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STRUCTURE OF THE ROOT SUBSPACES AND OSCILLATION PROPERTIES OF THE EIGENFUNCTIONS OF COMPLETELY REGULAR STURMIAN SYSTEMS

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Abstract. A spectral problem for a fourth order ordinary differential operator with self-adjoint boundary conditions is considered. The structure of the root subspaces and oscillation properties of the eigenfunctions of this problem is studied.

1. Introduction

Consider the following boundary value problem

$$\ell(y) \equiv (p(x)y'')'' - (q(x)y')' + r(x)y(x) = \lambda\tau(x)y,\ 0 < x < l, \tag{1.1}$$

$$\begin{array}{l} (py'')(0) - C_0Ty(0) - C_1y'(0) = 0, \\ y(0) + C_2Ty(0) - C_0y'(0) = 0, \\ (py'')(l) + D_0Ty(l) + D_1y'(l) = 0, \\ y(0) - D_2Ty(0) + D_0y'(0) = 0, \end{array} \tag{1.2}$$

where $\lambda \in$ C is a spectral parameter, $Ty \equiv (py'')' - qy'$,$p(x)$, $\tau(x) > 0$, $x \in [0, l]$, $p(x) \in C^2[0, l]$, $q(x) \in C^1[0, l]$, $r(x), \tau(x) \in C[0, l]$, C_i, D_i, $i = \overline{0, 2}$ are non-negative constants and the case $C_1 = \infty$ or $D_1 = \infty$ is not excluded.

Problem (1.1), (1.2) was investigated in the papers [7] (case $q \equiv 0$) and [2]. In these papers the classes of regular and completely regular Sturmian systems were introduced and studied. For completely regular Sturmian systems (1.1), (1.2) it is established that the eigenvalues of these systems are real and form an unbounded monotonically increasing sequence, and in the case $r\equiv 0$, all of them are positive and simple, and the corresponding eigenfunctions posses the Sturmian oscillation properties, i.e. the number of zeros of eigenfunctions behaves in a common way (equal to the number of eigenvalue decreased by 1) (see also [3] (case $q \equiv 0$) and [4]). In the case when $r(x)$ doesn't vanish identically on any interval composing the part $[0, l]$, it is shown that the eigenvalues are simple except may be the first m ones and the corresponding eigenfunctions with ordinal numbers greater than m posses Sturmian oscillation properties (see the definition of the number m in the context).

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Note that the structure of the root subspaces and oscillation properties of the eigenfunctions corresponding to eigenvalues of problem (1.1), (1.2) with ordinal number is no greater than m have not been studied so far.

The present paper studies these issues.

The main result of this paper is the following theorem.

Theorem 1.1. *The eigenvalues of completely regular Sturm system* (1.1), (1.2) *are real, simple and form a infinitely increasing sequence* $\lambda_1 < \lambda_2 < ... < \lambda_m < ...$. *Furthermore, the eigenfunction* $y_k(x)$, $k\in\mathrm{N}$, *corresponding to the eigenvalue* λ_k, *has* $k-1$ *simple zeros in* $(0, l)$.

Recall that at completely regular boundary conditions (1.2), the eigenvalues of equation (1.1) without the potential are positive and simple, the corresponding eigenfunctions are Sturmian oscillation properties. These properties allow us to show that under these boundary conditions (1.2), the eigenvalues of equation (1.1) in the case $r \neq 0$ remain simple, and the corresponding eigenfunctions are Sturmian oscillation properties.

2. Preliminary results

Based on maximal-minimal properties of eigenvalues [6, p. 343], the eigenvalues of problem (1.1), (1.2) are determined from the relation:

$$\lambda_k = \max_{V^{(k-1)}} \min_{y\in B.C.} \left\{ R\ [y] | \int_0^l \tau(x) y(x) \varphi(x) dx = 0,\ \varphi \in V^{(k-1)} \right\}, \tag{2.1}$$

where $R\ [y]$ is Reley ratio

$$R[y] = \left\{ \int_0^l (py''^2 + qy'^2 + r\, y^2) dx + N[y] \right\} \Big/ \int_0^l \tau\, y^2 dx, \tag{2.2}$$

$$N[y] = C_1 y'^2(0) + D_1 T^2 y(0) + C_2 y'^2(l) + D_2 T^2 y(l),$$

$V^{(k-1)}$ is an arbitrary set of linear-independent functions $\varphi_j \in B.C.$, $1 \leq j \leq k-1$, $B.C.$ is the set of functions satisfying boundary conditions (1.2).

