Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Number 1, 2014, Pages 93–103

WEIGHTED INEQUALITIES FOR DUNKL FRACTIONAL MAXIMAL FUNCTION AND DUNKL FRACTIONAL INTEGRALS

YAGUB Y. MAMMADOV

Abstract. We consider the generalized shift operator, associated with the Dunkl operator $\Lambda_\alpha(f)(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right)$. In this paper, the boundedness of the Dunkl maximal operator, the Dunkl fractional maximal operator and Dunkl fractional integrals in weighted $L_{p,w,\alpha}(\mathbb{R})$ spaces is established.

1. Preliminaries

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) \tag{1.1}$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem:

$$\Lambda_\alpha(f)(x) = \lambda f(x),\ f(0) = 1,\ x \in \mathbb{R}$$

has a unique analytic solution $E_\alpha(\lambda x)$ called Dunkl kernel [4, 16, 19] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x),\ x \in \mathbb{R},$$

where j_α is the normalized Bessel function of the first kind and order α, which is defined as (see [20])

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)},\ z \in \mathbb{C}.$$

We can write for $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ (see Dunkl [3])

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}\,(1-t)\,e^{i\lambda x t}\,dt.$$

2010 *Mathematics Subject Classification.* Primary 42B25, 42B35; Secondary 43A62, 44A20, 47G10.

Key words and phrases. Dunkl operator, Dunkl maximal function, Dunkl fractional maximal function, Dunkl fractional integral.

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx.$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(\mathbb{R}, d\mu_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} := \|f\|_{L_{p,\alpha}} = \left(\int_{\mathbb{R}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty \text{ if } p \in [1,\infty),$$

and

$$\|f\|_{\infty,\alpha} := \|f\|_{L_\infty} = ess \sup_{x\in\mathbb{R}} |f(x)| \text{ if } p = \infty.$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,\alpha}} := \sup_{r>0} r \left(\mu_\alpha \{x \in \mathbb{R} \ : \ |f(x)| > r\}\right)^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \text{ and } \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha}, \text{ for all } f \in L_{p,\alpha}(\mathbb{R}).$$

The Dunkl transform $\mathcal{F}_\alpha$ of a function $f \in L_{1,\alpha}(\mathbb{R})$, is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x)\, f(x) d\mu_\alpha(x), \ \lambda \in \mathbb{R}.$$

Here the integral makes sense since $|E_\alpha(ix)| \le 1$ for every $x \in \mathbb{R}$ (see [15], p. 295).

For all $x,\ y,\ z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x,y,z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x,\ y \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{otherwise} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) := \begin{cases} d_\alpha \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-1/2}}{|xyz|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2}))$ and $A_{x,y} = [\||x|-|y|\|, |x|+|y|]$.

Properties 1.1. *(see Rösler [15]) The signed kernel W_α is even and satisfies the following properties*

$$W_\alpha(x,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y),$$

$$W_\alpha(x,y,z) = W_\alpha(-z,y,-x) = W_\alpha(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|\, d\mu_\alpha(z) \le 4.$$

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} := \begin{cases} W_\alpha(x,y,z)\,d\mu_\alpha(z) & \text{if } x,\ y \in \mathbb{R}\setminus\{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases}$$

Definition 1.1. For $x,\ y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put

$$\tau_x f(y) := \int_{\mathbb{R}} f(z)\,d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form (see [15])

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left((x,y)_\theta\right)\, h_1(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta$$

$$+c_\alpha \int_0^\pi f_o\left((x,y)_\theta\right)\, h_2(x,y,\theta)(\sin\theta)^{2\alpha}\,d\theta,$$

where $(x,y)_\theta = \sqrt{x^2+y^2-2|xy|\cos\theta}$, $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with

$$c_\alpha := \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$

$$h_1(x,y,\theta) = 1 - \operatorname{sgn}(xy)\cos\theta$$

and

$$h_2(x,y,\theta) = \begin{cases} \frac{(x+y)\,[1-\operatorname{sgn}(xy)\cos\theta]}{(x,y)_\theta}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0. \end{cases}$$

Proposition 1.1. *(see Soltani* [17]*)*

(i) If f is an even positive continuous function, then $\tau_x f$ is positive.

