Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Number 1, 2014, Pages 68–79

A NEW ITERATIVE PROJECTION METHOD FOR APPROXIMATING FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

IBRAHIM KARAHAN AND MURAT OZDEMIR

Abstract. In this paper, we introduce and study a new extragradient iterative process for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a variational inequality for an inverse strongly monotone mapping in a real Hilbert space. Also, we prove that under quite mild conditions the iterative sequence defined by our new extragradient method converges strongly to a solution of the fixed point problem for an infinite family of nonexpansive mappings and the classical variational inequality problem. In addition, utilizing this result, we provide some applications of the considered problem not just giving a pure extension of existing mathematical problems.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, C is a nonempty closed convex subset of H, and I is the idendity mapping on C. Below, we gather some basic definitions and results which are needed in the subsequent sections. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|\,,\ \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T. For a mapping $A : C \to H$, it is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in C;$$

(ii) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\,\|x - y\|\,,\ \forall x, y \in C;$$

(iii) α-inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\,\|Ax - Ay\|^2\,,\ \forall x, y \in C.$$

2000 *Mathematics Subject Classification.* 49J30, 47H09, 47J20.
Key words and phrases. Variational inequalities, fixed point problems, strong convergence.

Remark 1.1. It is obvious that any α-inverse strongly monotone mapping A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Remark 1.2. Every L-Lipschitzan mapping is $\frac{2}{L}$-inverse strongly monotone mapping.

For a mapping $A : C \to H$, the classical variational inequality problem $VI(C, A)$ is to find a $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall y \in C, \tag{1.1}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle . \tag{1.2}$$

The set of solutions of $VI(C, A)$ is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \ \langle Ax, y - x \rangle \geq 0, \forall y \in C\} .$$

In the context of the variational inequality problem it is easy to check that

$$x \in \Omega \Leftrightarrow x \in F(P_C(I - \lambda A)), \ \forall \lambda > 0.$$

Variational inequalities were initially studied by Stampacchia [7], [8]. Such a problem is connected with convex minimization problem, the complementarity problem, the problem of finding point $x \in C$ satisfying $0 \in Ax$ and etc. Fixed point problems are also closely related to the variational inequality problems. Based on this relationship, iterative methods for nonexpansive mappings have recently been applied to find the common solution of fixed point problems and variational inequality problems; see, for example [2, 5, 16, 18, 19] and the references therein. Below, we give some of them.

In 2005, Iiduka and Takahashi [4] proposed an iterative process as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) T P_C (I - \lambda_n A) x_n, \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where A is an α-inverse strongly monotone mapping, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \in (0, 2\alpha)$ satisfy some parameters controlling conditions. They showed that if $F(T) \cap \Omega$ is nonempty, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to some $z \in F(T) \cap \Omega$.

One year later, in 2006, by a narrow margin from the iterative process (1.3), Takahashi and Toyoda [13] introduced the following iterative process which is based on the Mann iteration [9]:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C (I - \lambda_n A) x_n, \ \forall n \geq 0, \end{cases} \tag{1.4}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $P_C : H \to C$ is a metric projection, $A : C \to H$ is an α-inverse strongly monotone mapping, and $T : C \to C$ is a nonexpansive mapping. They proved that if the set of fixed points of T is nonempty, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to some $z \in F(T) \cap \Omega$ where $z = \lim_{n\to\infty} P_{F(T)\cap\Omega} x_n$. In the same year, Yao et. al. [19], introduced following iterative scheme for a nonexpansive mapping S, and a monotone k-Lipschitzian continuous mapping A.

Under the suitable conditions, they proved the strong convergence of $\{x_n\}$ for a fixed $u \in H$ and a given $x_0 \in H$ arbitrary.

$$\begin{cases} x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C\left(x_n - \lambda_n y_n\right), \\ y_n = P_C\left(I - \lambda_n A\right) x_n,\ \forall n \geq 0. \end{cases} \tag{1.5}$$

Lastly, Khan [6] and Sahu [10], individually, introduced the following iterative process which Khan referred to as Picard-Mann hybrid iterative process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n, \\ y_n = \alpha_n x_n + \left(1 - \alpha_n\right) Tx_n,\ \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Picard-Mann hybrid iterative process is independent of all Picard, Mann and Ishikawa iterative processes. Khan [6] showed that the process (1.6) converges faster than all of Picard, Mann and Ishikawa iterative processes for contractions. Moreover, he proved a strong and a weak convergence theorems in Banach space for iterative process (1.6) with a T nonexpansive mapping under the suitable conditions.

