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MAXIMAL OPERATORS ASSOCIATED WITH GEGENBAUER EXPANSIONS ON THE HALF-LINE. I

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Abstract. In this paper we consider the generalized shift operator, generated by the Gegenbauer differential operator

$$G=\left(x^2-1\right)^{\frac{1}{2}-\lambda}\frac{d}{dx}\left(x^2-1\right)^{\lambda+\frac{1}{2}}\frac{d}{dx}.$$

Maximal function ($G-$ maximal function) generated by the Gegenbauer differential operator G is investigated. The $L_{p,\lambda}$ -boundedness for the $G-$ maximal function is obtained. The concept of potential of Riesz-Gegenbauer is introduced and for it the theorem of Sobolev type is proved.

1. Introduction

The Hardy-Lettlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 (see [22]) for $2\pi-$ periodical functions and later it was extended to the Euclidean spaces, some weighted measure spaces (see [26, 27, 29]), symmetric spaces (see [6, 7]), various Lie groups [11], for the Jacobi-type hypergroups [8, 9], for Chebli-Trimeche hypergroups [2], for the one-dimensional Bessel-Kingman hypergroups [28], for the $n-$ dimensional Bessel-Kingman hypergroups ($n\geq 1$) [13, 14, 16], for Morrey-Bessel spaces [3, 4, 5, 15, 16], for Laguerre hypergroup [1, 18, 19, 23, 24]. The structure of the paper is as follows. In Section 1 we present some definitions, notation and auxiliary results. In Section 2 the $L_{p,\lambda}$ boundedness of the maximal function, associated with the Gegenbauer differential operator ($G-$ maximal function) is proved. In Section 3 we introduce and study some embeddings into the Morrey spaces, associated with the Gegenbauer differential operator.

2. Definitions, notation and auxiliary results

Let $H(x,r)=(x-r,x+r)\cap[0,\infty), r\in(0,\infty), x\in[0,\infty)$. For all measurable set $E\subset[0,\infty)$ $\mu E\equiv|E|_\lambda=\int\limits_E sh^{2\lambda}t dt$. For $1\leq p\leq\infty$ let $L_p([0,\infty),G)\equiv$

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$L_{p,\lambda}[0,\infty)$ be the space of functions measurable on $[0,\infty)$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \left(\int_0^\infty |f(ch\,t)|^p sh^{2\lambda}\,t dt\right)^{\frac{1}{p}}, 1 \le p < \infty,$$

$$\|f\|_{\infty,\lambda} = \operatorname*{ess\,sup}_{t\in[0,\infty)} |f(ch\,t)|,\ p=\infty$$

In analogy to [10] we define the Gegenbauer maximal functions as follows:

$$M_G f\,(ch\,x) = \sup_{r>0} \frac{1}{|H(0,r|_\lambda} \int\limits_0^r A^\lambda_{ch\,t} \,|f\,(ch\,x)|\, d\mu\,(t)\,,$$

$$M_\mu f\,(ch\,x) = \sup_{r>0} \frac{1}{|H\,(x,r)|_\lambda} \int\limits_{H(x,r)} |f(ch\,t|\, d\mu\,(t)\,,\ d\mu(t) = sh^{2\lambda}\,tdt,$$

$$|H(0,r)|_\lambda = \int\limits_0^r sh^{2\lambda}\,tdt,\ |H(x,r)|_\lambda = \int\limits_{H(x,r)} sh^{2\lambda}\,tdt.$$

Here

$$A^\lambda_{ch\,t} f(ch\,x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma\,(\lambda)} \int\limits_0^\pi f(ch\,xch\,t - sh\,xsh\,t\cos\varphi)(\sin\varphi)^{2\lambda-1} d\varphi$$

denotes the generalized shift operator associated with the Gegenbauer differential operator

$$G = \left(x^2-1\right)^{1/2-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+1/2} \frac{d}{dx}.$$

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

Further we'll need some auxiliary assertions.

Lemma 2.1. *For $0<\lambda<1/2$ the following correlations are true:*

$$|H(0,r)|_\lambda \approx \begin{cases} (sh\,\frac{r}{2})^{2\lambda+1}, 0<r\le c; & (a) \\ (ch\,\frac{r}{2})^{4\lambda}, \quad c<r<\infty, & (b) \end{cases}$$

where c denotes a positive constant.