(ii) For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}. \tag{1.2}$$

(iii) For all $x,\ \lambda \in \mathbb{R}$ and $f \in L_{1,\alpha}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha\left(\tau_x f\right)(\lambda) = E_\alpha(i\lambda x)\,\mathcal{F}_\alpha f(\lambda).$$

Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_\alpha$ of f and g by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y)\,g(y)\,d\mu_\alpha(y),\ x \in \mathbb{R}.$$

The generalized convolution $*_\alpha$ is associative and commutative [15]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

2. Weighted inequalities for Dunkl maximal functions

Let $B(x,r) = \{y \in \mathbb{R} : |y| \in]\max\{0, |x|-r\}, |x|+r[\,\}$ and $r>0$. Then $B_r = B(0,r) =]-r, r[$ and $\mu_\alpha(B_r) = b_\alpha\, r^{2\alpha+2}$, where $b_\alpha = \left(2^{\alpha+1}\,(\alpha+1)\,\Gamma(\alpha+1)\right)^{-1}$.

An almost everywhere positive and locally integrable function $\omega:\ \mathbb{R}\to\mathbb{R}$ will be called a weight. Denote by $L_{p,\omega,\alpha}(\mathbb{R})$ the set of measurable functions $f(x)$, $x\in\mathbb{R}$, with finite norm

$$\|f\|_{L_{p,\omega,\alpha}} := \left(\int_{\mathbb{R}} |f(x)|^p w(x)\, d\mu_\alpha(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

Definition 2.1. The weight function ω belongs to the class $A_{p,\alpha}(\mathbb{R})$ for $1<p<\infty$, if

$$\sup_{x,\,r}\left(\mu_\alpha(B(x,r))^{-1}\int\limits_{B(x,r)} \omega(y)d\mu_\alpha(y)\right) \times\left(\mu_\alpha(B(x,r))^{-1}\int\limits_{B(x,r)} \omega^{-\frac{1}{p-1}}(y)d\mu_\alpha(y)\right)^{p-1} < \infty$$

and ω belongs to $A_{1,\alpha}(\mathbb{R})$, if there exists a positive constant C such that for any $x\in\mathbb{R}$ and $r>0$

$$\mu_\alpha(B(x,r))^{-1}\int\limits_{B(x,r)} w(y)d\mu_\alpha(y) \le C\ \operatorname*{ess\,sup}_{y\in B(x,r)} \omega(y).$$

The properties of the class $A_{p,\alpha}(\mathbb{R})$ are analogous to those of the Muckenhoupt classes. In particular, if $w\in A_{p,\alpha}(\mathbb{R})$, then $w\in A_{p-\varepsilon,\alpha}(\mathbb{R})$ for a certain sufficiently small $\varepsilon>0$ and $w\in A_{p_1,\alpha}(\mathbb{R})$ for any $p_1>p$.

Note that, $|x|^\beta \in A_{p,\alpha}(\mathbb{R})$, $1<p<\infty$, if and only if $-(2\alpha+2) < \alpha < (2\alpha+2)(p-1)$ and $|x|^\beta \in A_{1,\alpha}(\mathbb{R})$, if and only if $-2\alpha-2<\beta\le 0$.

We define the Dunkl type maximal function by

$$Mf(x) := \sup_{r>0}\big(\mu_\alpha(B_r)\big)^{-1}\int_{B_r} \tau_x|f|(y)\, d\mu_\alpha(y). \tag{2.1}$$

Note that M is the Hardy-Littlewood maximal operator associated with the Dunkl operator (see [1, 7, 12, 18]).

The following theorem was proved in [12].

Theorem 2.1. *1) If $f\in L_{1,\alpha}(\mathbb{R})$, then for every $\beta>0$*

$$\mu_\alpha\left\{x\in\mathbb{R} : M_\alpha f(x) > \beta\right\} \le \frac{C_1}{\beta}\int_{\mathbb{R}} |f(x)|\, d\mu_\alpha(x),$$

where $C_1>0$ is independent of f.

2) If $f\in L_{p,\alpha}(\mathbb{R})$, $\quad 1<p\le\infty$, then $Mf\in L_{p,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{p,\alpha} \le C_2\|f\|_{p,\alpha},$$

where $C_2>0$ is independent of f.

