu} \psi \right\| \leq 1 \right\} \equiv S. \tag{3.6}$$

It follows from the Schwarz inequality that the set S is a subset of $F^1_{\mathrm{H}_{\mathrm{V},\mu}}$:

$$F^1_{\mathrm{H}_{\mathrm{V},\mu}} \equiv \left\{ \psi \in D\left(\mathrm{H}_{\mathrm{V},\mu}\right) : \left\| \psi \right\| \leq 1, \ \ \left(\mathrm{H}_{\mathrm{V},\mu} \psi, \psi \right) \leq 1 \right\} \supset S. \tag{3.7}$$

From Lemma 3.2 it follows that the set $F^1_{\mathrm{H}_{\mathrm{V},\mu}}$ is compact and therefore by (3.6) and (3.7), the set L is compact. This means that the operator $\left(\mathrm{H}_{\mathrm{V},\mu} + E \right)^{-1}$ is compact. The theorem is proved. □

4. Expansion in terms of eigenfunctions of operator H_{V}

Let $\varphi_1\left(x\right), \varphi_2\left(x\right), \varphi_3\left(x\right), ...$ be orthonormal eigenfunctions of the operator $\left(\mathrm{H}_{\mathrm{V},m} + E \right)^{-1}$ responding to positive eigenvalues

$$\tau_1 \geq \tau_2 \geq \tau_3 \geq ... \geq \tau_k \geq ... \ \to 0 \text{ as } k \to +\infty.$$

Then for each k, $\varphi_k\left(x\right)$ is an eigenfunction of the operator H_{V}, responding to the eigenvalue

$$\lambda_k = \frac{1}{\tau_k} - m - 1 \ (k = 1, 2, 3, ...),$$

i.e.

$$\mathrm{H}_{\mathrm{V}} \varphi_k\left(x\right) = \lambda_k \varphi_k\left(x\right), \ \ k = 1, 2, 3, ...$$

Then according to Hilbert-Schmidt expansion theorem (see [13, p. 172, Theorem 3.1]) for any $f\left(x\right) \in D\left(\mathrm{H}_{\mathrm{V}}\right)$ we have

$$\left(\mathrm{H}_{\mathrm{V}} + \left(m+1\right) E \right) f\left(x\right) = \sum_{k=1}^{+\infty} \frac{1}{\tau_k} \left(f, \ \varphi_k\right) \varphi_k\left(x\right). \tag{4.1}$$

Hence it follows

$$\left(\mathrm{H}_{\mathrm{V}} + \left(m+1\right) E \right) \left(f\left(x\right) - \sum_{k=1}^{+\infty} \left(f, \ \varphi_k\right) \varphi_k\left(x\right) \right) = 0.$$

Since $\mathrm{H_V} + (m+1)\,E > 0$, we get

$$f\,(x) = \sum_{k=1}^{\infty} (f,\ \varphi_k)\,\varphi_k\,(x)\,.$$

Expansion (4.1) means that the orthonormal system of eigenfunctions

$$\{\varphi_k\,(x)\}_{k=1}^{\infty}$$

forms a complete system in the space $\overline{(\mathrm{H_V} + (m+1)\,E)\,D\,(\mathrm{H_V})}$.

Since by Theorem 3.1 the spectrum of the operator $\mathrm{H_V}$ is purely discrete, it holds the following

Theorem 4.1. *For any element $f\,(x) \in L_2\,(\mathbb{R}_n)$ it is valid the equality*

$$f\,(x) = \sum_{k=1}^{+\infty} a_k \varphi_k\,(x)\,, \tag{4.2}$$

where

$$a_k = (f,\ \varphi_k) = \int_{\mathbb{R}_n} f\,(x)\,\overline{\varphi_k\,(x)}dx, k = 1, 2, 3, ...,$$

and series (4.2) converges in the sense of $L_2\,(\mathbb{R}_n)$.

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