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# ON ATOMIC DECOMPOSITION FOR HARDY CLASSES WITH RESPECT TO DEGENERATE EXPONENTIAL SYSTEMS

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**Abstract**. Part of a system of exponents with degenerate coefficient is considered in this work. An atomic decomposition on this system in Hardy classes is studied in case when the coefficient may not satisfy the Muckenhoupt condition.

## 1. Introduction

Basis properties of classical system of exponents $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ ($\mathbb{Z}$ is the set of all integers) in Lebesgue spaces $L_p\left(-\pi,\pi\right)$, $1\leq p<+\infty$, have been well studied (see [7;8;24;25]). In [2], N.K.Bari raised the issue of the existence of normalized basis for $L_2$ which is not a Riesz basis. The first example was given by K.I.Babenko [1]. He proved that the degenerate system of exponents $\left\{\ |x|^{\alpha}\, e^{int}\right\}_{n\in\mathbb{Z}}$ with $|\alpha|<\frac{1}{2}$ forms a basis for $L_2\left(-\pi,\pi\right)$ but is not a Riesz basis when $\alpha\neq 0$. This result has been generalized by V.F.Gaposhkin [9]. In [12], the condition on the weight $\rho$ was found which makes the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ form a basis for the weight space $L_{p,\rho}\left(-\pi,\pi\right)$ with a norm $\|f\|_{p,\rho}=\left(\int_{-\pi}^{\pi}|f\left(t\right)|^{p}\rho\left(t\right)dt\right)^{\frac{1}{p}}$. It should be noted that the Riesz basicity of a degenerate system of exponents is considered in detail in [11].

Basis properties of a degenerate system of exponents are closely related to the similar properties of an ordinary system of exponents in a corresponding weight space. All the above-mentioned works consider the cases when the weight or the degenerate coefficient satisfies the Muckenhoupt condition (see, for example, [10]). Similar results are true for the systems of sines and cosines, too.

Basis properties of the systems of exponents and sines with linear phase in weighted Lebesgue spaces have been studied in [16;17;20]. Similar questions have been studied in [4;5] when the systems of exponents have degenerate coefficients. In case when the Muckenhoupt condition does not hold, then these systems have finite defects: some parts of them are complete and minimal, but they (i.e. these parts) do not form a basis. The questions then arise: are these systems an atomic decompositions or frames?

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In addition, it should be note that when solving many problems in mathematical physics by Fourier method (see, e.g.[3;18;19]), one has to deal with the basicity of the systems of the form $\{\sin[(n+\alpha)\,x+\beta]\}_{n\in\mathbb{N}}$ in the Lebesgue space $L_{p,\rho}$, with weight $\rho$, where $\alpha,\beta\in\mathbb{R}$ are real parameters. This is directly related to the study of the same issue for a part of the system of exponents $\left\{\rho\,(t)\,e^{int}]\right\}_{n\geq 0}$ $(\left\{\rho\,(t)\,e^{int}]\right\}_{n\leq m})$ in Hardy classes. Our work is dedicated to the study of frame properties (an atomic decomposition, frameness) on these systems in Hardy classes in the case when the weight is given in the form of power function. Note that the similar issues concerning degenerate systems of sines, cosines and exponents in Lebesgue spaces have been studied in [11;14].

The paper is organized as follows. In Section 2 we present the necessary notation, concepts, and some facts from the theory of bases and frames used throughout this paper. In Section 3 some considerations regarding frame sequences are introduced in Banach spaces. In Section 4 we consider the parts of a system of exponent with degenerate coefficient corresponding to positive values of the index. We prove that if the coefficients satisfy the Muckenhoupt condition, these parts form a basis for the corresponding classes of analytic functions of Hardy. Section 5 is dedicated to the studying of the frame properties (an atomic decomposition and frameness) of the part of a system corresponding to the positive values of the index when the coefficient, in general, does not satisfy the Muckenhoupt condition. Finally, in Section 6, the frame property, i.e. an atomic decomposition on this system and its frame in Hardy classes are studied.

## 2. Needful information

We will use the standard notation. $\mathbb{N}$ will be the set of all positive integers, $\exists$ will mean "there exist(s)", $\Rightarrow$ will mean "it follows", $\Leftrightarrow$ will mean "if and only if", $\exists!$ will mean "there exists a unique", $\mathbb{Z}_+=\{0\}\bigcup\mathbb{N}$; $\mathbb{C}$ is the complex plane; $\delta_{nk}$ will be the Kronecker symbol, and $\bar{(\cdot)}$ will stand for conjugation.

Let $X$ be some Banach space with a norm $\|\cdot\|_X$, and $X^*$ denote its conjugate with the corresponding norm $\|\cdot\|_{X^*}$. By $L\,[M]\equiv span[M]$ we denote the linear span of the set $M\subset X$, and $\overline{M}$ will stand for the closure of $M$.

We will also use the symbol " $\sim$ ". The expression $f\sim g\,,\ t\to a$, means that in sufficiently small neighborhood of the point $t=a$ there holds the inequality $0<\delta\leq\left|\frac{f(t)}{g(t)}\right|\leq\delta^{-1}<+\infty$.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ is said to be complete in $X$ if $\overline{L\left[\{x_n\}_{n\in\mathbb{N}}\right]}=X$. It is called minimal in $X$ if $x_k\notin\overline{L\left[\{x_n\}_{n\neq k}\right]}$, $\forall k\in\mathbb{N}$.

System $\{x_n\}_{n\in\mathbb{N}}\subset X$ is said to be uniformly minimal in $X$ if $\exists\delta>0$ : $\inf\limits_{\forall u\in L\left[\{x_n\}_{n\neq k}\right]}\|x_k-u\|_X\geq\delta\,\|x_k\|_X$ , $\forall k\in\mathbb{N}$.