The following theorem is valid.

Theorem 2.2. *1) If $f \in L_{1,w,\alpha}$ and $w \in A_{1,\alpha}$, then $M_\alpha f \in WL_{1,w,\alpha}$ and*

$$\|M_\alpha f\|_{WL_{1,w,\alpha}} \le C_{1,\omega,\alpha}\|f\|_{L_{1,w,\alpha}},$$

where $C_{1,\omega,\alpha} > 0$ is independent of f.

2) If $f \in L_{p,w,\alpha}(\mathbb{R})$, $\quad 1 < p < \infty$ and $w \in A_{p,\alpha}$, then $M_\alpha f \in L_{p,w,\alpha}(\mathbb{R})$ and

$$\|M_\alpha f\|_{L_{p,w,\alpha}} \le C_{p,w,\alpha}\|f\|_{L_{p,w,\alpha}},$$

where $C_{p,w,\alpha} > 0$ is independent of f.

Proof. We need to introduce other maximal function defined on a space of homogeneous type (X, d, μ). By this we mean a topological space $X = \mathbb{R}$ equipped with a continuous pseudometric d and a positive measure μ satisfying

$$\mu(E(x, 2r)) \le C\mu(E(x, r)) \tag{2.2}$$

with a constant C independent of x and $r > 0$. Here $E(x, r) = \{y \in X \; : \; d(x, y) < r\}$, $d\mu(y) = |y|^{2\alpha+1}dy$, $d(x, y) = |x - y|$.

Let M_μ be the maximal function on the space of homogeneous type (X, d, μ):

$$M_\mu g(x) := \sup_{r>0} \mu(E(x, r))^{-1} \int_{E(x,r)} |g(y)|d\mu(y).$$

It is well known that the maximal function M_μ is weighted weak type $(1, 1)$ with $w \in A_{1,\alpha}$, and is bounded on $L_{p,w}(X, d, \mu)$ for $1 < p < \infty$, $w \in A_{p,\alpha}$ (see [2, 13]). Here we are concerned with the maximal operator defined by $d\mu(y) = |y|^{2\alpha+1}dy$. It is clear that this measure satisfies the doubling condition (2.2).

Also it is known that (see [12])

$$M_\alpha f(x) \le CM_\mu g(x), \tag{2.3}$$

where the constant $C > 0$ independent of f.

Then for $p = 1$ and $w \in A_{1,\alpha}$

$$\begin{aligned}\int_{\{x\in\mathbb{R}:\, M_\alpha f(x)>\tau\}} w(x)\, d\mu_\alpha(x) &\le \int_{\{x\in X \;:\; M_\mu f(x)>\frac{\tau}{C}\}} w(x)\, d\mu(x)\\ &\le \frac{C_{1,\omega,\alpha}}{\tau} \int_{\mathbb{R}} |f(x)|w(x)\, d\mu(x).\end{aligned} \tag{2.4}$$

and for $1 < p < \infty$ and $w \in A_{p,\alpha}$

$$\|M_\alpha f\|_{p,w,\alpha} \le C\|M_\mu f\|_{p,w,\alpha} \le C_{p,\omega,\alpha}\|f\|_{p,w,\alpha}.$$

□

Denote by $K_\nu(z)$ the following McDonald function (modified Bessel function, see [11])

$$K_\nu(z) = \frac{(2z)^\nu}{2} \int_0^\infty t^{\nu-1} e^{\frac{1}{4t} - tz^2} dt \tag{2.5}$$

and for $t > 0$, $x \in \mathbb{R}$ let

$$m(x, t) = \sqrt{\frac{2}{\pi}}\, t \frac{K_{1+\alpha}\left(\sqrt{t^2 + x^2}\right)}{\left(\sqrt{t^2 + x^2}\right)^{1+\alpha}}. \tag{2.6}$$

We define the D-metaharmonic semigroup $M_{t,\alpha}f$ as the convolution with the kernel $m(x,t)$ in the form

$$(M_{t,\alpha}f)(x) := f *_\alpha (m(\cdot,t))(x) = \int_{\mathbb{R}} m(y,t)\tau_x f(-y)d\mu_\alpha(y). \qquad (2.7)$$