Remark 2.1. By virtue of maximal-minimal properties of eigenvalues and relations (2.2), the eigenvalues of problem (1.1), (1.2) are continuous and strongly increasing functions of parameters $C_i, D_i \in [0, +\infty)$, $i = 1,\ 2$. Therewith, when $C_1(D_1)$ everywhere monotonically increases from zero to infinity, then each eigenvalue of λ monotonically increases from the value accepted by them under the boundary condition

$$(py'')(0) - C_0 Ty(0) = 0((py'')(l) + D_0 Ty(l) = 0)$$

to its value under the boundary condition $y'(0) = 0$ $(y'(l) = 0)$ (see [6, p. 347]).

Assume:

$$r_0 = \min_{x\in[0,l]} r(x),\ r_1 = \max_{x\in[0,l]} r(x),\ \tau_0 = \min_{x\in[0,l]} \tau(x),\ \tau_1 = \max_{x\in[0,l]} \tau(x).$$

Denote by (Σ) a completely regular Sturmian system obtained from the system (1.1) (1.2) by changing $r(x)$ by r_0 and $\tau(x)$ by τ_1. By changing $\lambda' = \lambda\tau_1 - r_0$

the system (Σ) goes into the equivalent system (Ψ) for which the statement of theorem 1 from [2] (see also [7, Theorem 1]) is valid.

Let $\lambda'_k,\ k\in \mathrm{N}$, be the k–th eigenvalue of the system (Ψ), that is positive by virtue of [2, Corollary 1], and $\tilde{\lambda}_k=(\lambda'_k+r_0)/\tau_1,\ \ k\in \mathrm{N}$, be the k–th eigenvalue of the system (Σ). Then from [2, Theorem 1]) the eigenfunction $\tilde{y}_k(x)$ corresponding to the eigenvalue $\tilde{\lambda}_k\,,\ k\in \mathrm{N}$, has exactly $k-1$ simple zeros in the interval $(0,l)$.

Now we go from the system (Σ) to the system (1.1), (1.2) by using "μ-process" (see [7], [2]) by means of deformation

$$r(x,\mu)\equiv(1-\mu')\,r_0+\mu' r(x),$$

$$\tau(x,\mu)=\ (1-\mu'')\tau_1+\mu''\tau(x),\ \ x\in[0,l],\ \ \mu',\mu''\in[0,1].$$

As $r(x,\mu)$ increases, and $\tau(x,\mu)$ decreases, then the parameters μ' and μ'' increase from 0 to 1 independent each other, then by virtue of [2, Lemma 4] the positive eigenvalues don't decrease.

Define the non-negative integer m_0 from the relations

$$\lambda'_{m_0+1}>(r_1\tau_1-r_0\tau_0)/r_0\geq\lambda'_{m_0}\ \text{and}\ \tilde{\lambda}_{m_0+1}>0.$$

It is known [2] that if $k>m=\max\{2,\,m_0\}$, then the following inequality is fulfilled

$$r(x,\,\mu)-\lambda_k(\mu)\,\tau(x,\mu)<0,\ \ x\in[0,l],\ \ \mu\in[0,1]$$

where $\lambda_k(\mu)$ is the k–th eigenvalue of the Sturmian system obtained from the system (1.1), (1.2) changing $r(x)$ by $r(x,\mu)$ and $\tau(x)$ by $\tau(x,\mu)$. Consequently, by virtue of corollary 1, lemma 7 and remark 1 from [2], the eigenvalues $\lambda_1,\lambda_2,...,\lambda_m$, $\lambda_{m+1},...$, of completely regular system (1.1), (1.2) except may be the m first ones that are positive and simple, and the eigenfunction $\vartheta_k(x)$ corresponding to the eigenvalue λ_k for $k>m$ has $k-1$ simple zeros in the interval $(0,\ l)$.