Proof. Let first $0<r\le c$. Then

$$|H(0,r)|_\lambda = \int\limits_0^r sh^{2\lambda}\,tdt = \int\limits_0^r (sh\,t)^{2\lambda-1} d(ch\,t) = \int\limits_0^r (ch^2\,t-1)^{\lambda-\frac{1}{2}} d(ch\,t)$$

$$= \int\limits_1^{ch\,r} (t-1)^{\lambda-\frac{1}{2}}(t+1)^{\lambda-\frac{1}{2}} dt \ge (ch\,r+1)^{\lambda-\frac{1}{2}} \int\limits_1^{ch\,r} (t-1)^{\lambda-\frac{1}{2}} dt$$

$$\ge (ch1+1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}} \Big|_1^{ch\,r} = \frac{2(ch\,r-1)^{\lambda+\frac{1}{2}}}{(2\lambda+1)(1+ch1)^{\frac{1}{2}-\lambda}}$$

$$= \frac{2^{2\lambda+2}}{(2\lambda+1)(1+ch1)^{\frac{1}{2}-\lambda}}(sh\,\frac{r}{2})^{2\lambda+1}. \tag{2.1}$$

On the other hand,

$$|H(0,r)|_\lambda = \int\limits_0^r sh^{2\lambda}\,tdt = \int\limits_1^{ch\,r}(t-1)^{\lambda-\frac{1}{2}}(t+1)^{\lambda-\frac{1}{2}}dt \le 2^{\lambda-\frac{1}{2}}\int\limits_1^{ch\,r}(t-1)^{\lambda-\frac{1}{2}}dt$$

$$= \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}(t-1)^{\lambda+\frac{1}{2}}\Big|_1^{ch\,r} = \frac{2^{\lambda+\frac{1}{2}}}{2\lambda+1}(ch\,r-1)^{\lambda+\frac{1}{2}} = \frac{2^{2\lambda+1}}{2\lambda+1}\left(sh\,\frac{r}{2}\right)^{2\lambda+1}. \tag{2.2}$$

Let now $c < r < \infty$. Then

$$|H(0,r)|_\lambda = \int\limits_0^r sh^{2\lambda}\,tdt = \int\limits_0^r (sh\,t)^{2\lambda-1}d(ch\,t) = \int\limits_0^r (ch^2\,t-1)^{\lambda-\frac{1}{2}}d(ch\,t)$$

$$= \int\limits_1^{ch\,r}\frac{(t-1)^{\lambda-\frac{1}{2}}}{(t+1)^{\frac{1}{2}-\lambda}}dt \ge (ch\,r+1)^{\lambda-\frac{1}{2}}\int\limits_1^{ch\,r}(t-1)^{\lambda-\frac{1}{2}}dt$$

$$= (ch\,r+1)^{\lambda-\frac{1}{2}}\frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}}\Bigg|_1^{ch\,r} = \frac{2}{2\lambda+1}\frac{(ch\,r-1)^{\lambda+\frac{1}{2}}}{(ch\,r+1)^{\frac{1}{2}-\lambda}}$$

$$= \frac{2}{2\lambda+1}\frac{\left(2sh^2\,\frac{r}{2}\right)^{\lambda+\frac{1}{2}}}{\left(2ch^2\,\frac{r}{2}\right)^{\frac{1}{2}-\lambda}} = \frac{2^{2\lambda+2}}{(2\lambda+1)2^{1-2\lambda}}\frac{\left(sh\,\frac{r}{2}\right)^{2\lambda+1}}{\left(ch\,\frac{r}{2}\right)^{1-2\lambda}}$$

$$\ge \frac{2^{4\lambda+1}}{(2\lambda+1)3^{2\lambda+1}}\left(ch\,\frac{r}{2}\right)^{4\lambda} \Leftrightarrow \frac{4^{2\lambda+1}}{2\lambda+1}\left(3sh\,\frac{r}{2}\right)^{2\lambda+1}$$

$$\ge \frac{4^{2\lambda+1}}{2\lambda+1}\left(ch\,\frac{r}{2}\right)^{2\lambda+1} \Leftrightarrow 3sh\,\frac{r}{2} \ge ch\,\frac{r}{2} \Leftrightarrow 3\frac{e^{r/2}-e^{-r/2}}{2}$$

$$\ge \frac{e^{r/2}+e^{-r/2}}{2} \Leftrightarrow 3(e^r-1) \ge e^r+1 \Leftrightarrow 2e^r \ge 4,$$

which takes place for $r \ge c \ge 1$.