We also define the Dunkl-Poisson semigroup $P_{t,\alpha}f$ as the convolution with the kernel $P_\alpha(x,t)$ in the form

$$P_{t,\alpha}f(x) := \int_{\mathbb{R}} f(y)\big(\tau_x P_{t,\alpha}\big)(-y))d\mu_\alpha(y), \; t>0, \; x \in \mathbb{R},$$

where $P_{t,\alpha}(x) = m_\alpha t(t^2+|x|^2)^{-\frac{2\alpha+3}{2}}$, $m_\alpha = \frac{2^{\alpha+1}\Gamma\left(\alpha+\frac{3}{2}\right)}{\sqrt{\pi}}$. Note that, Dunkl-Poisson integral is an integral operator of convolution type generated by the generalized D-translation $P_{t,\alpha}f(x) = (P_{t,\alpha} *_\alpha f)(x)$.

Corollary 2.1. *1) If $f \in L_{1,w,\alpha}$ and $w \in A_{1,\alpha}$, then for all $t>0$ $M_t^\alpha f$, $P_{t,\alpha}f \in WL_{1,w,\alpha}$ and*

$$\left\| \sup_{t>0} |M_t^\alpha f| \right\|_{WL_{1,w,\alpha}} \le C \left\| \sup_{t>0} |P_{t,\alpha} * f| \right\|_{WL_{1,w,\alpha}} \le C_1 \|f\|_{L_{1,w,\alpha}},$$

where $C_{1,\omega,\alpha} > 0$ is independent of f and t.

2) If $f \in L_{p,w,\alpha}(\mathbb{R})$, $\quad 1<p<\infty$ and $w \in A_{p,\alpha}$, then $M_t^\alpha f$, $P_{t,\alpha}f \in L_{p,w,\alpha}(\mathbb{R})$ and

$$\left\| \sup_{t>0} |M_t^\alpha f| \right\|_{L_{p,w,\alpha}} \le C \left\| \sup_{t>0} |P_{t,\alpha} * f| \right\|_{L_{p,w,\alpha}} \le C_2 \|f\|_{L_{p,w,\alpha}},$$

where $C_{p,w,\alpha} > 0$ is independent of f and t.

3. Weighted $(L_{p,\alpha}, L_{q,\alpha})$ boundedness of the Dunkl fractional maximal functions

Definition 3.1. The weight function ω belongs to the class $A_{p,q,\alpha}$ for $1<p<\infty$, $1 \le q < \infty$, if

$$\sup_{x,\,r} \left(\mu_\alpha(B(x,r))^{-1} \int_{B(x,r)} \omega(y)^q d\mu_\alpha(y)\right)^{1/q} \times \left(\mu_\alpha(B(x,r))^{-1} \int_{B(x,r)} \omega^{-p'}(y) d\mu_\alpha(y)\right)^{1/p'} < \infty$$

and ω belong to $A_{1,q,\alpha}$, if there exists a positive constant C such that for any $x \in \mathbb{R}$ and $r>0$

$$\left(\mu_\alpha(B(x,r))^{-1} \int_{B(x,r)} \omega(y)^q d\mu_\alpha(y)\right)^{1/q} \left(\operatorname*{ess\,sup}_{y \in B(x,r)} \frac{1}{w(y)}\right) \le C.$$

Remark 3.1. (see [5, 6]). If $w \in A_{p,q,\alpha}$, $1<p<q<\infty$, then the following properties are valid:

(a) $w^q \in A_{t,\alpha}$ with $t = 1 + q/p'$.
(b) $w^{-p'} \in A_{t',\alpha}$ with $t' = 1 + p/q'$.
(c) $w \in A_{q,p,\alpha}$.

(d) $w^p \in A_{s,\alpha}$ with $s = 1 + p/q'$.
(e) $w^{-q'} \in A_{s',\alpha}$ with $s' = 1 + q'/p$.

For the D-fractional maximal functions $M_{\beta,\alpha}f$ (see [8, 9])

$$M_{\beta,\alpha}f(x) = \sup_{r>0}\big(\mu_\alpha(B(0,r))\big)^{\frac{\beta}{2\alpha+2}-1}\int_{B(0,r)}\tau_x|f|(y)\,d\mu_\alpha(y),\ 0\le\beta<2\alpha+2$$

the following theorem is valid.