Obviously, these statements are valid also for the completely regular Sturmian system

$$\left.\begin{array}{l}\ell(y)+\mu\,r(x)y=\lambda\tau(x)y,\ \ x\in(0,l)\,,\\ y(x)\in B.C.,\mu\in[0,l].\end{array}\right\}\tag{2.3}$$

Lemma 2.1. *It is valid the following relation*

$$\mu_k-\mu\,M/\tau_0\leq\nu_k(\mu)\leq\mu_k+\mu\,M/\tau_0,\tag{2.4}$$

where $\nu_k(\mu)$ *is the k-th eigenvalue of problem* (2.3), $M=\max\{|r_0|,\,|r_1|\}$, μ_k *is the k-th eigenvalue of the completely regular Sturmian system*

$$\left.\begin{array}{l}\ell(y)=\lambda\tau(x)\,y,\ \ x\in(0,l),\\ y\in B.C..\end{array}\right\}\tag{2.5}$$

Proof. From the maximal-minimal property of eigenvalues (see (2.1)) we have

$$\lambda_k\,(\mu)=\max_{V^{(k-1)}}\ \min_{y(x)\in B.C.}\left\{R_\mu[y]\mid\int_0^l r(x)y(x)\varphi(x)dx=0,\ \ \varphi(x)\in\ V^{(k-1)}\right\},\tag{2.6}$$

where

$$R_\mu[y] = \left(\int_0^l \left(py''^2 + qy'^2 + \mu\, r\, y^2 \right) dx + N\,[y] \right) \Big/ \int_0^l \tau\, y^2 dx. \qquad (2.7)$$

For an arbitrary choice of $V^{(k-1)}$ from (2.6) we get

$$R_\mu[y] = R_0[y] + \mu \left(\int_0^l r\, y^2 dx \Big/ \int_0^l \tau\, y^2 dx \right), \qquad (2.8)$$

where

$$R_0[y] = \left(\int_0^l \left(py''^2 + qy'^2 \right) dx + N[y] \right) \Big/ \int_0^l \tau\, y^2 dx. \qquad (2.9)$$

(2.7) yields the relation

$$R_0[y] - \mu\, M/r_0 \le R_\mu[y] \le R_0[y] + \mu\, M/r_0,$$

from which we get

$$\max_{V^{(k-1)}} \min_{y \in B.C.} R[y] - \mu\, M/r_0 \le \max_{V^{(k-1)}} \min_{y \in B.C.} R_\mu\,[y] \le \max_{V^{(k-1)}} \min_{y \in B.C.} R[y] \; + \mu\, M/r_0. \qquad (2.10)$$

From (2.6)-(2.10) follows relation (2.4). The lemma is proved.

Let in boundary conditions (1.2) $C_0 = D_0 = 0$. Introduce the following denotation:

$$\alpha = \arcsin C_1/(1+C_1^2), \;\; \beta = \arcsin C_2/(1+C_2^2),$$
$$\gamma = \arcsin D_1/(1+D_1^2), \;\; \delta = \arcsin D_2/(1+D_2^2).$$

Then boundary conditions (1.2) takes the form:

$$\begin{array}{l} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \\ y(l)\cos\delta - Ty(l)\sin\delta = 0. \end{array} \qquad (2.11)$$

Let $E = C^3[0,l] \bigcap B.\,C_1.$ supplied with the norm $||y||_j = \sum_{i=0}^j |y^{(i)}|_0$, where $B.\,C_1.$, is the set of functions satisfying boundary conditions (2.11), $|\cdot|_0$ be an ordinary sup-norm in $C\,[0,\,l]$.