The following criteria of completeness and minimality are available.

**Criterion 2.1.** *System $\{x_n\}_{n\in\mathbb{N}}\subset X$ is complete in $X$ if $f\,(x_n)=0$, $\forall n\in\mathbb{N}$, $f\in X^*\Rightarrow f=0$.*

**Criterion 2.2.** *System $\{x_n\}_{n\in\mathbb{N}}\subset X$ is minimal in $X\Leftrightarrow$ it has a biorthogonal system $\{f_n\}_{n\in\mathbb{N}}\subset X^*$, i.e. $f_n\,(x_k)=\delta_{nk}$, $\forall n,k\in\mathbb{N}$.*

With respect to the uniformly minimality there is the following criteria.

**Criterion 2.3.** *Complete system* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *is uniformly minimal in* $X \Leftrightarrow \sup_n \|x_n\|_X \|y_n\|_{X^*} < +\infty$, *where* $\{y_n\}_{n\in\mathbb{N}} \subset X^*$ *is a system biorthogonal to it.*

System $\{x_n\}_{n\in\mathbb{N}} \subset X$ is said to be a basis for $X$ if $\forall x \in X$, $\exists! \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ : $x = \sum_{n=1}^{\infty} \lambda_n x_n$. If system $\{x_n\}_{n\in\mathbb{N}} \subset X$ forms a basis for $X$, then it is uniformly minimal.

We will also need some facts about an atomic decomposition and frames in Banach spaces.

**Definition 2.1.** Let $X$ be a Banach space and $\mathscr{K}$ be a Banach sequence space indexed by $\mathbb{N}$. Let $\{f_k\}_{k\in\mathbb{N}} \subset X$, $\{g_k\}_{k\in\mathbb{N}} \subset X^*$. Then $\left(\{g_k\}_{k\in\mathbb{N}};\ \{f_k\}_{k\in\mathbb{N}}\right)$ is an atomic decomposition of $X$ with respect to $\mathscr{K}$ if :
(i) $\{g_k(f)\}_{k\in\mathbb{N}} \in \mathscr{K}$, $\forall f \in X$;
(ii) $\exists A, B > 0 : A\|f\|_X \le \left\| \{g_k(f)\}_{k\in\mathbb{N}} \right\|_{\mathscr{K}} \le B\|f\|_X$, $\forall f \in X$;
(iii) $f = \sum_{k=1}^{\infty} g_k(f)\, f_k$, $\forall f \in X$.

This fact will be denoted as a quadruple $\left\{\{f_k\}_{k\in\mathbb{N}}; \{g_k\}_{k\in\mathbb{N}}; K; \mathscr{K}\right\}$, where $K : \mathscr{K} \to X$ is a coefficient operator defined by the expression

$$K\vec{\lambda} = \sum_{n=1}^{\infty} \lambda_n f_n\,,\ \forall \vec{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K}.$$

**Definition 2.2.** Let $X$ be a Banach space and $\mathscr{K}$ be a Banach sequence space indexed by $\mathbb{N}$. Let $\{g_k\}_{k\in\mathbb{N}} \subset X^*$ and $S : \mathscr{K} \to X$ be a bounded operator. Then $\left(\{g_k\}_{k\in\mathbb{N}};\ S\right)$ is a Banach frame for $X$ with respect to $\mathscr{K}$ if :
(i) $\{g_k(f)\}_{k\in\mathbb{N}} \in \mathscr{K}$, $\forall f \in X$;
(ii) $\exists A, B > 0 : A\|f\|_X \le \left\| \{g_k(f)\}_{k\in\mathbb{N}} \right\|_{\mathscr{K}} \le B\|f\|_X$, $\forall f \in X$;
(iii) $S\ \left[\{g_k(f)\}_{k\in\mathbb{N}}\right] = f$, $\forall f \in X$.

It is appropriate to note that, in connection with the applications in various fields of natural sciences there is great interest in studying properties of frames. The review articles and monographs of various mathematicians are dedicated to this matter. More details can be found in [6;13;21;23].

Let us recall the definition of the Hardy classes. By $H_p^+$ we denote the usual Hardy class of analytical functions inside the unit circle $\omega \equiv \{z \in \mathbb{C} : |z| < 1\}$, furnished with the norm

$$\|f\|_{H_p^+} = \sup_{0<r<1} \left( \int_{-\pi}^{\pi} \left| f\left(re^{it}\right) \right|^p dt \right)^{1/p},\ p \ge 1.$$

Similarly we define the weighted Hardy class of functions that are analytical outside the unit circle $\omega$ and have a pole of order $k \le m$ at the infinitely remote point. The norm in this space is defined as

$$\|f\|_{{}_mH^-_{p,\rho}} = \left\| f^-\left(e^{it}\right) \right\|_{p,\rho},$$

where $f^-(\tau)$ are non-tangential boundary values of function $f$ on the unit circumference from the outside of the unit circle.

Restrictions of classes $H_p^+$ and ${}_mH_p^-$ to the unit circumference $\partial\omega$ will be denoted by $L_p^+$ and ${}_mL_p^-$, respectively. Spaces $H_p^+$ and $L_p^+$ $\left({}_mH_p^-\right.$ and $\left.{}_mL_p^-\right)$ are isomorphic and isometric.