In addition to all these studies, the existence of common elements of the set of common fixed points of an infinite family of nonlinear mappings and the set of solutions of the variational inequality problem has been also considered by many authors (see [14, 15, 20]). In such articles, authors usually use a mapping generated by nonexpansive mappings such as the mapping W_n defined, as in Shimoji and Takahashi [12], by

$$\begin{aligned} U_{n,n+1} &= I \\ U_{n,n} &= \mu_n T_n U_{n,n+1} + \left(1 - \mu_n\right) I \\ U_{n,n-1} &= \mu_{n-1} T_{n-1} U_{n,n} + \left(1 - \mu_{n-1}\right) I \\ &\vdots \\ U_{n,k+1} &= \mu_{k+1} T_{k+1} U_{n,k+2} + \left(1 - \mu_{k+1}\right) I \\ U_{n,k} &= \mu_k T_k U_{n,k+1} + \left(1 - \mu_k\right) I \\ &\vdots \\ U_{n,2} &= \mu_2 T_2 U_{n,3} + \left(1 - \mu_2\right) I \\ W_n &= U_{n,1} = \mu_1 T_1 U_{n,2} + \left(1 - \mu_1\right) I \end{aligned} \tag{1.7}$$

where C is a nonempty closed convex subset of a Hilbert space H, $\mu_1, \mu_2, \ldots$ are real numbers such that $0 \leq \mu_n \leq 1$, and $T_1, T_2, \ldots$ is an infinite family of self-mappings on C. W_n is called W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\mu_n, \mu_{n-1}, \ldots, \mu_1$. It is clear that nonexpansivity of each T_i, $i \geq 1$, ensures the nonexpansivity of W_n.

In this paper, motivated and inspired by the above processes and independently from all of them, we introduce the following iterative process for an infinite family of nonexpansive mappings $\{T_n\}$ which is based on Picard-Mann hybrid iterative process:

$$\begin{cases} x_0 \in C \\ x_{n+1} = W_n P_C\left(I - \lambda_n A\right) y_n \\ y_n = \left(1 - \alpha_n\right) x_n + \alpha_n W_n P_C\left(I - \lambda_n A\right) x_n,\ \forall n \geq 0, \end{cases} \tag{1.8}$$

where $A : C \to H$ is an α-inverse strongly monotone mapping, W_n is a mapping defined by (1.7), $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Also, we prove that the sequence $\{x_n\}$ defined by (1.8) converge strongly to a common element of the set of common fixed points of the infinite family $\{T_n\}$ and the set of solutions of the variational inequality (1.1) which is the optimality condition for the minimization problem (1.2).

2. Preliminaries

In this section, we collect some useful lemmas that will be used for our main result in the next section. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, and $x_n \to x$ for the strong convergence.

It is well known that for any $x \in H$, there exists a unique point $y_0 \in C$ such that

$$\|x - y_0\| = \inf \{\|x - y\| : y \in C\}.$$

We denote y_0 by $P_C x$, where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. It is also known that P_C has the following properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$, for all $x, y \in H$,
(ii) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$, for all $x \in H$, $y \in C$,
(iii) $\langle x - P_C x, y - P_C x\rangle \leq 0$, for all $x \in H$, $y \in C$.

It is known that a Hilbert space H satisfies the Opial condition that, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. [13] *Let C be a nonempty closed convex subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H. Suppose that, for all $z \in C$,*

$$\|x_{n+1} - z\| \leq \|x_n - z\|$$

for every $n = 0, 1, 2, \ldots$. Then, $\{P_C x_n\}$ converges strongly to some $u \in C$.

Lemma 2.2. [13] *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-inverse strongly monotone mapping of C into H. Then, the solution of $VI(C, A)$, Ω, is nonempty.*

For a set-valued mapping $S : H \to 2^H$, if the inequality

$$\langle f - g, u - v\rangle \geq 0$$

holds for all $u, v \in C, f \in Su, g \in Sv$, then S is called monotone mapping. A monotone mapping $S : H \to 2^H$ is maximal if the graph $G(S)$ of S is not properly contained in the graph of any other monotone mappings. It is known that a monotone mapping S is maximal if and only if, for $(u, f) \in H \times H$, $\langle u - v, f - w\rangle \geq 0$ for every $(v, w) \in G(S)$ implies $f \in Su$. Let A be an inverse strongly monotone mapping of C into H, let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\},$$

and define

$$Sv = \begin{cases} Av + N_C v & v \in C \\ \emptyset & v \notin C. \end{cases}$$

Then, S is maximal monotone and $0 \in Sv$ if and only if $v \in \Omega$.