Denote: $S = \{y \in E\,|\, y^{(i)}(x) \neq 0\,,\, x \in (0,l),\, i = \overline{0\,,\,3}\,\} \bigcup \{y \in E\,|$ if $y\,(\xi) = 0$ or $y''\,(\xi) = 0$, then $y'(x)Ty(x) < 0$ in some neighborhood of the point $\xi \in (0,l)$; and if $y'\,(\eta) = 0$ or $Ty\,(\eta) = 0$, then $y(x)\,y''(x) \; < 0$ in some neighborhood of the point $\eta \in (0,l)\}$.

Note that if $y \in S$, then the Jacobian $J[y] = r^3 \cos\psi \sin\psi$ of the Prufer type transformation

$$\begin{cases} y(x) = r(x)\sin\psi(x)\cos\theta(x), \\ y'(x) = r(x)\cos\psi(x)\sin\varphi(x), \\ (py)''(x) = r(x)\cos\psi(x)\cos\varphi(x), \\ Ty(x) = r(x)\sin\psi(x)\sin\theta(x). \end{cases} \qquad (2.12)$$

differs from zero for $x \in (0,\,l)$.

Note that in [3], [4], using transformation (2.12), the oscillation properties of eigenfunctions and their derivatives of problem (1.1), (2.11) for $r \equiv 0$ were studied.

Denote by S_k^ν, $k \in N$, $\nu = +$ or $-$, the set of functions $y \in S$ that satisfy the following conditions:

1) $y(x)$ has exactly $k-1$ zeros in the interval $(0,l)$;

2) $\lim\limits_{x\to 0} \nu sgn y(x) = 1$;

3) the angular function ψ satisfies eigther the condition $\psi(x) \in (0,\, \pi/2)$ or $\psi(x) \in (\,\pi/2,\, \pi)$;

4) the boundary values of angular functions θ, φ and ψ from (2.12) are determined as follows:

$$\theta\,(0) = \pi/2 - \beta, \theta(l) = k\pi - \pi/2 - \delta;$$

$$\varphi(0) = \alpha, \varphi(l) = n_k\pi - \gamma, \;\; k \in \mathrm{N},$$

where $\alpha = 0$ in the case $\psi(0) = \pi\,/2$, $\gamma = 0$ in the case $\psi(l) = \pi\,/2$, $n_k = k$ either $k+1$ in the case $\psi(0) \in (0, \pi\,/2)$, $n_1 = 1$ and $n_k = k$ or $k-1$, $k \in \mathrm{N}\backslash\{1\}$, in the case $\psi(0) \in [\pi\,/2, \pi)$; $w(0) = ctg\psi(0)$ is determine at least by one of the following equalities

$$a)\;\; w(0) = \frac{y'(0)\sin\beta}{y(0)\sin\alpha}, \qquad b)\; w(0) = -\frac{(py'')(0)\cos\beta}{Ty(0)\cos\alpha},$$

$$c)\; w(0) = \frac{(py'')(0)\sin\beta}{y(0)\cos\alpha}, \qquad d)\; w(0) = -\frac{y'(0)\cos\beta}{Ty(0)\sin\alpha};$$

5) the graphs of the functions $\theta(x)$ and $\varphi(x), x \in [0,l]$, intersect the lines $\theta = (2m-1)\pi/2$, $\theta = m\pi$ and $\varphi = m\pi$, $m = 0,\, 1\,,\, 2,\, ...,$ strictly increasing, respectively;

6) if (i) $y(0)y'(0) > 0$, (ii) $y(0) = 0$, or (iii) $y'(0) = 0$ and $y(0)y''(0) > 0$, then $\psi(x) \in (0, \pi/2)$ for $x \in (0,l)$, and if (iv) $y(0)y'(0) < 0$, (v) $y'(0) = 0$ and $y(0)y''(0) < 0$, or (vi) $y'(0) = y''(0) = 0$, then $\psi(x) \in (\pi/2, \pi)$ for $x \in (0,l)$.