## 3. On Banach frames

In the sequel, we will use the following general considerations. Let $X$ be a $B$-space, $\vec{x} \equiv \{x_n\}_{n\in\mathbb{N}} \subset X$ be complete and minimal system in it . Denote by $\mathscr{K}_{\vec{x}}$ the space of coefficients of system $\vec{x}$ with the norm

$$\|\vec{\lambda}\|_{\mathscr{K}} = \sup_m \|\sum_{n=1}^{m} \lambda_n x_n\|_X \ , \ \vec{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K},$$

where $\|\cdot\|_X$ is a norm in $X$. According to the results of [15], the canonical system $\{\delta_n\}_{n\in\mathbb{N}}$ forms a basis for $\mathscr{K}_{\vec{x}}$. Let $K$ be a coefficient operator defined by the expression

$$K\vec{\lambda} = \sum_{n=1}^{\infty} \lambda_n x_n \ , \ \forall\vec{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\vec{x}}.$$

According to the results of [15]: $K \in L\left(\mathscr{K}_{\vec{x}}; X\right)$, moreover $\exists K^{-1}$. If the system $\vec{x}$ forms a basis for $X$, then it is obvious that $(\vec{x}; \vec{x}^*; K; \mathscr{K}_{\vec{x}})$ is an atomic decomposition of $X$, where $\vec{x}^* \equiv \{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ is biorthogonal system to $\vec{x}$. In this case, it is clear that $K^{-1} \in L\left(X; \mathscr{K}_{\vec{x}}\right)$. If $K^{-1} \notin L\left(X; \mathscr{K}_{\vec{x}}\right)$, then it is clear that $(\vec{x}^*; \vec{x}; K; \mathscr{K}_{\vec{x}})$ isn't an atomic decomposition of $X$. Thus, if the system $\vec{x}$ is complete and minimal in $X$, but doesn't form a basis for it, then $(\vec{x}^*; \vec{x}; K; \mathscr{K}_{\vec{x}}))$ isn't an atomic decomposition of $X$.

Consider the case of $\omega$-linearly independent system $\vec{x}$. Let the system $\vec{x}$ have a finite defect $d\left(\vec{x}\right) = k_0$, i.e. after removing $k_0$ elements from it (case $k_0 > 0$), or adding to it (case $k_0 < 0$), the resulting system proves complete and minimal in $X$. Denote by $\vec{x}_{(k_0)}$ the obtained system and let $\mathscr{K}_{\vec{x}_{(k_0)}}$ be the space of its coefficients. Let $\mathbb{C}^m$ be $m$-dimensional complex space, where $\mathbb{C}$ be the field of scalars. In the first case (i.e. $k_0 > 0$) $\vec{x}_{(k_0)}$ is a part of the system $\vec{x}$, and in the second case ( i.e. $k_0 < 0$), conversely. From the $\omega$-linear independence of system $\vec{x}$ it follows directly that we have the direct sum

$$\mathscr{K}_{\vec{x}} = \mathbb{C}^{k_0} \dot{+} \mathscr{K}_{\vec{x}_{(k_0)}}, k_0 > 0;$$

$$\mathscr{K}_{\vec{x}_{(k_0)}} = \mathbb{C}^{|k_0|} \dot{+} \mathscr{K}_{\vec{x}}, k_0 < 0.$$

Consider the case $k_0 > 0$. Let $L_{(k_0)} = L\left[\{x_k\}_1^{k_0}\right]$. It is clear that holds

$$X \equiv L_{(k_0)} \dot{+} X_{(k_0)}.$$

It is obvious that the system $\vec{x}_{(k_0)}$ is complete and minimal in $X_{(k_0)}$ if and only if the system $\vec{x}$ is complete and minimal in $X$. Let $\vec{x}^*_{(k_0)} \equiv \{x_k^*\}_{k\geq k_0}$ be a system biorthogonal to $\vec{x}_{(k_0)}$. Consequently, it is clear that $\left(\vec{x}^*_{(k_0)}; \vec{x}_{(k_0)}; K; \mathscr{K}_{\vec{x}_{(k_0)}}\right)$ is an atomic decomposition of $X_{(k_0)}$ if and only if $(\vec{x}^*; \vec{x}; K; \mathscr{K}_{\vec{x}})$ is an atomic decomposition of $X$.

Conversely, in the case of $k_0 < 0$, we have

$$X_{(k_0)} = \mathbb{C}^{|k_0|} \dot{+} X.$$

Let $\vec{x}^{*}_{(k_0)}$ be a system biorthogonal to $\vec{x}_{(k_0)}$. Similar considerations about frame are also valid in this case. Thus, it is valid

**Statement 3.1.** *Let $X$ be a $B$-space , $\vec{x}$ be a complete and minimal system in $X$, with the conjugate system $\vec{x}^{*}$. If the system $\vec{x}$ isn't an atomic decomposition of $X$, then $\left(\vec{x}^{*}_{(k_0)}; \vec{x}_{(k_0)}; K; \mathscr{K}_{\vec{x}_{(k_0)}}\right)$ isn't an atomic decomposition of $X_{(k_0)}$, with respect to the space of coefficients $\mathscr{K}_{\vec{x}_{(k_0)}}$ for $\forall k_0 \in \mathbb{Z}$.*

Under fulfilling all conditions of this statement we will say that the system $\vec{x}$ isn't an atomic decomposition of $X$ with respect to the space of coefficients $\mathscr{K}_{\vec{x}}$. In the future, anywhere by $\mathscr{K}(\vec{x})$ we will denote the space of coefficients of a system $\vec{x}$.