Theorem 3.1. *Let $0<\beta<2\alpha+2$, $1<p<\frac{2\alpha+2}{\beta}$, $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{2\alpha+2}$, $\omega\in A_{1+\frac{q}{p'},\alpha}(\mathbb{R})$, $p'=\frac{p}{p-1}$. Then there exists $C>0$ such that for all $f\in L_{p,\omega,\alpha}(\mathbb{R})$ the following inequality is valid:*

$$\left(\int_{\mathbb{R}}\left(M_{\beta,\alpha}(f\omega^\beta)(x)\right)^q\omega(x)\,d\mu_\alpha(x)\right)^{1/q}\le C\left(\int_{\mathbb{R}}|f(x)|^p\omega(x)\,d\mu_\alpha(x)\right)^{1/p}.$$

Proof. We need to introduce other fractional maximal function defined on a space of homogeneous type (X,d,μ) (see, the proof of Theorem 2.2).

Let M_μ be the maximal function on the space of homogeneous type (X,d,μ):

$$M_{\mu,\beta}g(x) := \sup_{r>0}\mu(E(x,r))^{-1+\frac{\beta}{2\alpha+2}}\int_{E(x,r)}|g(y)|d\mu(y),\ 0\le\beta<2\alpha+2.$$

It is well known that the fractional maximal operator $M_{\mu,\beta}$ is weighted weak type $(1,q)$ with $w\in A_{1,q,\alpha}$, and is bounded from $L_{p,w}(X,d,\mu)$ to $L_{q,w}(X,d,\mu)$ for $1<p<\infty$, $w\in A_{p,q,\alpha}$ (see [2, 10]). Here we are concerned with the maximal operator defined by $d\mu(y)=|y|^{2\alpha+1}dy$. It is clear that this measure satisfies the doubling condition (2.2).

Also it is known that (see [12])

$$M_{\beta,\alpha}f(x)\le CM_{\mu,\alpha}g(x),\tag{3.1}$$

where the constant $C>0$ independent of f.

Then for $p=1$ and $w\in A_{1,q,\alpha}$

$$\left(\int_{\{x\in\mathbb{R}:\ M_{\beta,\alpha}f(x)>\tau\}}w(x)\ d\mu_\alpha(x)\right)^{1/q}\le\left(\int_{\{x\in X\ :\ M_{\mu,\beta}f(x)>\frac{\tau}{C}\}}w(x)\ d\mu(x)\right)^{1/q}$$
$$\le\frac{C_{1,q,\omega,\alpha}}{\tau}\int_{\mathbb{R}}|f(x)|w(x)\ d\mu(x).\tag{3.2}$$

and for $1<p<\infty$ and $w\in A_{p,q,\alpha}$

$$\|M_{\beta,\alpha}f\|_{q,w,\alpha}\le C\|M_{\mu,\beta}f\|_{q,w,\alpha}\le C_{p,q,\omega,\alpha}\|f\|_{p,w,\alpha}.$$

□

Similarly, one can prove the following theorem.

Theorem 3.2. *Let $0<\beta<2\alpha+2$, $1-\frac{1}{q}=\frac{\beta}{2\alpha+2}$, $\omega\in A_{1,\alpha}(\mathbb{R})$. Then there exists $C>0$ such that for all $f\in L_{1,\omega,\alpha}(\mathbb{R})$ the following inequality is valid:*

$$\left(\int_{\{x\in\mathbb{R}:M_{\beta,\alpha}f(x)>\tau\}}\omega(x)\,d\mu_\alpha(x)\right)^{1/q}\le C\int_{\mathbb{R}}|f(x)|\omega(x)\,d\mu_\alpha(x).$$

4. Weighted $(L_{p,\alpha}, L_{q,\alpha})$ boundedness of the Dunkl fractional integral

We now consider the Dunkl fractional integral

$$I_{\beta,\alpha}\phi(x) := \int_{\mathbb{R}} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y), \qquad 0<\beta<2\alpha+2.$$

In this section, using the method of G.Welland [21], we give a full description of measures for which weighted estimates for the Dunkl fractional integral $I_{\beta,\alpha}f$ hold.