Lemma 2.3. [3] *Let C be a nonempty closed convex subset of a real Hilbert space H, and T be a nonexpansive self-mapping on C. If $F(T) \neq \emptyset$, then $I - T$ is demiclosed; that is whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here, I is the identity operator of H.*

Lemma 2.4. [11] *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, \ldots$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq c, \quad \limsup_{n\to\infty} \|y_n\| \leq c \text{ and } \lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = c,$$

for some $c > 0$. Then,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.5. [17] *Assume that $\{x_n\}$ is a sequence of nonnegative real numbers satisfying the conditions*

$$x_{n+1} \leq (1-\alpha_n) x_n + \alpha_n \beta_n, \ \forall n \geq 0$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently* $\prod_{n=0}^{\infty} (1-\alpha_n) = 0$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, *or* $\sum_n \alpha_n \beta_n < \infty$.

Then, $\lim_{n\to\infty} x_n = 0$.

Concerning the mapping W_n defined by (1.7), we have the following lemmas in a real Hilbert space which can be obtained from Shimoji and Takahashi [12].

Lemma 2.6. [12] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\{T_n\}$ be an infinite family of nonexpansive mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq 1$ for all $n \in \mathbb{N}$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

By using the Lemma 2.6, one can define the mapping W on C as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \ \forall x \in H.$$

Such a W is called the W-mapping generated by $T_1, T_2, T_3, \ldots$ and $\mu_1, \mu_2, \mu_3, \ldots$. Throughout this paper, we assume that $0 < \mu_n \leq b < 1$ for $n \geq 0$.

Lemma 2.7. [12] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\{T_n\}$ be an infinite family of nonexpansive mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\mu_1, \mu_2, \ldots$ be real numbers such that $0 \leq \mu_n \leq 1$ for $n \geq 0$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

3. Main result

Now, we are in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $A : C \to H$ be an α-inverse strongly monotone mapping and let $\{T_n\}$ be an infinite family of nonexpansive self-mappings on C such that $\mathcal{F} := \bigcap_{n=0}^{\infty} F(T_n) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.8), where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,2\alpha)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then, the sequence $\{x_n\}$ converges strongly to a point $z \in \mathcal{F}$ where z is the unique solution of the variational inequality (1.1).*

Proof. We devide our proof into five steps.

Step 1. First, we show that $\{x_n\}$ is a bounded sequence. Let

$$t_n = P_C (I - \lambda_n A) x_n$$

and $z \in \mathcal{F}$. Then, we have

$$\begin{aligned} \|t_n - z\|^2 &= \|P_C (I - \lambda_n A) x_n - z\|^2 \\ &\leq \|(I - \lambda_n A) x_n - (I - \lambda_n A) z\|^2 \\ &= \|x_n - z - \lambda_n (Ax_n - Az)\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, Ax_n - Az\rangle + \lambda_n^2 \|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 \end{aligned} \tag{3.1}$$

and from (3.1) we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|W_n P_C (I - \lambda_n A) y_n - z\|^2 \\ &= \|W_n P_C (I - \lambda_n A) y_n - W_n P_C (I - \lambda_n A) z\|^2 \\ &\leq \|y_n - z\|^2 \\ &= \|(1 - \alpha_n)(x_n - z) + \alpha_n (W_n t_n - z)\|^2 \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \|W_n t_n - z\|^2 \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \|t_n - z\|^2 \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 \\ &\quad + \alpha_n \left[\|x_n - z\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax_n - Az\|^2\right] \\ &= \|x_n - z\|^2 + \alpha_n \lambda_n (\lambda_n - 2\alpha) \|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 + da (b - 2\alpha) \|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 . \end{aligned} \tag{3.2}$$