## 4. Basicity

Consider the system $E_{+}^{(k)}(\rho) \equiv \left\{\rho(t) e^{int}\right\}_{n \geq k}$. We will assume that the degenerate coefficient $\rho$ is given in the form of power function

$$\rho(t) = \left(e^{it} - 1\right)^{\alpha_0} \prod_{k=1}^{r} \left(e^{it} - e^{it_k}\right)^{\alpha_k},$$

where $\{t_k\}_1^r \subset (-\pi, \pi] \setminus \{0\}$ are different points and $\{\alpha_k\}_0^r \subset \mathbb{R}$. By $\mathrm{M}_p$ we denote the class of weights $\nu(t)$ satisfying the Muckenhoupt condition (see e.g. [10])

$$\sup_{I \subset [-\pi, \pi]} \left(\frac{1}{|I|} \int_I \nu(t)\, dt\right) \left(\frac{1}{|I|} \int_I [\nu(t)]^{-\frac{1}{p-1}}\, dt\right)^{p-1} < +\infty,$$

where *sup* is taken over all intervals $I \subset [-\pi, \pi]$ and $|I|$ is the Lebesgue measure $I$. It is easy to see that $|\rho|^{\frac{1}{p}} \in \mathrm{M}_p$ if and only if the following inequalities are true

$$-\frac{1}{p} < \alpha_k < 1 - \frac{1}{p}, \quad k = \overline{0, r}. \tag{4.1}$$

Consider the system $E_{+}^{(k)}\left(\bar{\rho}^{-1}\right) \equiv \left\{\bar{\rho}^{-1}(t) e^{int}\right\}_{n \geq k}$. From condition (4.1) it directly follows that $E_{+}^{(0)}\left(\bar{\rho}^{-1}\right)$ belongs to the space $L_q \equiv L_q(-\pi, \pi)$, where $q$ is number conjugate to $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. As $\forall g \in L_q$ determines the functional $l_g \in \left(L_p^{+}\right)^{*}$ by the expression

$$l_g(f) = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt, \quad \forall f \in L_p^{+},$$

then it is clear that the system $E_{+}^{(0)}\left(\bar{\rho}^{-1}\right)$ is biorthogonal to $E_{+}^{(0)}$. Take $\forall f \in L_p^{+}$ and let $F/_{\partial\omega} = f$. Assume $\Omega(z) \equiv (z-1)^{-\alpha_0} \prod_{k=1}^{r} \left(z - e^{it_k}\right)^{-\alpha_k}$. $\Omega(z)$ is an analytic function in $\omega$, and it is easy to see that $\Omega/_{\partial\omega} = \rho$. Moreover, from condition (4.1) it follows that $\Omega^{+} \in L_q$, and, as a result, $\Omega \in H_q^{+}$. Thus, it is clear that $\Omega \in H_1^{+}$. Then it follows from the classical results (see e.g. [22]) that

$$\int_{-\pi}^{\pi} f(t)\,\Omega^{+}\left(e^{it}\right) e^{int}dt \;=0\,,\;\; \forall n\in\mathbb{N}. \tag{4.2}$$

Now let us consider the system of exponents $\left\{\rho(t)\,e^{int}\right\}_{n\in\mathbb{Z}}$. It is clear that this system forms a basis for $L_p$ if and only if the classical system of exponents $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ forms a basis for $L_{p,\nu}$, where $\nu=|\rho|^{\frac{1}{p}}$. From results of [10;12;16;17;20] and from condition (4.1) it follows the basicity of system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ in $L_{p,\nu}$, and, as a result, the basicity of system $\left\{\rho(t)\,e^{int}\right\}_{n\in\mathbb{Z}}$ in $L_p$. $\left\{\bar{\rho}^{-1}(t)\,e^{int}\right\}_{n\in\mathbb{Z}}$ is a system biorthogonal to it. From (4.2) it directly follows that the biorthogonal coefficients of function $f$, which correspond to the negative values of index $n\in\mathbb{Z}$, are equal to zero. As a result, $f$ has the expansion $f(t)=\sum_{n=0}^{\infty} f_n\rho(t)\,e^{int}$ in $L_p^{+}$. It is clear that such an expansion is unique. So we get the validity of

**Theorem 4.1.** *Let the inequalities* (4.1) *be fulfilled. Then the system* $E_{+}^{(0)}(\rho)$ *forms a basis for* $L_p^{+}$, $1<p<+\infty$.

## 5. Defect case

We will consider the case when $|\rho|^{\frac{1}{p}}\notin \mathrm{M}_p$. Let the following inequalities hold

$$1-\frac{1}{p}\le\alpha_0<2-\frac{1}{p}\,,\;\; -\frac{1}{p}<\alpha_k<1-\frac{1}{p}\,,\;\; k=\overline{1,r}. \tag{5.1}$$

In the sequel, we will use the following result of [22].

**Theorem 5.1.** *If* $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ *then* $H_p^{+}(0)$ *is the conjugate of the space* $L_q/H_q^{+}$, *and* $L^q/H_q^{+}(0)$ *is the conjugate of* $H_p^{+}$.

Thus, every functional $l\in\left(H_p^{+}\right)^{*}$ can be determined by $g\in L_q$ through the expression

$$l(f)=l_g(f)=\int_{-\pi}^{\pi}\left(g\left(e^{it}\right)+F\left(e^{it}\right)\right)f\left(e^{it}\right)dt\;,\;\;\forall f\in H_p^{+},$$

where $F\in H_q^{+}(0)$ is an arbitrary function, i.e. the result doesn't depend on $F\in H_q^{+}(0)$, and vice versa. Consequently, zero functional is generated by zero function.