Denote: $S_k = S_k^+ \bigcup S_k^-$. By virtue of [3], [4] the eigenfunction $\vartheta_k(x)$, corresponding to the eigenvalue η_k of problem (1.1), (2.11) for $r \equiv 0$ is contained in the set S_k, $k \in \mathrm{N}$. Consequently, the set S_k^ν, $k \in \mathrm{N}$, $\nu = +$ or $-$, are non-empty. The sets S_k^ν,$k \in \mathrm{N}$,$\nu = +$ or $-$, are open subsets in E [1].

Along with problem (1.1), (2.11) consider the following nonlinear approximation problem

$$\left.\begin{array}{l} (\ell y)(x) + r(x)\, \|y(x)\|_3^\varepsilon\, y(x) = \lambda\tau(x)y(x), \;\; x \in (0,l)\,, \\ y(x) \in B.C_1., \end{array}\right\} \tag{2.13}$$

where $\varepsilon \in (0,1]$. Such approximations are encountered in [5], [9]. Determine the function $g(y) \in C\,[0,l]$, $y \in E$ in the following way:

$$g(y)(x) = -r(x)y(x), \;\; x \in [0,l]. \tag{2.14}$$

As $r(x) \in C\,[0,l]$, then the function $g : E \to C\,[0,l]$ is continuous. We can rewrite problem (2.13) in the following equivalent form

$$\left.\begin{array}{l} (\ell(y) = \lambda r(x)y + g(\,\|y\|_3^\varepsilon\, y), x \in (0,l) \\ y \in B.C_1.\,. \end{array}\right\} \tag{2.15}$$

By virtue of (2.14) for each fixed $\varepsilon \in (0,\, 1]$ it is valid the relation

$$g(||y||_3^{\varepsilon} y) = o(||y||_3) \text{ for } ||y||_3 \to 0,$$

consequently, for problem (2.13) the statement of theorem 1 from [1] is valid. Then for each $k \in \mathrm{N}$ and $\nu = +$ or $-$, there exists an unbounded continuum of the set of solutions to problem (2.13), $C_{k,\varepsilon}^{\nu}$ such that

$$(\eta_k, 0) \in C_{k,\varepsilon}^{\nu} \subset (R \times S_k^{\nu}) \bigcup \{(\eta_k, 0)\}.$$

Lemma 2.2. *There exists* $\omega_0 \in (0,\, 1)$*, such that for any* $\varepsilon \in (0, \omega_0)$ *and* $\gamma_0 > 0$*, the problem* (2.13) *has no solution* (λ, w) *satisfying the conditions*

$$w \in S_k^{\nu},\ k \le m_0,\ ||w||_3 \le \omega_0 \ \textit{and}\ \textit{dist}\,(\lambda, I_0) = \gamma_0,$$

where $I_0 = [\eta_1 - M/\tau_0, \eta_m + M/\tau_0]$.

Proof. Assume that there exists the sequence $\{(\xi_n,\, \vartheta_n)\}_{n=1}^{\infty} \subset R \times E$ of the solutions of problem (2.13), corresponding to $\varepsilon_n < 1/n$ such that

$$\textit{dist}\,(\lambda_n, I_0) = \gamma_0 \text{ and } \vartheta_n \in S_k^{\nu},\ k \le m_0,\ ||\vartheta_n||_3 < 1/n.$$

Denote: $w_n = \vartheta_n / ||\vartheta_n||_3$. Then w_n satisfies the relations

$$\ell(w_n) + r\ \|w_n\|_3^{\varepsilon_n} w_n = \xi_n \tau\, w_n,\ w_n \in B.C_1.. \tag{2.16}$$