Therefore, the limit $\lim_{n\to\infty} \|x_n - z\|$ exists and $Ax_n - Az \to 0$. Hence $\{x_n\}$ is bounded and so are $\{t_n\}$ and $\{W_n t_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Before that, we shall show that $\lim_{n\to\infty} \|W_n t_n - x_n\| = 0$. From Step 1, we know that $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in \mathcal{F}$. Let $\lim_{n\to\infty} \|x_n - z\| = c$. From (3.2), since

$$\|x_{n+1} - z\| \leq \|y_n - z\| \leq \|x_n - z\| ,$$

we get

$$\lim_{n\to\infty} \|y_n - z\| = c. \tag{3.3}$$

On the other hand, since

$$\|W_n t_n - z\| \leq \|t_n - z\| \leq \|x_n - z\|,$$

we have

$$\limsup_{n\to\infty} \|W_n t_n - z\| \leq c. \tag{3.4}$$

Also, we know that

$$\limsup_{n\to\infty} \|x_n - z\| \leq c \tag{3.5}$$

and

$$\lim_{n\to\infty} \|y_n - z\| = \lim_{n\to\infty} \|(1-\alpha_n)(x_n - z) + \alpha_n (W_n t_n - z)\| = c. \tag{3.6}$$

Hence, from (3.4), (3.5), (3.6), and Lemma 2.4 , we get that

$$\lim_{n\to\infty} \|x_n - W_n t_n\| = 0. \tag{3.7}$$

We have also

$$\|x_n - y_n\| = \alpha_n \|W_n t_n - x_n\|.$$

So, from (3.7) we obtain that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.8}$$

Since A is Lipschitz continuous, we have $Ax_n - Ay_n \to 0$.

Step 3. Now, we show that $\lim_{n\to\infty} \|Wx_n - x_n\| = 0$. Using the properties of the metric projection, since

$$\begin{aligned}
\|t_n - z\|^2 &= \|P_C (I - \lambda_n A) x_n - P_C (I - \lambda_n A) z\|^2 \\
&\leq \langle t_n - z, (I - \lambda_n A) x_n - (I - \lambda_n A) z \rangle \\
&= \frac{1}{2} \Big[\|t_n - z\|^2 + \|(I - \lambda_n A) x_n - (I - \lambda_n A) z\|^2 \\
&\quad - \|t_n - z - [(I - \lambda_n A) x_n - (I - \lambda_n A) z]\|^2 \Big] \\
&\leq \frac{1}{2} \Big[\|t_n - z\|^2 + \|x_n - z\|^2 - \|(t_n - x_n) - \lambda_n (Ax_n - Az)\|^2 \Big] \\
&= \frac{1}{2} \Big[\|t_n - z\|^2 + \|x_n - z\|^2 - \|t_n - x_n\|^2 \\
&\quad -2\lambda_n \langle t_n - x_n, Ax_n - Az \rangle - \lambda_n^2 \|Ax_n - Az\|^2 \Big],
\end{aligned}$$

it follows that

$$\begin{aligned}
\|t_n - z\|^2 &\leq \|x_n - z\|^2 - \|t_n - x_n\|^2 \\
&\quad +2\lambda_n \langle t_n - x_n, Ax_n - Az \rangle - \lambda_n^2 \|Ax_n - Az\|^2.
\end{aligned} \tag{3.9}$$

So, by using the inequality (3.9) and (3.2), we get

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n)\|x_n-z\|^2+\alpha_n\|t_n-z\|^2\\ &\le \|x_n-z\|^2-\alpha_n\|t_n-x_n\|^2\\ &\quad +2\lambda_n\alpha_n\langle t_n-x_n, Ax_n-Az\rangle-\lambda_n^2\alpha_n\|Ax_n-Az\|^2\\ &\le \|x_n-z\|^2-d\|t_n-x_n\|^2\\ &\quad +2\lambda_n\alpha_n\langle t_n-x_n, Ax_n-Az\rangle-\lambda_n^2\alpha_n\|Ax_n-Az\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-z\|=\lim_{n\to\infty}\|x_n-z\|$ and $Ax_n-Az\to 0$, we obtain

$$\lim_{n\to\infty}\|x_n-t_n\|=0. \tag{3.10}$$

On the other hand, we have

$$\begin{aligned}\|W_nx_n-x_n\| &\le \|W_nx_n-W_nt_n\|+\|W_nt_n-x_n\|\\ &\le \|x_n-t_n\|+\|W_nt_n-x_n\|.\end{aligned}$$

So, it follows from (3.7) and (3.10) that

$$\lim_{n\to\infty}\|W_nx_n-x_n\|=0. \tag{3.11}$$

Hence, from (3.11) and by the same argument as in the [1, Remark 2.2], it follows that

$$\|Wx_n-x_n\|\le\|Wx_n-W_nx_n\|+\|W_nx_n-x_n\|\to 0, \tag{3.12}$$

as $n\to\infty$.