Now, let us assume that the functional $l_g\in\left(L_p^{+}\right)^{*}=\left(H_p^{+}\right)^{*}$ cancels the system $E_{+}^{(0)}(\rho)$ out, i.e.

$$\int_{-\pi}^{\pi}\left(g\left(e^{it}\right)+F\left(e^{it}\right)\right)\rho(t)\,e^{int}dt=0\;,\;\;\forall n\in\mathbb{Z}_{+}, \tag{5.2}$$

where $F\in H_q^{+}(0)$ is an arbitrary function. Take $\forall\beta\ge0$:$(\alpha_0-\beta)\in\left(-\frac{1}{p},1-\frac{1}{p}\right)$ and assume

$$\rho(t)\equiv\tilde{\rho}(t)\left(e^{it}-1\right)^{\beta}.$$

Let $\tilde{g}\left(e^{it}\right)=g\left(e^{it}\right)\left(e^{it}-1\right)^{\beta}$ and $\tilde{F}(z)=F(z)(z-1)^{\beta}$. It is clear that $\tilde{g}\in L_q$ and $\tilde{F}\in H_q^{+}(0)$, as, $\beta\ge0$ and $\tilde{F}(0)=0$. The relation (5.2) can be rewritten as follows

$$\int_{-\pi}^{\pi}\left(\tilde{g}\left(e^{it}\right)+\tilde{F}\left(e^{it}\right)\right)\tilde{\rho}\left(t\right)e^{int}dt=0\ ,\ \ \forall n\in\mathbb{Z}_{+}. \tag{5.3}$$

As $|\tilde{\rho}|^{\frac{1}{p}}\in \mathrm{M}_p$, it is clear that the system $E_{+}^{(0)}\left(\tilde{\rho}\right)$ forms a basis for $L_p^{+}$, and, moreover, is complete in $L_p^{+}$. Then from (5.3) it follows $\tilde{g}=0\Rightarrow g=0$. As a result, we obtain that the system $E_{+}^{(0)}\left(\rho\right)$ is complete in $L_p^{+}$. In a similar way we prove that, with the conditions

$$\alpha_k>-\frac{1}{p}\ ,\ \ k=\overline{0,r}, \tag{5.4}$$

fulfilled, the system $E_{+}^{(0)}\left(\rho\right)$ is complete in $L_p^{+}$.

Let us show that if the inequalities (5.1) hold, then the system $E_{+}^{(1)}\left(\rho\right)$ is minimal in $L_p^{+}$. Consider the system

$$\left\{\overline{\rho^{-1}\left(t\right)}\left(e^{int}-1\right)\right\}_{n\in\mathbb{N}}. \tag{5.5}$$

We have

$$\int_{-\pi}^{\pi}\rho\left(t\right)e^{int}\rho^{-1}\left(t\right)\ \left(e^{-imt}-1\right)\ dt=\int_{-\pi}^{\pi}e^{i(n-m)t}dt-\int_{-\pi}^{\pi}e^{int}dt=$$
$$=2\pi\delta_{nm}\ ,\ \ \forall n,m\in\mathbb{N}.$$

The relations

$$e^{int}-1\sim t\ ,\ \ t\to 0\ ,\ \ e^{it}-e^{it_k}\sim t-t_k\ ,\ \ t\to t_k,$$

imply

$$\left|\overline{\rho^{-1}\left(t\right)}\left(e^{int}-1\right)\right|^{q}\sim|t|^{q(1-\alpha_0)}\prod_{k=1}^{r}|t-t_k|^{-\alpha_k q}\ ,$$

on $(-\pi,\pi)$. Consequently

$$\left\|\overline{\rho^{-1}\left(t\right)}\left(e^{int}-1\right)\right\|_q^q=\int_{-\pi}^{\pi}\left|\overline{\rho^{-1}\left(t\right)}\left(e^{int}-1\right)\right|^{q}\ dt\sim$$
$$\sim\int_{-\pi}^{\pi}|t|^{q(1-\alpha_0)}\prod_{k=1}^{r}|t-t_k|^{-\alpha_k q}\ dt.$$

Hence, from (5.1) we obtain that the system (5.5) belongs to $L_q$. As a result, the previous relations yield the minimality of system $E_{+}^{(1)}\left(\rho\right)$ in $L_p^{+}$. Consider the completeness of this system in $L_p^{+}$. Let the functional $l\in\left(L_p^{+}\right)^{*}$ cancel the system $E_{+}^{(1)}\left(\rho\right)$ out

$$l\left(\rho\left(t\right)e^{int}\right)=0\ ,\ \ \forall n\in\mathbb{N}.$$

Let us assume that the functional $l$ is generated by $\vartheta\in L_q$ and denote it by $l_{\vartheta}$. Thus

$$\int_{-\pi}^{\pi}\left(\vartheta\left(t\right)+V^{+}\left(e^{it}\right)\right)e^{int}\rho\left(t\right)\ dt=0,\ \ \forall n\in\mathbb{N}, \tag{5.6}$$

where $V \in H_q^+(0)$ is an arbitrary function. Similarly to the previous case, we consider the function $\tilde{\rho}$ :

$$\tilde{\rho}(t) = \left(e^{it} - 1\right)^{-\beta} \rho(t),$$

where $\beta \geq 0 \,:\, (\alpha_0 - \beta) \in \left(-\frac{1}{p}, 1 - \frac{1}{p}\right)$. Assume $\tilde{\vartheta}(t) = \vartheta(t)\left(e^{it} - 1\right)^{\beta}$, $\tilde{V}(z) = V(z)(z-1)^{\beta}$. From (5.6) it follows

$$\int_{-\pi}^{\pi} \left(\tilde{\vartheta}(t) + \tilde{V}\left(e^{it}\right)\right) \tilde{\rho}(t)\, e^{int} dt = 0, \;\; \forall n \in \mathbb{N}.$$