A the sequence $\{(\xi_n,\, w_n)\}_{n=1}^{\infty}$ is bounded in $R \times E$, then from (2.16) it follows that it is bounded in $R \times C^4[0, l]$ as well. Then there exists the subsequence $\{(\xi_{n_s},\, w_{n_s})\}_{s=1}^{\infty}$ of the sequence $\{(\xi_n,\, w_n)\}_{n=1}^{\infty}$, that converges to $\left(\bar{\lambda}, w\right)$ in $R \times E$. By (2.16), $\{(\xi_{n_s},\, w_{n_s})\}_{s=1}^{\infty}$ converges to $\left(\bar{\lambda}, w\right)$ and in $R \times C^4[0, l]$ as well. Furthermore, we have $||w||_3 = 1$, $w \in S_k^{\nu}$, $||w_n||_3^{\varepsilon_n} \to \gamma_1$, $n \to \infty$, $0 \le \gamma_1 \le 1$ and

$$\left.\begin{array}{l} \ell(w) + \gamma_1 r\, w = \bar{\lambda}\, \tau\, w, \\ w \in B.C_1.\,. \end{array}\right\} \tag{2.17}$$

Problem (2.17) has the form of (2.3) and as $w \in S_k^{\nu}$, $k \le m$, then based on (2.4) we get $\bar{\lambda} \in I_0$. But this contradicts the inequality $\textit{dist}\left(\bar{\lambda}, I_0\right) = \gamma_0 > 0$. The lemma is proved.

3. Proof of Main Results

Proof of theorem 1.1. Assume $k \le m$. As $C_{k,\varepsilon}^{\nu}$ is a connected set, then for any $\varepsilon \in (0, \omega_0)$ there exists the solution $(\lambda_\varepsilon,\, y_\varepsilon)$ to problem (2.13) such that $\lambda_\varepsilon \in I_0$ and $||y_\varepsilon||_3 = \omega_0$. Following the reasonings carried out in proving lemma 2, one can find the sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, \omega_0)$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$ such that the sequence $\{(\lambda_{\varepsilon_n},\, y_{\varepsilon_n})\}_{n=1}^{\infty}$ converges to the solution $(\hat{\lambda}, \hat{y})$ of problem (1.1), (2.11), i.e. of the problem

$$\left.\begin{array}{l} \ell(y) + r(x)\, y = \lambda\, \tau(x)\, y,\ x \in (0, l), \\ y \in B.C_1.\,, \end{array}\right\} \tag{3.1}$$

where $\hat{\lambda} \in I_0, \hat{y} \in S_k$. So, for each $k \in \{\,1, 2, ..., m\,\}$ there exists the eigenfunction $\hat{y}_k \in S_k$ of problem (3.1), corresponding to the eigenvalue $\lambda_s,\ s \in \{\,1, 2, ..., m\}$.

Using the system

$$\left.\begin{array}{l}\ell(y)+\mu\, r(x)\, y=\lambda\, \tau(x)\, y,\ x\in(0,l),\\ y\in B.C_1.,\end{array}\right\}$$

by applying the "$\mu-$process", we go from the Sturmian system

$$\left.\begin{array}{l}\ell(y)=\lambda\, \tau(x)\, y,\\ y\in B.C_1.\ ,\end{array}\right\}$$

to the Sturmian system (3.1). As the eigenvalues move from the initial value (origin) η_k to which the eigenfunction $\vartheta_k\in S_k$ corresponds, we can assume that $s=k$.

Thus, the eigenfunction $\hat{y}_k(x)$ of problem (1.1), (2.11) corresponding to the eigenvalue $\hat{\lambda}_k$, $k\in\{1,\,2,\,...,\,m\}$, has exactly $k-1$ simple zeros in the interval $(0,l)$, and $\hat{\lambda}_1\le\hat{\lambda}_2\le\,...\,\le\hat{\lambda}_m<\hat{\lambda}_{m+1}$. Then by virtue of [7, Lemma 3], [2, Lemma 3] this statement is valid for problem (1.1), (1.2) (see also [8]) as well.

