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IMPROVED INVERSE THEOREMS IN WEIGHTED SMIRNOV CLASSES

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Abstract. Let $G\subset\mathbb{C}$ be a finite simple connected domain and let $\Gamma:=\partial G$ be a Carleson curve. In this work the improved inverse theorems of approximation theory by algebraic polynomials in the weighted Smirnov classes $E^p(G,\omega)$ and $E^p(G^-,\omega)$, $1<p<\infty$, in term of the fractional modulus of smoothness are obtained.

1. Introduction and the main result

Let G be a finite domain in the complex plane, bounded by a rectifiable Jordan curve Γ and let $G^-:=ext\ \Gamma$. Further let $\mathbb{U}:=\{w\in\mathbb{C}:|w|<1\}$, $\partial\mathbb{U}:=\mathbb{T}$ and $\mathbb{U}^-:=\mathbb{C}\setminus\overline{\mathbb{U}}$. By $L^p(\Gamma)$ and $E^p(G)\,,1\leq p<\infty$, we denote the set of measurable complex valued functions f such that $|f|^p$ is Lebesgue integrable with respect to arclength on Γ and the Smirnov class of analytic functions in G, respectively. Let's remember that a function f analytic in G is said to be of class $E^p(G)$ if there exists a sequence of rectifiabe Jordan curves (γ_n) in G, tending to the boundary in the sense that γ_n eventually surrounds each compact subdomain of G, such that

$$\int\limits_{\gamma_n}|f(z)|^p\,|dz|\leq M<\infty.$$

Each function $f\in E^p(G)$ has a non-tangential limit almost everywhere (a.e.) on Γ and if we use the same notation for the non-tangential limit of f, then $f\in L^p(\Gamma)$.

$L^p(\Gamma)$ and $E^p(G)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(G)}:=\|f\|_{L^p(\Gamma)}:=\left(\int\limits_{\Gamma}|f(z)|^p\,|dz|\right)^{1/p},\ 1\leq p<\infty.$$

For more information see, [6, p. 438-453] and [4, p. 168-185].

Let ω is a weight function on Γ, that is $\omega:\Gamma\to[0,\infty]$ is measurable and the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero.

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We define a weighted Lebesgue space consisting of all measurable functions on Γ such that

$$\|f\|_{L^p(\Gamma,\omega)} := \left(\int_{\Gamma} |f(z)\,\omega(z)|^p\,|dz| \right)^{1/p} < \infty,\ 1 \le p < \infty.$$

Definition 1.1. A rectifiable Jordan curve Γ is called a Carleson curve if the condition

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0} |\Gamma(z,\varepsilon)|/\varepsilon < \infty$$

holds, where $\Gamma(z,\varepsilon)$ is portion of Γ in the open disk of radius ε centered at z, and $|\Gamma(z,\varepsilon)|$ its length.

We denote by S the set of all Carleson curves in the complex plane.

Definition 1.2. Let $1 < p < \infty$, $1/p + 1/q = 1$ and ω be a weight function on Γ. ω is said to satisfy the Muckenhoupt A_p-condition on Γ if

$$\sup_{z\in\Gamma}\sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} \omega(\tau)\,|d\tau| \right) \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} [\omega(\tau)]^{-q}\,|d\tau| \right)^{1/q} < \infty.$$

Let us denote by $A_p(\Gamma)$ the set of all weight functions satisfying the Muckenhoupt A_p-condition on Γ.

Let $T := [0, 2\pi]$ and let $\omega : T \to [0, \infty]$ be a weighted function on the T. By $L^p(T,\omega)$ we denote the class of the Lebesgue integrable 2π periodic functions, such that

$$\|f\|_{L^p(T,\omega)} := \left(\int_0^{2\pi} |f(x)\,\omega(x)|^p\,dx \right)^{1/p} < \infty.$$