We start with the lemma.

Lemma 4.1. *For any ε, $0<\varepsilon<\min(\beta, 2\alpha+2-\beta)$, there exists a constant $c_\varepsilon>0$ such that for any nonnegative function $\phi : R_+ \to R$ and for any point $x\in R_+$ the following inequality holds:*

$$I_{\beta,\alpha}\phi(x) \le c_\varepsilon\sqrt{M_{\beta-\varepsilon,\alpha}\phi(x)M_{\beta+\varepsilon,\alpha}\phi(x)}. \tag{4.1}$$

Proof. Let r be an arbitrary positive real number. We write the integral as the sum of two integrals:

$$I_{\beta,\alpha}\phi(x) = \int_{B(0,r)} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y) + \int_{\mathbb{R}\setminus B(0,r)} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y).$$

For $0<\varepsilon<\beta$ we have

$$\begin{aligned}
&\int_{B(0,r)} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y)\\
&= \sum_{k=0}^{\infty}\int_{B(0,2^{-k}r)\setminus B(0,2^{-k-1}r)} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y)\\
&\le \sum_{k=0}^{\infty}\left(2^{-k-1}r\right)^{\beta-2\alpha-2}\int_{B(0,2^{-k}r)}\tau_y\phi(x)d\mu_\alpha(y)\\
&= Cr^{\varepsilon}\sum_{k=0}^{\infty}2^{-k\varepsilon}\left(2^{-k}r\right)^{\beta-\varepsilon-2\alpha-2}\int_{B(0,2^{-k}r)}\tau_y\phi(x)d\mu_\alpha(y)\\
&\le Cr^{\varepsilon}M_{\beta-\varepsilon,\alpha}\phi(x).
\end{aligned}$$

On the other hand, for $0<\varepsilon<2\alpha+2-\beta$ we have

$$\begin{aligned}
&\int_{\mathbb{R}\setminus B(0,r)} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y)\\
&= \sum_{k=0}^{\infty}\int_{B(0,2^{k+1}r)\setminus B(0,2^{k}r)} |y|^{\beta-2\alpha-2}\tau_y\phi(x)d\mu_\alpha(y)\\
&\le \sum_{k=0}^{\infty}\left(2^{k}r\right)^{\beta-2\alpha-2}\int_{B(0,2^{k+1}r)}\tau_y\phi(x)d\mu_\alpha(y)\\
&\le Cr^{-\varepsilon}\sum_{k=0}^{\infty}\left(2^{k}r\right)^{\beta+\varepsilon-2\alpha-2}\int_{B(0,2^{k+1}r)}\tau_y\phi(x)d\mu_\alpha(y)\\
&\le Cr^{-\varepsilon}M_{\beta+\varepsilon,\alpha}\phi(x).
\end{aligned}$$

Consequently, we obtain that for any ε, $0 < \varepsilon < \min\{\beta, 2\alpha + 2 - \beta\}$, there exists a constant $c_\varepsilon > 0$ such that for every nonnegative function ϕ and for any $x \in \mathbb{R}$ and $r > 0$ we have

$$I_{\beta,\alpha}\phi(x) \leq c_\varepsilon \left(r^\varepsilon M_{\beta-\varepsilon,\alpha}\phi(x) + r^{-\varepsilon} M_{\beta+\varepsilon,\alpha}\phi(x)\right). \tag{4.2}$$

Taking

$$r^\varepsilon = \left(\frac{M_{\beta+\varepsilon,\alpha}\phi(x)}{M_{\beta-\varepsilon,\alpha}\phi(x)}\right)^{1/2}$$

in (4.2), we obtain (4.1). □

Theorem 4.1. *Suppose that* $1 < p < \frac{2\alpha+2}{\beta}$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{2\alpha+2}$. *Then the inequality*

$$\left(\int_{\mathbb{R}} \left|I_{\beta,\alpha}(f\omega^\beta)(x)\right|^q \omega(x) d\mu_\alpha(x)\right)^{1/q} \leq c \left(\int_{\mathbb{R}} |f(x)|^p \omega(x) d\mu_\alpha(x)\right)^{1/p} \tag{4.3}$$

holds for any $f \in L_{p,\omega,\alpha}(\mathbb{R})$ *with a constant* $c > 0$ *independent of* f *if and only if*

$$\omega \in A_{\beta,\alpha}(\mathbb{R}), \qquad \beta = 1 + \frac{q}{p'}. \tag{4.4}$$