So,

$$|H(0,r)|_\lambda \ge \frac{2^{4\lambda+1}}{(2\lambda+1)3^{2\lambda+1}}\left(ch\,\frac{r}{2}\right)^{4\lambda}. \tag{2.3}$$

Estimate above $|H(0,r)|_\lambda$.

$$|H(0,r)|_\lambda = \int\limits_0^r sh^{2\lambda}\,tdt = \int\limits_0^r\left(2sh\,\frac{t}{2}ch\,\frac{t}{2}\right)^{2\lambda}dt$$

$$= 2^{2\lambda+1}\int\limits_0^r\left(sh\,\frac{t}{2}\right)^{2\lambda}\left(ch\,\frac{t}{2}\right)^{2\lambda-1}d\left(sh\,\frac{t}{2}\right) \le 2^{2\lambda+1}\int\limits_0^r\left(sh\,\frac{t}{2}\right)^{4\lambda-1}d\left(sh\,\frac{t}{2}\right)$$

$$= \frac{2^{2\lambda+1}}{4\lambda}\left(sh\,\frac{t}{2}\right)^{4\lambda}\Bigg|_0^r = \frac{4^\lambda}{2\lambda}\left(sh\,\frac{r}{2}\right)^{4\lambda} \le \frac{4^\lambda}{2\lambda}\left(ch\,\frac{r}{2}\right)^{4\lambda}. \tag{2.4}$$

Combine (1.1)-(1.4) we obtain assertion of Lemma 1.1. □

Lemma 2.2. *Let $0 < \lambda < 1/2$ and $x \in [0,\infty)$, $r \in (0,\infty)$. Then the following estimates are true: for $0 < r \le c$*

$$|H(x,r)|_\lambda \le c_\lambda \begin{cases} r^{2\lambda+1}, \ 0 \le x \le r \le c; \\ \\ ch^{2\lambda} x, \quad r < x < \infty \ (r \le c < x < \infty). \end{cases} \tag{a}$$

And for $c < r < \infty$

$$|H(x,r)|_\lambda \le c_\lambda \begin{cases} ch^{2\lambda} r, \ 0 < x \le 2r \ (0 < x \le 2c < 2r); \\ \\ ch^{2\lambda} x ch^{2\lambda} r, \ 2r < x < \infty \ (2c < 2r < x < \infty). \end{cases} \tag{b}$$

Here and further $c_\lambda, c_{\alpha,\lambda}, c_{\alpha,\lambda,p}$ will denote some constants depending only of subscribed indexes and generally speaking different in different formulas.

Proof. First we consider case when $0 < r \le c$ and $x \in [0,\infty)$.

Let $0 \le t \le 2c$, then we have

$$t \le sh\, t \le e^{2c} t. \tag{2.5}$$

We prove the left-hand part of this estimate. We consider the function $f(t) = sh\, t - t$. As, $f'(t) = ch\, t - t \ge 0$, then $f(t)$ increases on $[0,\infty)$, and that takes the smallest value for $t = 0$, $f(0) = 0$, consequently $f(t) \ge 0$ equivalent to $sh\, t \ge t$.

We prove the right-hand part of estimate (1.5).

$$\frac{e^t - e^{-t}}{2} \le e^{2c} \cdot t \quad \Leftrightarrow e^{2t} - 1 \le 2 \cdot e^{2c+t} \cdot t \ \Leftrightarrow e^{2t} \le 2 \cdot e^{2c+t} \cdot t + 1.$$

We consider the function $f(t) = 2 \cdot e^{2c+t} \cdot t + 1 - e^{2t}$.

$$f'(t) = 2 \cdot e^{2c+t} + 2 \cdot e^{2c+t} \cdot t - 2e^{2t} = 2e^t \left(e^{2c} + t \cdot e^{2c} - e^t \right)$$
$$\ge e^{2c} (t+1) - e^t \ge e^{2c} - e^t \ge 0, \ \text{as}, \ t \le 2c.$$

Thus, the estimate (1.5) is proved.