Let $f \in L^p(\mathbb{T},\omega)$ with $1 < p < \infty$ and $\omega \in A_p(\mathbb{T})$. For a given $w \in \mathbb{T}$, $r \in \mathbb{R}^+$ and $t \in \mathbb{R}$, we set

$$\Delta_t^r f(w) := \sum_{k=0}^{\infty} (-1)^k [C_k^r]\, f\left(we^{i(r-k)t}\right),$$

where $[C_k^r] := \frac{r(r-1)...(r-k+1)}{k!}$ for $k > 1$, $[C_k^r] := r$ for $k = 1$ and $[C_k^r] := 1$ for $k = 0$. Since [24, p. 14]

$$|[C_k^r]| = \left| \frac{r(r-1)...(r-k+1)}{k!} \right| \le \frac{c(r)}{k^{r+1}}, \quad k \in \mathbb{Z}^+$$

we have $C(r) := \sum_{k=0}^{\infty} |[C_k^r]| < \infty$. Therefore, the series converges absolutely and $\Delta_t^r f(w)$ is measurable on T. We define an operator

$$\sigma_\delta^r f(w) := \frac{1}{\delta} \int_0^{\delta} |\Delta_t^r f(w)|\,dt.$$

By the boundedness of the Hardy-Littlewood maximal function in $L^p(\mathbb{T},\omega)$ [23], we get

$$\sup_{|\delta|\leq h}\left\|\sigma_\delta^r f(w)\right\|_{L^p(\mathbb{T},\omega)}\leq c(p,r)\left\|f\right\|_{L^p(\mathbb{T},\omega)},$$

which implies the correctness of the following definition:

Definition 1.3. Let $f\in L^p(\mathbb{T},\omega)$ with $1<p<\infty$ and $\omega\in A_p(\mathbb{T})$, and let $r\in\mathbb{R}^+$. The function $\Omega_r(f,.)_{p,\omega}:[0,\infty)\to[0,\infty)$ defined by

$$\Omega_r(f,h)_{p,\omega}:=\sup_{|\delta|\leq h}\left\|\sigma_\delta^r f(w)\right\|_{L^p(\mathbb{T},\omega)}$$

is called r-th mean modulus of $f\in L^p(\mathbb{T},\omega)$.

This modulus in case of $r\in\mathbb{N}$ coincides with the mean modulus of smoothness defined in [21] and is easy to verify that the modulus $\Omega_r(f,h)_{p,\omega}$, $r\in\mathbb{R}^+$ has the following properties :

i) $\Omega_r(f,h)_{p,\omega}$ is non-negative and non-decreasing function of $h>0$,

ii) $\Omega_r(f_1+f_2,\cdot)_{p,\omega}\leq\Omega_r(f_1,\cdot)_{p,\omega}+\Omega_r(f_2,\cdot)_{p,\omega}$,

iii) $\lim_{h\to0}\Omega_r(f,h)_{p,\omega}=0$.

The main functional spaces of analytic functions, where we will realize the approximation are defined as following:

Definition 1.4. The set

$$E^p(G,\omega):=\left\{f\in E^1(G):f\in L^p(\Gamma,\omega)\right\},1\leq p<\infty$$

is called the $\omega-$*weighted Smirnov class of order* p *of analytic functions in* G.

Similarly, we define

$$E^p(G^-,\omega):=\left\{f\in E^1(G^-):f\in L^p(\Gamma,\omega)\right\},1\leq p<\infty$$

$\omega-$*weighted Smirnov class of order* p *of analytic functions in* G^-.

Let $f\in E^p(G,\omega)$ with $1<p<\infty$ and $E_n(f)_{G,p,\omega}:=\inf_{P_n\in\mathcal{P}_n}\left\|f-P_n\right\|_{L^p(\Gamma,\omega)}$, $n=0,1,2,...$, where $\mathcal{P}_n$ is the class of algebraic polynomials of degree not exceeding n. In the case of $G=\mathbb{U}$, we also denote $E_n(f)_{p,\omega}:=E_n(f)_{\mathbb{U},p,\omega}$. Similarly we define the quantity $E_n(f)_{G^-,p,\omega}:=\inf_{P_n^*\in\mathcal{P}_n^*}\left\|f-P_n^*\right\|_{L^p(\Gamma,\omega)}$, $n=0,1,2,..$, for $f\in E^p(G^-,\omega)$ with $1<p<\infty$, where $\mathcal{P}_n^*$ is the class of polynomials of degree not exceeding n with respect to $1/z$.

We denote by φ and φ_1 the conformal mappings of G^- and G onto $\mathbb{U}^-$, respectively, normalized by

$$\varphi(\infty)=\infty\text{ and }\lim_{z\to\infty}\frac{\varphi(z)}{z}>0$$

and

$$\varphi_1(0)=0\text{ and }\lim_{z\to0}z\varphi_1(z)>0.$$

ψ and ψ_1 will be the inverse mappings of φ and φ_1, respectively. The functions φ and ψ have continuous extensions to Γ and $\mathbb{T}$, their derivatives φ' and ψ' have definite nontangential limit values a.e. on Γ and $\mathbb{T}$, and they are integrable with respect to Lebesgue measure on Γ and $\mathbb{T}$, respectively [6, p. 419 - 438].