It is clear that

$$\int_{-\pi}^{\pi} \left(\vartheta_1(t) + F\left(e^{it}\right)\right) \tilde{\rho}(t)\, e^{int} dt = 0, \;\; \forall n \in \mathbb{N}, \tag{5.7}$$

where $\vartheta_1(t) = \tilde{\vartheta}(t) + \tilde{V}\left(e^{it}\right)$ and $F \in H_q^+(0)$ is an arbitrary function, because $\tilde{\rho}(t)e^{int} \in L_p^+$. By Theorem 4.1, the system $E_+^{(0)}(\tilde{\rho})$ forms a basis for $H_p^+$. It is not difficult to see that the system biorthogonal to this basis is generated by the system $\left\{\tilde{\rho}^{-1}(t)\, e^{int}\right\}_{n\in\mathbb{Z}_+}$. Then from (5.7) we directly obtain that there exists a function $\vartheta_1^0$ of the form $\vartheta_1^0(t) = \tilde{\vartheta}^0(t) + F^0\left(e^{it}\right)$ such that $\vartheta_1^0(t) = c\tilde{\rho}^{-1}(t)$, where $F^0\left(e^{it}\right) \in H_q^+(0)$ is some function, and $\tilde{\vartheta}^0$ has the form $\tilde{\vartheta}^0(t) = \vartheta(t)\left(e^{it} - 1\right)^{\beta} + V^0\left(e^{it}\right)\left(e^{it} - 1\right)^{\beta}$ for some function $V^0 \in H_q^+(0)$. Consequently

$$\vartheta(t) + V^0\left(e^{it}\right) + F^0\left(e^{it}\right)\left(e^{it} - 1\right)^{-\beta} = c\rho^{-1}(t).$$

It is absolutely clear that if the function $\Phi(t) \equiv F^0\left(e^{it}\right)\left(e^{it} - 1\right)^{-\beta}$ belongs to $L_q$, then $c = 0$, as it is not difficult to see that $\rho^{-1} \notin L_q$. And if $\Phi(t) \notin L_q$, then $F^0\left(e^{it}\right) = c\tilde{\rho}^{-1}(t)$, and, as a result, $F^0(z) = c\prod_{k=1}^{r}\left(z - e^{it_k}\right)^{\alpha_k}(z-1)^{\tilde{\alpha}_0}$, where $\tilde{\alpha}_0 = \alpha_0 - \beta$. As $F^0(0) = 0$, we obtain $c = 0$. Thus, $\vartheta \in H_q^+(0)$, and this means that the system $E_+^{(1)}(\rho)$ is complete in $L_p^+$. So, under condition (5.1), the system $E_+^{(1)}(\rho)$ is complete and minimal in $L_p^+$. Let us show that in this case it doesn't form a basis for $L_p^+$. For this purpose, it suffices to prove that $E_+^{(1)}(\rho)$ is not uniformly minimal in $L_p^+$. Denote $E_n(t) \equiv \rho(t)\, e^{int}$ and $V_n(t) \equiv \overline{\rho^{-1}(t)}\left(e^{int} - 1\right)$. We have

$$\|E_n\|_{L_p^+} = \left(\int_{-\pi}^{\pi} |\rho(t)|^p\, dt\right)^{\frac{1}{p}} > 0\,, \;\; \forall n \in \mathbb{N}. \tag{5.8}$$

Consider the system of exponents $\{E_n(t)\}_{n\neq 0}$. From the results of [11] it follows that under condition (5.1) this system is complete and minimal in $L_p$ and $\{V_n(t)\}_{n\neq 0}$ is the system biorthogonal to it. Suppose that the system $E_+^{(1)}(\rho)$ is uniformly minimal in $L_p^+$, i.e. $\exists \delta > 0$ :

$$\left\| E_n(\cdot) - \sum_{k\neq n; k\in\mathbb{N}} a_k E_k(\cdot) \right\|_p \geq \delta \left\|E_n(\cdot)\right\|_p\,, \;\; \forall n \in \mathbb{N}.$$

Taking into account (5.8), we obtain

$$\left\| E_n(\cdot) - \sum_{k\neq n; k\in\mathbb{N}} a_k E_k(\cdot) \right\|_p \geq \delta_0 , \ \forall n \in \mathbb{N}, \tag{5.9}$$

where $\delta_0 > 0$ is some constant. Denote $G(z) = (z-1)^{\alpha_0} \prod_{k=1}^{r} \left(z - e^{it_k}\right)^{\alpha_k}$ and let $m = \min\left\{ n \in \mathbb{Z}_+ : \lim\limits_{z\to\infty} z^{-n} G(z) = 0 \right\}$. Assume $L^{(m)} = span\left[E_{-n} : n = \overline{m+1, \infty}\right]$, $L_{(m)} = span\left[E_{-n} : n = \overline{1, m}\right]$. It is easy to see that $L^{(m)} \subset_{-1} L_p^-$, and $L_{(m)}$ is the finite-dimensional subspace of $L_p$. It is absolutely clear that the direct decomposition

$$L_p = \overline{L^{(m)}} \dot{+} L_{(m)} \dot{+} L_p^+ , \tag{5.10}$$

holds. This decomposition follows directly from the completeness and minimality of system $\{E_n\}_{n\neq 0}$ in $L_p$ and from completeness and minimality of system $E_+^{(1)}(\rho)$ in $L_p^+$. Denote by $P_m$ the projection operator on $L_{(m)}$ generated by the expansion (5.10). $P_m$ is a continuous projector. The same is true about the projector $(I - P_m)$, where $I$ is an identity operator. Take arbitrary $f \in L_p$. Then

$$f = f^{(m)} + f_{(m)} + f^+ ,$$

where $f^{(m)} \in \overline{L^{(m)}}$, $f_{(m)} \in L_{(m)}$ and $f^+ \in L_p^+$ . It is absolutely clear that the inequality

$$\left\| f^{(m)} + f^+ \right\|_p \leq c_0 \left\| f \right\|_p , \tag{5.11}$$

holds, where $c_0 > 0$ is a constant independent of $f$ . Take $\forall n \in \mathbb{N}$ and fix it. Let $\varepsilon > 0$ be an arbitrary number. It is clear that $\exists f_\varepsilon^{(m)} \in L^{(m)}$ , $\exists f_{(m)}^\varepsilon \in L_{(m)}$, $f_\varepsilon^+ \in span\left[E_k : k \geq 1 \quad and \quad k \neq n\right]$, $\exists \lambda_\varepsilon \in \mathbb{C}$ :

$$\left\| f - \left( f_\varepsilon^{(m)} + f_{(m)}^\varepsilon + f_\varepsilon^+ \right) - \lambda_\varepsilon E_n \right\|_p < \varepsilon.$$