Hence it follows that for $0 \le x \le r \le c$

$$|H(x,r)|_\lambda = \int\limits_0^{x+r} sh^{2\lambda} t dt \le e^{2c} \int\limits_0^{2r} t^{2\lambda} dt = \frac{e^{2c} \cdot 2^{2\lambda}}{2\lambda + 1} \cdot r^{2\lambda+1}. \tag{2.6}$$

$r < x \le c$

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda} t dt \le e^{2c} \int\limits_{x-r}^{x+r} t^{2\lambda} dt \le e^{2c} \cdot r \cdot (x+r)^{2\lambda}$$
$$\le e^{2c} \cdot r \cdot (2x)^{2\lambda} \le c_\lambda ch^{2\lambda} x. \tag{2.7}$$

Let now $0 < r \le c \le x < \infty$, then we have

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda} t dt \le 2r \cdot sh^{2\lambda}(x+r) = 2r(sh\, x ch\, r + ch\, x sh\, r)^{2\lambda}$$
$$\le 2r(sh\, x ch\, c + ch\, x sh\, c)^{2\lambda} \le 2r(2ch\, x ch\, c)^{2\lambda} \le c_\lambda ch^{2\lambda} x.$$

Now we consider case when $c < r < \infty$, $x \in [0,\infty)$.

Let $0 < x \leq 2c < 2r$. Similarly to the proof of the estimate (1.6) we obtain

$$|H(x,r)|_\lambda = \int\limits_0^{x+r} sh^{2\lambda}\, t dt = \frac{4^\lambda}{2\lambda} sh^{4\lambda} \frac{t}{2} \Big|_0^{x+r} = \frac{4^\lambda}{2\lambda} sh^{4\lambda}\, \frac{x+r}{2}$$

$$= \frac{4^\lambda}{2\lambda}\left(sh\,\frac{x}{2} ch\,\frac{r}{2} + ch\frac{x}{2} sh\,\frac{r}{2}\right)^{4\lambda}$$

$$\leq c_\lambda (sh\, c\, ch\, \frac{r}{2} + ch\, c\, sh\, \frac{r}{2})^{4\lambda} \leq c_\lambda ch^{4\lambda}\, \frac{r}{2}. \tag{2.8}$$

Let now $2c < 2r < x < \infty$, then

$$|H(x,r)|_\lambda = \int\limits_{x-r}^{x+r} sh^{2\lambda}\, t dt \leq \frac{4^\lambda}{2\lambda}\, sh^{4\lambda}\, \frac{t}{2}\Big|_{x-r}^{x+r} = \frac{4^\lambda}{2\lambda}\left(sh^{4\lambda}\, \frac{x+r}{2} - sh^{4\lambda}\, \frac{x-r}{2}\right)$$

$$\leq \frac{4^\lambda}{2\lambda} sh^{4\lambda}\, \frac{x+r}{2} \leq c_\lambda ch^{4\lambda}\, \frac{x}{2} ch^{4\lambda}\, \frac{r}{2} \leq c_\lambda ch^{2\lambda}\, x ch^{2\lambda}\, r. \tag{2.9}$$

From (1.9) and (1.10) it follows that at $c < r < \infty$ and $0 < x < \infty$

$$|H(x,r)|_\lambda \leq c_\lambda \begin{cases} ch^{2\lambda}\, r, & 0 < x \leq 2c < 2r; \quad (1.11) \\ ch^{2\lambda}\, x ch^{2\lambda}\, r, & 2c < 2r < x < \infty. \quad (1.12) \end{cases}$$

Assertion of Lemma 1.2 follows from (1.6)-(1.8), (1.11) and (1.12). □

3. $L_{p,\lambda}$ -boundedness of G- maximal operator

Theorem 3.1. *For $0 \leq x < \infty$ and $0 < r < \infty$ the inequality is valid*

$$M_G f(ch\, x) \leq c_\lambda M_\mu f(ch\, x),$$

where c_λ is some positive constant.

Proof. Consider the integral

$$I(x,r) = \int\limits_0^r A^\lambda_{ch\, t} \left|f(ch\, x)\right| sh^{2\lambda}\, t dt$$

$$= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int\limits_0^r \left\{ \int\limits_0^\pi |f(ch\, x \cdot ch\, t - sh\, x \cdot sh\, t \cos\,\varphi| \, (\sin\,\varphi)^{2\lambda-1} d\varphi \right\} sh^{2\lambda}\, t dt.$$