For an arbitrary function $f\in L^p(\Gamma,\omega)$ and a weight ω on Γ, we set

$$f_0(w):=f[\psi(w)]\left(\psi'(w)\right)^{1/p},\quad f_1(w):=f[\psi_1(w)]\left(\psi_1'(w)\right)^{1/p}w^{2/p},$$

$$\omega_0(w) := \omega[\psi(w)], \quad \omega_1(w) := \omega[\psi_1(w)].$$

If $f \in L^p(\Gamma, \omega)$, then $f_0 \in L^p(\mathbb{T}, \omega_0)$ and $f_1 \in L^p(\mathbb{T}, \omega_1)$.

For $f \in L^p(\Gamma, \omega)$ the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\psi(w))\psi'(w)}{\psi(w) - z}dw, \quad z \in G, \tag{1.1}$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\psi_1(w))\psi_1'(w)}{\psi_1(w) - z}dw, \quad z \in G^-, \tag{1.2}$$

are analytic in G and G^-, respectively and $f^-(\infty) = 0$.

Furthermore, if Γ is a Carleson curve and $\omega \in A_p(\Gamma)$, then $f^+ \in E^p(G, \omega)$ and $f^- \in E^p(G^-, \omega)$ for $f \in L^p(\Gamma, \omega)$, $1 < p < \infty$ (see, [10, Lemma 3]). Hence, if $f_0 \in L^p(\mathbb{T}, \omega_0)$ and $f_1 \in L^p(\mathbb{T}, \omega_1)$, then the condition ω_0, $\omega_1 \in A_p(\mathbb{T})$ implies that $f_0^+ \in E^p(\mathbb{U}, \omega_0)$ and $f_1^+ \in E^p(\mathbb{U}, \omega_1)$.

We define *r-th mean modulus of smoothness* of $f \in E^p(G, \omega)$ by

$$\Omega_r(f, \delta)_{G,p,\omega} := \Omega_r\left(f_0^+, \delta\right)_{p,\omega_0}, \ \delta > 0.$$

In the case of $f \in E^p(G^-, \omega)$ we define the *r-th mean modulus of smoothness* by

$$\Omega_r(f, \delta)_{G^-,p,\omega} := \Omega_r\left(f_1^+, 1/n\right)_{p,\omega_1}, \ \delta > 0.$$

In this work we investigate the inverse problems of approximation theory in the weighted Smirnov classes $E^p(G, \omega)$. The direct and inverse problems in the weighted and nonweighted Lebesgue spaces were studied, especially in case of $T = [0, 2\pi]$, wide enough; for positive integer r, inverse theorem and its improvement in $L^p(T, \omega)$, with $\omega \in A_p(T)$ by using the $r-$th Butzer-Wehrens type modulus of smoothness, which is equivalent in our terms to $\Omega_{2r}(f, \delta)_{p,\omega}$ was obtained in [8] and [7], respectively. The direct and inverse theorems in the weighted Lebesgue spaces $L^p(T, \omega)$, with $\omega \in A_p(T)$ and $1 < p < \infty$, using the modulus $\Omega_r(f, \delta)_{p,\omega}$ was proved by N. X. Ky [21]. The generalization and the improvement of this result in the fractional case were obtained in [1] and [29], respectively. Note that, in the nonweighted Lebesgue space $L^p(T)$ direct and inverse theorems in the fractional case earlier were proved by R. Taberski in [26].

The approximation problems were also investigated in the complex plane, especially in the Smirnov classes of analytic functions; firstly in the nonweighted case, the direct and inverse theorems of the polynomial approximation in $E^p(G)$, $p > 1$, classes, when Γ is an analytic curve was obtained by Walsh and Russel [27]. When Γ is a smooth Jordan curve and $\theta(s)$, the angle between the tangent and the positive real axis expressed as a function of arclength s has modulus of continuity $\Omega(\theta, s)$ satisfying the Dini-smooth condition:

$$\int_0^\delta \Omega(\theta, s)/sds < \infty, \ \delta > 0, \tag{1.3}$$

the direct and inverse theorems for $p > 1$, were obtained by S. Y. Alper [2]. The improvements of the last results were proved by Kokilashvili in [19]. In case of Carleson curves the direct theorem was stated in [20] and later by J. E. Andersson

[3] was proposed a method for obtaining the direct and inverse theorems in case of $p \geq 1$.