Step 4. Next, we show that

$$\limsup_{n\to\infty}\left[\langle W_nt_n-z, x_n-z\rangle+\|W_nt_n-z\|^2\right]\le 0,$$

where $z\in\mathcal{F}$. But first, we need to show that the variational inequality (1.1) has unique solution. Indeed, suppose both $p\in C$ and $q\in C$ are solutions to (1.1), then

$$\langle Ap, p-q\rangle\le 0 \tag{3.13}$$

and

$$\langle Aq, q-p\rangle\le 0. \tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\langle Aq-Ap, q-p\rangle\le 0. \tag{3.15}$$

Since the mapping A is an inverse strongly monotone mapping, (3.15) implies $p=q$. So, the uniqueness of the solution of the variational inequality (1.1) is proved. Next, we need to show that $\{x_n\}$ converges weakly to an element of $\mathcal{F}$. Since $\{x_n\}$ and $\{W_nt_n\}$ are bounded sequences, there exist subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{W_nt_{n_i}\}$ of $\{W_nt_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\left[\langle W_nt_n-z, x_n-z\rangle+\|W_nt_n-z\|^2\right]\\ =\ &\limsup_{i\to\infty}\left[\langle W_nt_{n_i}-z, x_{n_i}-z\rangle+\|W_nt_{n_i}-z\|^2\right].\end{aligned} \tag{3.16}$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup p$. From (3.7), we have $W_n t_{n_i} \rightharpoonup p$. Hence, (3.16) reduces to

$$\limsup_{n\to\infty} \left[\langle W_n t_n - z, x_n - z\rangle + \|W_n t_n - z\|^2\right] = 2\|p - z\|^2$$

Now, it is sufficient to show that p belongs to $\mathcal{F}$, i.e., $p = z$. First, we show that $p \in \Omega$. Let

$$Sv = \begin{cases} Av + N_C v & , \ v \in C, \\ \emptyset & , \ v \notin C. \end{cases}$$

Then, S is maximal monotone mapping. Let $(v, w) \in G(S)$. Since $w - Av \in N_C v$ and $t_n \in C$, we get

$$\langle v - t_n, w - Av\rangle \geq 0. \tag{3.17}$$

On the other hand, from the definiton of t_n, we have that

$$\langle x_n - \lambda_n A x_n - t_n, t_n - v\rangle \geq 0$$

and hence,

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ax_n \right\rangle \geq 0.$$

Therefore, using (3.17), we get

$$\begin{aligned} \langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i}\right\rangle \\ &= \left\langle v - t_{n_i}, Av - Ax_{n_i} - \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ax_{n_i}\rangle \\ &\quad - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ax_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle. \end{aligned}$$

Hence, for $i \to \infty$, we have

$$\langle v - p, w\rangle \geq 0.$$

Since S is maximal monotone, we have $p \in S^{-1}0$ and hence $p \in \Omega$. Next, we show that $p \in F(W)$. From (3.12), Lemma 2.3 and by using $x_{n_i} \rightharpoonup p$, we have that $p \in F(W)$. So, from Lemma 2.7, we get $p \in \mathcal{F}$.

Also, Opial's condition guarantee that the weakly subsequential limit of $\{x_n\}$ is unique. Hence, this implies that $x_n \rightharpoonup p \in \mathcal{F}$. From the uniqueness of the solution of the variational inequality, we obtain $p = z \in \mathcal{F}$. So, the desired conclusion

$$\limsup_{n\to\infty} \left[\langle W_n t_n - z, x_n - z\rangle + \|W_n t_n - z\|^2\right] \leq 0$$

is obtained.

Furthermore, $p = \lim_{n\to\infty} P_{\mathcal{F}} x_n$. Indeed, since $p \in \mathcal{F}$, we have

$$\langle p - P_{\mathcal{F}} x_n, P_{\mathcal{F}} x_n - x_n\rangle \geq 0.$$

By Lemma 2.1, $\{P_{\mathcal{F}} x_n\}$ converges strongly to $u_0 \in \mathcal{F}$. Then, we get

$$\langle p - u_0, u_0 - p\rangle \geq 0,$$

and hence $p = u_0$.