We have

$$V_n\left( f - f_\varepsilon^{(m)} - f_{(m)}^\varepsilon - f_\varepsilon^+ - \lambda_\varepsilon E_n \right) = V_n(f) - \lambda_\varepsilon .$$

From $V_n \in L_q = (L_p)^*$ it directly follows that $\lambda_\varepsilon \to V_n(f)$ as $\varepsilon \to 0$. Taking into account the inequality (5.11), we obtain

$$\varepsilon > \left\| f - f_\varepsilon^{(m)} - f_{(m)}^\varepsilon - f_\varepsilon^+ - \lambda_\varepsilon E_n \right\|_p \geq \left\| f_\varepsilon^{(m)} + f_{(m)}^\varepsilon + f_\varepsilon^+ + \lambda_\varepsilon E_n \right\|_p - \left\| f \right\|_p \geq$$

$$\geq \frac{1}{c_0} \left\| f_\varepsilon^{(m)} + f_\varepsilon^+ + \lambda_\varepsilon E_n \right\|_p - \left\| f \right\|_p . \tag{5.12}$$

In what follows, attention should be paid to the fact that $f_\varepsilon^{(m)} \in_{-1} L_p^-$ and $(f_\varepsilon^+ + \lambda_\varepsilon E_n) \in L_p^+$. As is known, the spaces $L_p^+$ and ${}_{-1}L_p^-$ are complementable in $L_p$ for $1 < p < +\infty$, and $L_p = L_p^+ \dot{+}_{-1} L_p^-$ (see e.g. [15]). Then $\exists c_1 > 0$ : $\left\| g^\pm \right\|_p \leq c_1 \left\| g \right\|_p$, where the function $g \in L_p$ has the representation $g = g^+ +$

$g^-$ , $g^+ \in L_p^+$, $g^- \in_{-1} L_p^-$. Taking into account this fact, from (5.12) we obtain $\varepsilon > \frac{1}{c_0 c_1} \|f_\varepsilon^+ + \lambda_\varepsilon E_n\|_p - \|f\|_p$. From (5.9) it directly follows

$$\left\|f_\varepsilon^+ + \lambda_\varepsilon E_n\right\| \geq |\lambda_\varepsilon| \, \delta_0.$$

Consequently

$$\varepsilon > \frac{\delta_0}{c_0 c_1} |\lambda_\varepsilon| - \|f\|_p \Rightarrow |\lambda_\varepsilon| < \frac{c_0 c_1}{\delta_0} \left(\|f\|_p + \varepsilon\right) = M \left(\|f\|_p + \varepsilon\right),$$

where $M > 0$ is a constant independent of $f$ and $\varepsilon$. Passing to the limit as $\varepsilon \to 0$, we have

$$|V_n(f)| \leq M \|f\|_p \,, \ \forall f \in L_p,$$

i.e.

$$\|V_n\|_q \leq M \,, \ \forall n \in \mathbb{N}. \tag{5.13}$$

So we obtained that the uniform minimality of system $E_+^{(1)}(\rho)$ in $L_p^+$ implies the uniform boundedness of functionals $\{V_n\}_{n\in\mathbb{N}}$ in $L_p$ defined by

$$V_n(f) = \int_{-\pi}^{\pi} f(t) \, V_n(t) \, dt.$$

We have

$$I_n \equiv \|V_n\|_q^q = \int_{-\pi}^{\pi} |\rho(t)|^{-q} \left|e^{int} - 1\right|^q dt = c \int_{-\pi}^{\pi} \frac{\left|\sin \frac{nt}{2}\right|^{\frac{q}{2}-\pi}}{|\rho(t)|^q} dt,$$

where (and further) $c$ is an absolute constant. Take $\varepsilon > 0$ such that the interval $[0, \varepsilon]$ doesn't contain the points $\{\lambda_k\}_1^r$. Then it is clear that $\exists c > 0$ :

$$\prod_{k=1}^{r} \left|e^{it} - e^{it_k}\right|^{-\alpha_k q} \geq c, \forall t \in [0, \varepsilon].$$

Thus

$$I_n \geq c \int_0^{\varepsilon} \frac{\left|\sin \frac{nt}{2}\right|^{\frac{q}{2}}}{\left|e^{it} - 1\right|^{\alpha_0 q}} dt \geq c \int_0^{\varepsilon} \frac{\left|\sin \frac{nt}{2}\right|^{\frac{q}{2}}}{t^{\alpha_0 q}} dt.$$

For sufficiently large values of $n$ we have $n\varepsilon > 2$. As a result

$$I_n \geq c n^{\alpha_0 q - 1} \int_0^1 \frac{|\sin \tau|^{\frac{q}{2}}}{\tau^{\alpha_0 q}} d\tau. \tag{5.14}$$

By assumption we have $0 \leq \alpha_0 q - 1 < 1$. If $\alpha_0 q > 1$, then from (5.14) it directly follows that $\lim\limits_{n\to\infty} I_n = +\infty$. And this contradicts the relation (5.13). Consequently, in this case the system $E_+^{(1)}(\rho)$ is not uniformly minimal in $L_p^+$, and, moreover, it doesn't form a basis for $L_p^+$.