In the weighted and nonweighted cases, applying the special modulus of smoothness the general approximation and constructive description problems in the some subclasses of Smirnov classes were also studied by Ibragimov and Mamedkhanov [9] and Mamedkhanov [22]. When the boundary of G is a Radon domain using the method of pseudoanalytic extension of functions the constructive description of some subclasses of weighted Smirnov classes was obtained by Dyn'kin [5]. When Γ is a Carleson curve using the Butzer-Wehrens type modulus of smoothness, which is equivalent to the modulus $\Omega_{2r}(f,\cdot)_{G,p,\omega}$, the direct theorems in the classes $E^p(G,\omega)$ and $L^p(\Gamma,\omega)$ were proved in [10] and [11], respectively. Here in [10] was considered only the case of $r=2$. Later, proving the adequate inverse theorems for the even integers r, the constructive descriptions of some generalized Lipschitz subclasses of $E^p(G,\omega)$ were obtained in [14].

When Γ is a Carleson curve in term of the modulus $\Omega_r(f,\cdot)_{G,p,\omega}$, $r \in \mathbb{N}$, the direct and inverse problems in the weighted Smirnov classes were investigated in [16] and were proved in particular the following inverse theorems:

Theorem 1.1 ([16]). *Let $\Gamma \in S$ and let $\omega \in A_p(\Gamma)$, $\omega_0 \in A_p(\mathbb{T})$ with $1<p<\infty$. For $f \in E^p(G,\,\omega)$ and $r \in \mathbb{N}$ the estimate*

$$\Omega_r\left(f,\frac{1}{n}\right)_{G,p,\omega} \leq \frac{c}{n^r}\sum_{v=0}^{n}(v+1)^{r-1}E_v(f)_{G,p,\omega.} \qquad n=1,2,...,$$

holds whit a constant $c>0$ independent of n.

Theorem 1.2 ([16]). *Let $\Gamma \in S$ and let $\omega \in A_p(\Gamma)$, $\omega_1 \in A_p(\mathbb{T})$ with $1<p<\infty$. For $f \in E^p(G^-,\omega)$ and $r \in \mathbb{N}$ the estimate*

$$\Omega_r\left(f,\frac{1}{n}\right)_{G^-,p,\omega} \leq \frac{c}{n^r}\sum_{v=0}^{n}(v+1)^{r-1}E_v(f)_{G^-,p,\omega.} \qquad n=1,2,...,$$

holds whit a constant $c>0$ independent of n.

Note that, Theorems 1.1 and 1.2, in particular, when Γ satisfies the condition (1.3) were stated in [1].

In this work we give some improvements of these inverse estimations. Our main results can be formulated as following.

Theorem 1.3. *Let $\Gamma \in S$ and let $\omega \in A_p(\Gamma)$, $\omega_0 \in A_p(\mathbb{T})$ with $1<p<\infty$. If $f \in E^p(G,\omega)$ and $r \in \mathbb{R}^+$, then the estimate*

$$\Omega_r\left(f,\frac{\pi}{n+1}\right)_{G,p,\omega} \leq \frac{c}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_v^{\beta}(f)_{G,p,\omega}\right\}^{1/\beta} \qquad n=1,2,...,$$

holds with $\beta=\min\{p,2\}$ and a constant $c>0$ independent of n.

Theorem 1.4. *Let $\Gamma \in S$ and let $\omega \in A_p(\Gamma)$, $\omega_1 \in A_p(\mathbb{T})$ with $1<p<\infty$. If $f \in E^p(G^-,\omega)$ and $r \in \mathbb{R}^+$, then the estimate*

$$\Omega_r\left(f,\frac{\pi}{n+1}\right)_{G^-,p,\omega} \leq \frac{c}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_v^{\beta}(f)_{G^-,p,\omega}\right\}^{1/\beta} \qquad n=1,2,...,$$

holds with $\beta = \min\{p, 2\}$ *and a constant* $c > 0$ *independent of* n.

Here the quantity $E_v(f)_{G^-,p,\omega}$ defines in the class of polynomials with respect to $1/z$.