Step 5. Let $z \in \mathcal{F}$. Then, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|W_n P_C (I - \lambda_n A) y_n - z\|^2 \\
&= \|W_n P_C (I - \lambda_n A) y_n - W_n P_C (I - \lambda_n A) z\|^2 \\
&\leq \|y_n - z\|^2 = \langle y_n - z, y_n - z\rangle \\
&= \langle (1 - \alpha_n)(x_n - z) + \alpha_n (W_n t_n - z), y_n - z\rangle \\
&= (1 - \alpha_n) \langle x_n - z, y_n - z\rangle + \alpha_n \langle W_n t_n - z, y_n - z\rangle \\
&\leq (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \langle W_n t_n - z, y_n - z\rangle \\
&= (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n^2 \langle W_n t_n - z, x_n - z\rangle \\
&\quad + \alpha_n (1 - \alpha_n) \langle W_n t_n - z, W_n t_n - z\rangle \\
&= (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \beta_n
\end{aligned}$$

where $\beta_n = \alpha_n \langle W_n t_n - z, x_n - z\rangle + (1 - \alpha_n) \|W_n t_n - z\|^2$. Thus an application of Lemma 2.5 combined with Step 4 yields that the sequence $\{x_n\}$ defined by (1.8) converges strongly to the unique element $z \in \mathcal{F}$. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $A : C \to H$ be an α-inverse strongly monotone mapping and let T be a nonexpansive self-mappings on C such that $F(T) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 = x \in C \\ x_{n+1} = T P_C (I - \lambda_n A) y_n \\ y_n = (1 - \alpha_n) x_n + \alpha_n T P_C (I - \lambda_n A) x_n, \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges strongly to a point $z \in F(T) \cap \Omega$ where z is the unique solution of the variational inequality (1.1).

4. Applications

In the first section, we state that the convex minimization problem is one of the application area of the variational inequality problems and the fixed point problems. One of the relationships between a convex minimization problem and a variational inequality problem is as follows: Let f be a convex differentiable function on a nonempty closed convex subset C of a real Hilbert space H and $\mathrm{Argmin}_{x \in C} f(x)$ be the set of minimizers of f relative to the set C. Then, it is known that element $x^* \in C$ is a minimizer of $f(x)$ if and only if x^* satisfies the variational inequality (1.1). On the other hand, iterative processes are often used to minimize a convex differentiable function. Also, it is stated in Remark 1.2 that every L-Lipschitzian mapping is $2/L$-inverse strongly monotone mapping. Therefore, we can give the following strong convergence theorem.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a convex differentiable function on an open set D containing the set C and let $\{T_n\}$ be an infinite family of nonexpansive self mappings on C such that $\mathcal{G} = \bigcap_{n=0}^{\infty} F(T_n) \cap \mathrm{Argmin}_{x \in C} f(x) \neq \emptyset$. Suppose that the gradient vector* *of f, ∇f, is a L-Lipschitz continuous operator on D. For an arbitrarily initial value $x_0 \in C$, let $\{x_n\}$ be a sequence in C defined by*

$$\begin{cases} x_{n+1} = W_n P_C \left(I - \lambda_n \nabla f\right) y_n \\ y_n = \left(1 - \alpha_n\right) x_n + \alpha_n W_n P_C \left(I - \lambda_n \nabla f\right) x_n, \forall n \geq 0, \end{cases}$$

where W_n is a mapping defined by (1.7), $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 4/L)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to an element of $\mathcal{G}$.

Proof. Considering the Remark 1.2, as in the proof of Theorem 3.1, if we take $A = \nabla f$, then we obtain the desired conclusion. □

Next, we give another theorem for a pair of nonexpansive mapping and strictly pseudocontractive mapping. A mapping $S : C \to C$ is called k-strictly pseudocontractive mapping if there exists k with $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k \left\|(I - S)\,x - (I - S)\,y\right\|^2$$

for all $x, y \in C$. Let $A = I - S$. Then, it is known that the mapping A is inverse strongly monotone mapping with $(1 - k)/2$, i.e.,

$$\langle Ax - Ay, x - y\rangle \geq \frac{1 - k}{2} \|Ax - Ay\|^2 .$$

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be an infinite family of nonexpansive self mappings on C and $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathcal{H} = \bigcap_{n=0}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. For an arbitrarily initial value $x_0 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_{n+1} = W_n \left(\left(I - \lambda_n\right) y_n + \lambda_n S y_n\right) \\ y_n = \left(1 - \alpha_n\right) x_n + \alpha_n W_n \left(\left(I - \lambda_n\right) x_n + \lambda_n S x_n\right), \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1 - k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to a point $p \in \mathcal{H}$.