Proof. If $\omega \in A_{\beta,\alpha}(\mathbb{R})$, then $w \in A_{\beta-\eta,\alpha}(\mathbb{R})$ for sufficiently small positive η. Therefore it is possible to choose ε, $0 < \varepsilon < \min\{\beta, 2\alpha + 2 - \beta\}$, in such a way simultaneously $\omega \in A_{\beta_1,\alpha}(\mathbb{R})$, with $\beta_1 = 1 + \frac{p}{p'(1-p(\beta+\varepsilon))}$ and $w \in A_{\beta_2,\alpha}(\mathbb{R})$, with $\beta_2 = 1 + \frac{p}{p'(1-p(\beta-\varepsilon))}$. If we now take

$$\frac{1}{q_\varepsilon} = \frac{1}{p} - (\beta + \varepsilon), \qquad \frac{1}{\overline{q_\varepsilon}} = \frac{1}{p} - (\beta - \varepsilon),$$

then we obtain that $\omega \in A_{1+\frac{q_\varepsilon}{p'}}$ and $\omega \in A_{1+\frac{\overline{q_\varepsilon}}{p'}}$.

Denoting $p_1 = \frac{2q_\varepsilon}{q}$ and $p_2 = \frac{2\overline{q_\varepsilon}}{q}$. we have

$$\frac{1}{p_1} + \frac{1}{p_2} = 1.$$

Put

$$F_1(x) = \left(M_{\beta+\varepsilon,\alpha}(f\omega^\beta)(x)\right)^{q/2} \omega(x)^{1/p_1}$$

and

$$F_2(x) = \left(M_{\beta-\varepsilon,\alpha}(f\omega^\beta)(x)\right)^{q/2} \omega(x)^{1/p_2}.$$

Further, (4.1) together with Holder's inequality imply the estimate

$$\int_{\mathbb{R}} \left| I_{\beta,\alpha}(f\omega^{\beta})(x)\right|^{q} \omega(x)d\mu_{\alpha}(x) \leq c_{\varepsilon} \int_{\mathbb{R}} F_1(x)F_2(x)d\mu_{\alpha}(x) \leq$$

$$\leq c_{\varepsilon} \left(\int_{\mathbb{R}} \left(M_{\beta+\varepsilon,\alpha}(f\omega^{\beta})(x)\right)^{qp_1/2} \omega(x)d\mu_{\alpha}(x)\right)^{1/p_1} \times$$

$$\times \left(\int_{\mathbb{R}} \left(M_{\beta-\varepsilon,\alpha}(f\omega^{\beta})(x)\right)^{qp_2/2} \omega(x)d\mu_{\alpha}(x)\right)^{1/p_2} =$$

$$= c_{\varepsilon} \left(\int_{\mathbb{R}} \left(M_{\beta+\varepsilon,\alpha}(f\omega^{\beta})(x)\right)^{q_{\varepsilon}} \omega(x)d\mu_{\alpha}(x)\right)^{1/p_1} \times$$

$$\times \left(\int_{\mathbb{R}} \left(M_{\beta-\varepsilon,\alpha}(f\omega^{\beta})(x)\right)^{\overline{q_{\varepsilon}}} \omega(x)\, d\mu_{\alpha}(x)\right)^{1/p_2} .$$

Finally, using Theorem 3.1 we conclude that

$$\|I_{\beta,\alpha}(f\omega^{\beta})\|_{L_{q,\omega,\alpha}} \leq c\|f\|_{L_{p,\omega,\alpha}}$$

The implication (4.3) $\Longrightarrow$ (4.4) follows from the pointwise inequality

$$M_{\beta,\alpha}(f\omega^{\beta})(x) \leq c_1 I_{\beta,\alpha}(|f|\omega^{\beta})(x)$$

and Theorem 3.1. □

Remark 4.1. For the Riesz potentials, Theorem 4.1 is due to B.Muckenhoupt and R.L. Wheeden [14]. It was proved by using the above described method by G.Welland [21].