In the weighted Orlicz–Smirnov classes, weighted rearrangement invariant spaces and Lorentz spaces defined on the interval of $\mathbb{R}$ and on the domains of complex plane the similar problems for the even integers r were also investigated in the works [28], [12], [13], [15], [17], [18].

2. Auxiliary results

Definition 2.1. Let Γ be a rectifiable Jordan curve and $f \in L^1(\Gamma)$. The limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\Gamma\setminus\Gamma(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

existing for almost all $z \in \Gamma$ *is called the Cauchy singular integral of* f at $z \in \Gamma$.

According to the Privalov theorem [6, p. 431], if one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit a.e. on Γ, then $S_\Gamma(f)(z)$ exist a.e. on Γ, and also the other one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exist a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have nontangential limits a.e. on Γ. In both cases, the formulas

$$f^+(z) = S_\Gamma(f)(z) + f(z)/2 \quad \text{and} \quad f^-(z) = S_\Gamma(f)(z) - f(z)/2 \tag{2.1}$$

and hence

$$f(z) = f^+(z) - f^-(z)$$

hold a.e. on Γ [6, p. 431].

In our discussions we use the polynomials $F_{k,p}(z)$ and $\widetilde{F}_{k,p}(1/z)$ with respect to z and $1/z$, respectively, which have the integral representations:

$$F_{k,p}(z) := \frac{1}{2\pi i} \int_{|w|=R} \frac{w^k (\psi'(w))^{1-1/p}}{\psi(w) - z} dw, \quad z \in G, \quad R > 1,$$

$$\widetilde{F}_{k,p}(1/z) := -\frac{1}{2\pi i} \int_{|w|=R} \frac{w^k w^{-2/p} (\psi_1'(w))^{1-1/p}}{\psi_1(w) - z} dw, \quad z \in G^-, \quad R > 1,$$

for every $k \in \mathbb{N}$.

The polynomials $F_{k,p}(z)$ and $\widetilde{F}_{k,p}(1/z)$ are called the *p-Faber polynomials* for G and G^-, respectively (see, [10] and [25]).

Let $\mathcal{P}$ be the set of all algebraic polynomials $P(w)$ (with no restriction on the degree), and let $\mathcal{P}(\mathbb{U})$ be set of traces of members of $\mathcal{P}$ on $\mathbb{U}$. If we define the operators $T_p : \mathcal{P}(\mathbb{U}) \to E^p(G, \omega)$ and $\widetilde{T}_p : \mathcal{P}(\mathbb{U}) \to E^p(G^-, \omega)$ as

$$T_p(P)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)(\psi'(w))^{1-1/p}}{\psi(w) - z} dw, \quad z \in G \tag{2.2}$$

and

$$\widetilde{T}_p(P)(z) := -\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\, w^{-2/p}\left(\psi_1'(w)\right)^{1-1/p}}{\psi_1(w)-z}dw, \quad z\in G^-, \tag{2.3}$$

then

$$T_p\left(\sum_{k=0}^{n}\alpha_k w_k\right)=\sum_{k=0}^{n}\alpha_k F_{k,p}(z), \qquad \widetilde{T_p}\left(\sum_{k=0}^{n}\alpha_k w_k\right)=\sum_{k=1}^{n}\alpha_k \widetilde{F}_{k,p}(1/z).$$

Since by (1.1)

$$T_p(P)(z')=\left[(P\circ\varphi)(\varphi')^{1/p}\right]^+(z') \text{ for } z'\in G,$$

taking the limit $z'\to z\in\Gamma$, over all nontangential paths inside Γ in (2.2), we get

$$T_p(P)(z)=\frac{1}{2}\left[(P\circ\varphi)(\varphi')^{1/p}\right](z)+S_L\left[(P\circ\varphi)(\varphi')^{1/p}\right](z)$$

for almost all $z\in\Gamma$. Similarly, by (1.2)

$$\widetilde{T}_p(P)(z'')=\left[(P\circ\varphi_1)\,\varphi_1^{-2/p}\left(\varphi_1'\right)^{1/p}\right]^-(z'') \quad \text{for } z''\in G^-,$$

and taking the limit $z''\to z\in\Gamma$, along all nontangential paths outside Γ in (2.3), we get

$$\widetilde{T}_p(P)(z)=S_L\left[(P\circ\varphi_1)\,\varphi_1^{-2/p}(\varphi_1')^{1/p}\right](z)-\frac{1}{2}\left[(P\circ\varphi_1)\,\varphi_1^{-2/p}(\varphi_1')^{1/p}\right](z).$$