Proof. Let $A = I - S$. Then, we know that A is inverse strongly monotone mapping. Also, it is clear that $F(S) = VI(C, A)$. Since, A is a mapping from C into itself, we get

$$\left(I - \lambda_n\right) x_n + \lambda_n S x_n = x_n - \lambda_n \left(I - S\right) x_n = P_C \left(I - \lambda_n A\right) x_n.$$

So, from Theorem 3.1, we obtain the desired conclusion. □

References

[1] L.C. Ceng and J.C. Yao, "Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings", *Appl. Math. Comput.,* **198** (2008), 729–741.

[2] T. Chamnarnpan and P Kumam, "A new iterative method for a common solution of fixed points for pseudo-contractive mappings and variational inequalities", *Fixed Point Theory Appl.*, 2012, 2012:67,.doi:10.1186/1687-1812-2012-6

[3] K. Goebel, W. A. Kirk, "Topics on Metric Fixed-Point Theory", *Cambridge University Press*, Cambridge, England, 1990.

[4] H. Iiduka and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings", *Nonlinear Analysis,* **61** (2005), 341 – 350.

[5] H. Iiduka and W. Takahashi, "Weak convergence of projection algorithm for variational inequalities in Banach spaces", *J. of Math. Analysis and Applications*, **339** (2008), 668–679.
[6] S. H. Khan, "A Piccard-Mann hybrid iterative process", *Fixed Point Theory and Appl.*, doi:10.1186/1687-1812-2013-69
[7] D. Kinderlehrer and G. Stampaccia, "An Iteration to Variational Inequalities and Their Applications", *Academic Press*, New York, NY, USA, 1990
[8] J.-L. Lions and G. Stampacchia," Variational inequalities", *Communications on Pure and Applied Mathematics*, **20** (1967), 493–519.
[9] W. R. Mann, "Mean value methods in iteration", Proceedings of the American Mathematical Society, **4** (1953), 506-510.
[10] D. R. Sahu, "Applications of S iteration process to constrained minimization problems and split feasibility problems", *Fixed Point Theory*, **12** (2011), 187–204.
[11] J. Schu, "Weak and strong convergence to fixed points of Asymptotically nonexpansive mappings", *Bull. of the Australian Math. Soc.*, **43** (1991), 153–159.
[12] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications", *Taiwanese J. Math.,* **5** (2001), 387–404.
[13] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings", *J. Optim. Theory Appl.* **118** (2003), 417–428.
[14] S. Wang, "A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces", Applied Mathematics Letter, **24** (2011), 901–907.
[15] R. Wangkeeree and U. Kamraksa, "A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces", *Fixed Point Theory and Appl.,* (2009) Article ID 369215, doi:10.1155/2009/369215
[16] N. C. Wong, D. R. Sahu, and J. C. Yao, "Solving variational inequalities involving nonexpansive type mappings", *Nonlinear Analysis, Theory, Methods and Applications*, **69** (2008), 4732–4753.
[17] H. K. Xu and T. H. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optim. Theory Appl.*, **119** (2003), 185–201.
[18] Y. Yao, Y-C. Liou and N. Shahzad, "Construction of iterative methods for variational inequality and fixed point problems", *Numer. Func. Ana. Optim.*, **33** (2012), 1250–1267.
[19] Y. Yao, Y-C. Liou and J-C. Yao, "An extragradient method for fixed point problems and variational inequality problems", *J. of Inequalities and Applications*, doi:10.1155/2007/38752
[20] Y. Yao, Y. C. Liou and J. C. Yao, "An iterative algorithm for approximating convex minimization problem", *Appl. Math. Comput.,* **188** (2007), 648–656.

Ibrahim Karahan
Department of Mathematics, Erzurum Technical University, Erzurum 25240, Turkey
E-mail address: `ibrahimkarahan@erzurum.edu.tr`

Murat Ozdemir
Department of Mathematics, Ataturk University, Erzurum 25240, Turkey
E-mail address: `mozdemir@atauni.edu.tr`

Received: March 17, 2014; Accepted: May 19, 2014