Acknowledgments The author would like to express their gratitude to the referees for their very valuable comments and suggestions.

References

[1] C. Abdelkefi and M. Sifi, Dunkl, Translation and Uncentered Maximal Operator on the Real Line, *International Journal of Mathematics and Mathematical Sciences* Volume 2007, Article ID 87808, 9 pages doi:10.1155/2007/87808.

[2] R. R. Coifman, G. Weiss, *Analyse harmonique non commutative sur certains expaces homogenes*, Lecture Notes in Math., v. 242, Springer- Verlag. Berlin, 1971.

[3] C. F. Dunkl, *Integral kernels with reflection group invariance*, Canad. J. Math. **43** (1991), 1213–1227.

[4] C. F. Dunkl, *Differential-difference operators associated with reflections groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.

[5] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*. North-Holland Math. 16, Amsterdam, 1985.

[6] L. Grafakos, *Classical and Modern Fourier Analysis*. Pearson Education, Inc. Upper Saddle River, New Jersey, 2004.

[7] V. S. Guliyev, Y. Y. Mammadov, Function spaces and integral operators for the Dunkl operator on the real line, *Khazar Journal of Mathematics*, **2** (2006), no.4 , 17–42.

[8] V. S. Guliyev, Y. Y. Mammadov, On fractional maximal function and fractional integrals associated with the Dunkl operator on the real line, *Journal of Mathematical Analysis and Applications*, **353** (2009), no.1, 449–459.

[9] V. S. Guliyev, Y. Y. Mammadov, (L_p, L_q) boundedness of the fractional maximal operator associated with the Dunkl operator on the real line, *Integral Transforms and Special Functions*, **21** (2010), no.8, 629–639.

[10] V. M. Kokilashvili, A. Kufner, Fractional integrals on spaces of homogeneous type, *Comment. Math. Univ. Carolinae.* **30** (1989), no.3, 511–523.

[11] S. M. Nikolski, *Approximation of Functions of Several Variables and Imbedding Theorems.* Nauka (Moscow, 1969), English translation: Springer-Verlag, 1975.

[12] Y. Y. Mammadov, On maximal operator associated with the Dunkl operator on $\mathbb{R}$, *Khazar Journal of Mathematics* **2** (2006), no.4, 59–70.

[13] R. A. Macias, C. Segovia, A well behaved quasi-distance for spaces of homogeneous type, *Trab. Mat. Inst. Argent. Mat.* **32** (1981), 18 p.

[14] B. Muckenhaupt, R. L. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. A.M.S.* **192** (1974), 261–274.

[15] M. Rösler, Bessel-type signed hypergroups on $\mathbb{R}$, in *Probability measures on groups and related structures, XI (Oberwolfach, 1994), H. Heyer and A. Mukherjea, Eds.*, pp. 292-304, World Scientific, River edge, NJ, USA, 1995.

[16] M. Sifi, F. Soltani, Generalized Fock spaces and Weyl relations for the Dunkl kernel on the real line, *J. Math. Anal. Appl.* **270** (2002), 92–106.

[17] F. Soltani, Lp-Fourier multipliers for the Dunkl operator on the real line, *J. Funct. Anal.* **209** (2004), 16–35.

[18] F. Soltani, Littlewood-Paley operators associated with the Dunkl operator on $\mathbb{R}$, *J. Funct. Anal.* **221** (2005), 205–225.

[19] K. Trimeche, Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators, *Int. Trans. Spec. Funct.* **13** (2002), 17–38.

[20] G.N. Watson, *A treatise on theory of Bessel functions*, Cambridge University Press, Cambridge (1966).

[21] G. Welland, Weighted norm inequalities for fractional integrals, *Proc. Amer. Math. Soc.* **51** (1975), no.1, 143–148.

Yagub Y. Mammadov
Institute of Mathematics and Mechanics, Baku, Azerbaijan, 9 F. Agayev str., AZ1141, Baku, Azerbaijan
Nakhchivan Teacher-Training Institute, Azerbaijan
E-mail address: yagub52@yandex.ru

Received: April 11, 2014; Revised: May 29, 2014; Accepted: June 3, 2014