Since the set of algebraic polynomials are dense in $E^p(\mathbb{U},\omega_0)$ and $E^p(\mathbb{U},\omega_1)$ the operators T_p and $\widetilde{T}_p$ can be extended as bounded linear operators to $E^p(\mathbb{U},\omega_0)$ and $E^p(\mathbb{U},\omega_1)$ respectively and we have the representations

$$T_p(f)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(w)\left(\psi'(w)\right)^{1-1/p}}{\psi(w)-z}dw, \qquad f\in E^p(\mathbb{U},\omega_0)$$

and

$$\widetilde{T}_p(f)(z) := -\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(w)\, w^{-2/p}\left(\psi_1'(w)\right)^{1-1/p}}{\psi_1(w)-z}dw, \qquad f\in E^p(\mathbb{U},\omega_1).$$

The following theorem was proved in [16].

Theorem 2.1 ([16]). *Let $\Gamma\in S$, $1<p<\infty$ and ω be a weight function on Γ such that $\omega\in A_p(\Gamma)$ and ω_0, $\omega_1\in A_p(\mathbb{T})$. Then the operators $T_p: E^p(\mathbb{U},\omega_0)\to E^p(G,\omega)$ and $\widetilde{T_p}: E^p(\mathbb{U},\omega_1)\to E^p(G^-,\omega)$ are linear and bounded. Moreover, the operators T_p and $\widetilde{T_p}$ are one-to-one and onto, and we have $T_p(f_0^+)=f$ for $f\in E^p(G,\omega)$ and $\widetilde{T}_p(f_1^+)=f$ for $f\in E^p(G^-,\omega)$.*

We also will use the following result given in [29].

Theorem 2.2 ([29])**.** *Let $f \in L^p(T,\omega)$ with $1<p<\infty$ and let $\omega \in A_p(\mathbb{T})$. Then for a given $r \in \mathbb{R}^+$*

$$\Omega_r\left(f,\frac{\pi}{n+1}\right)_{p,\omega} \leq \frac{c}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_v^{\beta}(f)_{p,\omega}\right\}^{1/\beta} \qquad n=1,2,...,$$

where $\beta=\min\{2,p\}$ holds with a constant $c>0$ independent of n and f.

3. Proofs of Main Results

Let $f \in E^p(\mathbb{U},\omega)$. Applying Theorem 2.2 for the boundary values of $f \in E^p(\mathbb{U},\omega)$, we have:

Lemma 3.1. *If $f \in E^p(\mathbb{U},\omega)$ with $\omega \in A_p(\mathbb{T})$, $1<p<\infty$ and $r \in \mathbb{R}^+$, then the estimate*

$$\Omega_r\left(f,\frac{\pi}{n+1}\right)_{\mathbb{U},p,\omega} \leq \frac{c}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_v^{\beta}(f)_{p,\omega}\right\}^{1/\beta} \qquad n=1,2,...,$$

where $\beta=\min\{p,2\}$, holds with a constant $c>0$ independent of n and f.

Proof of Theorem 1.3. Let $f \in E^p(G,\omega)$. Then $f_0 := (f\circ\psi)(\psi')^{1/p} \in L^p(\mathbb{T},\omega_0)$, because $\|f_0\|_{L^p(\mathbb{T},\omega_0)} = \left\|(f\circ\psi)(\psi')^{1/p}\right\|_{L^p(\mathbb{T},\omega_0)} = \|f\|_{L^p(\Gamma,\omega)} < \infty$. Hence, if T_p^{-1} be the inverse operator of $T_p : E^p(\mathbb{U},\omega_0) \to E^p(G,\omega)$, then by Theorem 2.1 we have $T_p^{-1}(f) = f_0^+ \in E^p(\mathbb{U},\omega_0)$. Let $P_n \in \mathcal{P}_n$ $(n=0,1,2,...,)$ be the polynomials of the best approximation to f in $E^p(G,\omega)$, that is

$$E_n(f)_{G,p,\omega} = \|f-P_n\|_{L^p(\Gamma,\omega)}.$$

Then $T_p^{-1}(P_n) \in \mathcal{P}(\mathbb{U})$ and by the boundedness of T_p^{-1}

$$\begin{aligned} E_n(f_0^+)_{p,\omega_0} &\leq \left\|f_0^+ - T_p^{-1}(P_n)\right\|_{L^p(\mathbb{T},\omega_0)} \\ &= \left\|T_p^{-1}(f) - T_p^{-1}(P_n)\right\|_{L^p(\mathbb{T},\omega_0)} \\ &\leq \left\|T_p^{-1}\right\| \|f-P_n\|_{L^p(\Gamma,\omega)} = \left\|T_p^{-1}\right\| E_n(f)_{G,p,\omega}. \end{aligned}$$