Determine the numbers $d_0>0$ and $d_1\ge 0$ from the following relations:

$$d_0=\min_{k=\overline{1,m}}\{\mu_{k+1}-\mu_k\},$$

$$d_1=\inf\{z\in R_+|r(x)+z\tau(x)>0,\ x\in[0,l]\}.$$

By changing $\xi=\lambda+d_1$, system (1.1), (1.2) goes into the equivalent system

$$\left.\begin{array}{l}\ell(y)+\bar{r}\, y=\xi\,\tau\, y\\ y\in B.C.,\end{array}\right\}\tag{3.2}$$

where $\bar{r}=r+d_1\tau$. Now we go from system (2.5) to system (3.2) by using the "μ – process":

$$\left.\begin{array}{l}\ell(y)+\mu\,\bar{r}\, y=\xi\,\tau\, y\\ y\in B.C..\end{array}\right\}\tag{3.3}$$

As the coefficient $r(x,\mu)=\mu\,\bar{r}(x)$ increases by increasing the parameter μ from 0 to 1, then the eigenvalues don't decrease. Note that if the condition $r_1/\tau_0+d_1<d_0$ is fulfilled, then the eigenvalues $\xi_1(\mu),\ \xi_2(\mu),\,,...,\ \xi_m(\mu)$ of problem (3.3) don't coincide in "$\mu-$processes", and consequently all of them are simple.

Note that if the condition $r_1/\tau_0+d_1<d_0$ is not fulfilled, we can choose $\bar{\mu}\in(0,\,1)$ such that the inequality $\bar{\mu}\,(r_1/\tau_0+d_1)<d_0$ be fulfilled. Then obviously, the eigenvalues $\xi_1(\mu),\ \xi_2(\mu)\ ,...,\ \xi_m(\mu)$ of problem (3.3) for $\mu\in(0,\,\bar{\mu})$ are simple.

Now show that the eigenvalues $\xi_1(\mu),\ \xi_2(\mu),...,\ \xi_m(\mu)$ of problem (3.3) are simple for $\mu\in(\,\bar{\mu},\,1]$ as well. Indeed, if it is not so, then there exist $\mu_0\in(\,\bar{\mu},\,1]$ closer to $\bar{\mu}$ and $k\in\{\,1,\,2,\,...,m-1\,\}$ such that $\xi_k(\mu_0)=\xi_{k+1}(\mu_0)$. Take a rather small $\varepsilon>0$ ($\varepsilon<\bar{\mu}-\mu_0$) and consider the eigenvalues $\xi_k(\mu_0-\varepsilon)$ and $\xi_{k+1}(\mu_0-\varepsilon)$. Obviously, $\xi_k(\mu_0-\varepsilon)<\xi_{k+1}(\mu_0-\varepsilon)$.

Let $\bar{D}_1\in(0,\,+\infty)$ be such that $D_1<\bar{D}_1$, in the case $D_1\in[0,+\infty)$. By the maximal-minimal property of eigenvalues [6, pp. 342–347] and Remark 2.1, we have

$$\xi_k(\mu_0-\varepsilon,\,D_1)<\xi_k(\mu_0-\varepsilon,\,\bar{D}_1)<\xi_{k+1}(\mu_0-\varepsilon,\,D_1),\ \ if\ D_1\in[0,\infty),$$

$$\xi_k(\mu_0-\varepsilon,\,\bar{D}_1)<\xi_k(\mu_0-\varepsilon,\,D_1)<\xi_{k+1}(\mu_0-\varepsilon,\,\bar{D}_1)<\xi_{k+1}(\mu_0-\varepsilon,\,D_1),\ \ if\, D_1=\infty,$$

Recall that ([7, Lemma 2] or [2, Lemma 3]) the eigenvalues of problem (3.3) $\xi_1(\mu), \xi_2(\mu), ..., \xi_m(\mu)$ are continuous functions of the parameter μ. Now passing to limit at the last two inequalities, as $\varepsilon \to 0$ we get

$$\xi_k(\mu_0, D_1) = \xi_k(\mu_0, \bar{D}_1), \ if \ D_1 \in [0, \infty),$$

$$\xi_{k+1}(\mu_0, \bar{D}_1) = \xi_{k+1}(\mu_0, D_1), \ \ if \ \ D_1 = \infty,$$

that contradicts Remark 2.1.

By changing the variable $\lambda = \xi - d_1$, system (3.2) goes into the equivalent system (1.1), (1.2).

The proof of Theorem 1.1 is completed.

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