Consider the case $\alpha_0 q = 1$. We have

$$I_n \geq c \int_0^{\frac{n\varepsilon}{2}} \frac{|\sin \tau|^{\frac{q}{2}}}{\tau} d\tau.$$

Consequently

$$\sup_n I_n \geq c \int_0^{+\infty} \frac{|\sin \tau|^{\frac{q}{2}}}{\tau} d\tau. \tag{5.15}$$

It is absolutely clear that

$$|\sin \tau| \geq \frac{\sqrt{2}}{2}\,, \ \forall \tau \in \left[k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}\right]\,, \ \forall k \in \mathbb{N}.$$

Taking into account this relation, from (5.15) we have

$$\sup_n I_n \geq c \sum_{k=1}^{\infty} \int_{\left(k+\frac{1}{4}\right)\pi}^{\left(k+\frac{3}{4}\right)\pi} \frac{d\tau}{\tau} \geq c \sum_{k=1}^{\infty} \frac{1}{k+\frac{1}{4}} = +\infty.$$

The latter contradicts (5.13). As a result, the system $E_+^{(1)}(\rho)$ is not uniformly minimal in $L_p^+$, and, moreover, it doesn't form a basis for $L_p^+$. It is clear that in this case the system $E_+^{(0)}(\rho)$ has a defect equal to 1. We can similarly prove that if the inequalities

$$k - \frac{1}{p} \leq \alpha_0 < k + \frac{1}{q}\,, \ -\frac{1}{p} < \alpha_k < \frac{1}{q}\,, \ k = \overline{1, r}, \tag{5.16}$$

are fulfilled, then the system $E_+^{(k)}(\rho)$ is complete and minimal in $L_p^+$, but it doesn't form a basis for $L_p^+$. Consequently, in this case the system $E_+^{(0)}(\rho)$ has a defect equal to $(k)$. As a result, we get the validity of

**Theorem 5.2.** *Let the inequalities* (5.16) *hold. Then the system* $E_+^{(0)}(\rho)$ *has a defect equal to* $(k)$ *in* $L_p^+$*. In addition, the system* $E_+^{(k)}(\rho)$ *is complete and minimal in* $L_p^+$*, but is not uniformly minimal in it, and, consequently, it doesn't form a basis for* $L_p^+$*.*

## 6. An atomic decomposition in Hardy classes

Let us show that if the Muchenhoupt condition (4.1) doesn't hold for $\alpha_0$, then the system $E_+^{(0)}(\rho)$ isn't an atomic decomposition of $L_p^+$. To do so, it suffices to prove that there exists a function from $L_p^+$ which cannot be expanded with respect to the system $E_+^{(0)}(\rho)$. Assume that an arbitrary function from $L_p^+$ can be expanded with respect to the system $E_+^{(0)}(\rho)$. By Theorem 5.2, if $\alpha_0$ doesn't satisfy the inequality (4.1), then the system $E_+^{(0)}(\rho)$ is complete, but not minimal in $L_p^+$. Then it is clear that this system has a nontrivial expansion of zero, i.e. $\exists n_0 \in \mathbb{Z}_+\,,\ c_{n_0} \neq 0$ and

$$0 = \sum_{n=0}^{\infty} c_n E_n. \tag{6.1}$$

Assume $k_0 = \left[\alpha_0 + \frac{1}{p}\right]$, where $[x]$ is the integer part of the number $x$. First we consider the case $k_0 = 1$. In this case, by Theorem 5.2, the system $E_+^{(1)}(\rho)$ is complete and minimal in $L_p^+$. Then it is absolutely clear that in expansion (6.1) $n_0 = 0$, i.e. $c_0 \neq 0$.

$$E_0 = \sum_{n=1}^{\infty} \left(-\frac{c_n}{c_0}\right) E_n.$$

As a result, every function from $L_p^+$ can be expanded with respect to the system $E_+^{(1)}(\rho)$. Then it is clear that the system $E_+^{(1)}(\rho)$ forms a basis for $L_p^+$, and this contradicts Theorem 5.2. Let $k_0 = 2$. Assume that every function from $L_p^+$ is decomposable in $E_+^{(1)}(\rho)$. By Theorem 5.2, the system $E_+^{(2)}(\rho)$ is complete and minimal in $L_p^+$. Then it has a nontrivial expansion of zero (6.1). Obviously, $|c_0| + |c_1| > 0$. Let $c_0 \neq 0$. Then it is clear that an arbitrary function from $L_p^+$ can be expanded with respect to the system $E_+^{(1)}(\rho)$. As this system is complete, but not minimal in $L_p^+$, then it has a nontrivial expansion of zero, i.e. $\exists m \in \mathbb{N}$, $b_m \neq 0$ and

$$0 = \sum_{n=0}^{\infty} b_n E_n.$$

The minimality of the system $E_+^{(2)}(\rho)$ implies $m = 1$. As a result, we obtain that every function from $L_p^+$ can be expanded with respect to the system $E_+^{(2)}(\rho)$. This directly implies the basicity of $E_+^{(2)}(\rho)$ in $L_p^+$ which contradicts Theorem 5.2. The other cases are proved similarly. That the system $E_+^0(\rho)$ doesn't form a frame for $L_p^+$ with respect the space of coefficients, follows directly from Statement 3.1. So the following theorem is true.

**Theorem 6.1.** *System $E_+^{(0)}(\rho)$ is an atomic decomposition of $L_p^+$ with respect to the space of coefficients $\mathscr{K}\left(E_+^{(0)}(\rho)\right)$ if and only if the degeneration coefficient $\rho$ satisfies the Muchenhoupt condition* (4.1).

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