Hence, using Lemma 3.1 for $f_0^+ \in E^p(\mathbb{U},\omega_0)$ and applying the last inequality we conclude that

$$\begin{aligned} \Omega_r\left(f,\frac{\pi}{n+1}\right)_{G,p,\omega} &= \Omega_r\left(f_0^+,\frac{\pi}{n+1}\right)_{\mathbb{U},p,\omega_0} \\ &\leq \frac{c}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_v^{\beta}\left(f_0^+\right)_{p,\omega_0}\right\}^{1/\beta} \\ &\leq \frac{c}{(n+1)^r}\left\|T_p^{-1}\right\|\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_n^{\beta}(f)_{G,p,\omega}\right\}^{1/\beta} \\ &= \frac{c_1}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}E_n^{\beta}(f)_{G,p,\omega}\right\}^{1/\beta}. \end{aligned}$$

□

Proof of Theorem 1.4. Let $f \in E^p(G^-,\omega)$. Then $f_1(w) := f[\psi_1(w)]\, w^{-2/p} \times (\psi_1'(w))^{1/p} \in L^p(\mathbb{T},\omega_1)$. Hence, if $\widetilde{T_p}^{-1}$ be inverse operator of $\widetilde{T_p} : E^p(\mathbb{U},\omega_1) \to E^p(G^-,\omega)$, then using Theorem 2.1 again, we have $\widetilde{T_p}^{-1}(f) = f_1^+ \in E^p(\mathbb{U},\omega_1)$. Let $P_n^* \in \mathcal{P}_n^*$ $(n = 0,1,2,...,)$ be a polynomial of the best approximation to f in $E^p(G^-,\omega)$, with respect to $1/z$, that is

$$E_n(f)_{G^-,p,\omega} = \|f - P_n^*\|_{L^p(\Gamma,\omega)} .$$

Then $\widetilde{T_p}^{-1}(P_n^*) \in \mathcal{P}_n(U)$. By the boundedness of $\widetilde{T}_p^{-1}$:

$$\begin{aligned} E_n(f_1^+)_{p,\omega_1} &\le \left\| f_1^+ - \widetilde{T_p}^{-1}(P_n^*) \right\|_{L^p(\mathbb{T},\omega_1)} \\ &= \left\| \widetilde{T_p}^{-1}(f) - \widetilde{T_p}^{-1}(P_n^*) \right\|_{L^p(\mathbb{T},\omega_1)} \\ &\le \left\| \widetilde{T_p}^{-1} \right\| \|f - P_n^*\|_{L^p(\Gamma,\omega)} = \left\| \widetilde{T_p}^{-1} \right\| E_n(f)_{G^-,p,\omega}. \end{aligned}$$

Hence, by definition of $\Omega_r\left(f,\frac{\pi}{n+1}\right)_{G^-,p,\omega}$ and by Lemma 3.1 for $f_1^+ \in E^p(\mathbb{U},\omega_1)$ we get

$$\begin{aligned} \Omega_r(f,\pi/(n+1))_{G^-,p,\omega} &= \Omega_r\left(f_1^+,\pi/(n+1)\right)_{\mathbb{U},p,\omega_1} \\ &\le \frac{c}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}\, E_v^\beta\left(f_1^+\right)_{p,\omega_1}\right\}^{1/\beta} \\ &\le \frac{c}{(n+1)^r}\left\|\widetilde{T_p}^{-1}\right\|\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}\, E_n^\beta(f)_{G^-,p,\omega}\right\}^{1/\beta} \\ &= \frac{c_2}{(n+1)^r}\left\{\sum_{v=0}^{n}(v+1)^{\beta r-1}\, E_n^\beta(f)_{G^-,p,\omega}\right\}^{1/\beta}. \end{aligned}$$

□

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