Making in internal integral replacing
$z = ch\, x \cdot ch\, t - sh\, x \cdot sh\, t \cos\,\varphi$, we get that

$$\cos\,\varphi = \frac{ch\, x \cdot ch\, t - z}{sh\, x \cdot sh\, t}, \varphi = \arccos\frac{ch\, x \cdot ch\, t - z}{sh\, x \cdot sh\, t},$$

$$d\varphi = \frac{dz}{\sqrt{1 - (\frac{ch\, x \cdot ch\, t - z}{sh\, x \cdot sh\, t})^2} sh\, x \cdot sh\, t}$$

$$= (sh^2\, x \cdot sh^2\, t - ch^2\, x \cdot ch^2\, t + 2 \cdot z \cdot ch\, x \cdot ch\, t - z^2)^{-\frac{1}{2}} dz.$$

Since

$$sh^2\, x \cdot sh^2\, t - ch^2\, x \cdot ch^2\, t$$

$$= (ch^2 x - 1) sh^2 t - ch^2 x \cdot ch^2 t = ch^2 x \cdot sh^2 t - sh^2 t - ch^2 x \cdot ch^2 t$$
$$= -sh^2 t + ch^2 x (sh^2 t - ch^2 t) = -sh^2 t - ch^2 x,$$

we obtain that

$$d\varphi = (2z \cdot ch\, x \cdot ch\, t - sh^2 t - ch^2 x - z^2)^{-\frac{1}{2}} dz$$

and

$$(\sin \varphi)^{2\lambda - 1} = (2z \cdot ch\, x \cdot ch\, t - sh^2 t - ch^2 x - z^2)^{\lambda - \frac{1}{2}} (sh\, x \cdot sh\, t)^{1 - 2\lambda}.$$

Then $I(x,r)$ takes the form

$$I(x,r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \times$$

$$\int_0^r \left\{ \int_{ch\,(x-t)}^{ch\,(x+t)} |f(z)| \, (2z \cdot ch\, x \cdot ch\, t - sh^2 t - ch^2 x - z^2)^{\lambda - 1} (sh\, x)^{1 - 2\lambda} dz \right\} sh\, t dt. \tag{3.1}$$

Transform expansion

$$2z \cdot ch\, x \cdot ch\, t - sh^2 t - ch^2 x - z^2$$
$$= 2z \cdot ch\, x \cdot ch\, t - sh^2 t (ch^2 x - sh^2 x) - ch^2 x - z^2$$
$$= 2z \cdot ch\, x \cdot ch\, t - sh^2 t \cdot ch^2 t + sh^2 t \cdot sh^2 x - ch^2 x - z^2$$
$$= 2z \cdot ch\, x \cdot ch\, t + sh^2 t \cdot sh^2 x - (ch^2 t - 1) ch^2 x - ch^2 x - z^2$$
$$= 2z \cdot ch\, x \cdot ch\, t + sh^2 x \cdot (ch^2 t - 1) - ch^2 t \cdot ch^2 x - z^2 (ch^2 x - sh^2 x)$$
$$= 2z \cdot ch\, x \cdot ch\, t + sh^2 x \cdot ch^2 t - sh^2 x - ch^2 t \cdot ch^2 x - z^2 \cdot ch^2 x - z^2 \cdot sh^2 x$$
$$= 2z \cdot ch\, x \cdot ch\, t - sh^2 x - ch^2 t - z^2 ch^2 x - z^2 sh^2 x = (z^2 - 1) sh^2 x - (ch\, t - z \cdot ch\, x)^2$$
$$= \left(z^2 - 1\right) sh^2 x \left[1 - \left(\frac{ch\, t - z \cdot ch\, x}{\sqrt{z^2 - 1} \cdot sh\, x} \right)^2 \right]. \tag{3.2}$$

Taking into account (2.1) and (2.2) we get

$$I(x,r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch\,(x-t)}^{ch\,(x+t)} |f(z)| \, (z^2 - 1)^{\lambda - 1} \right.$$
$$\left. \times \left[1 - \left(\frac{ch\, t - z \cdot ch\, x}{\sqrt{z^2 - 1} \cdot sh\, x} \right)^2 \right]^{\lambda - 1} dz \right\} \frac{sh\, t}{sh\, x} dt. \tag{3.3}$$

Note that

$$\frac{sh\, t}{sh\, x} = (z^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial t} \left(\frac{ch\, t - z \cdot ch\, x}{\sqrt{z^2 - 1} \cdot sh\, x} \right),$$

rewrite (2.3) in the form

$$I(x,r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_{ch\,(x-t)}^{ch\,(x+t)} |f(z)| \, (z^2 - 1)^{\lambda - \frac{1}{2}} \right.$$

$$\times\left[1-\left(\frac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)^2\right]^{\lambda-1}\frac{\partial}{\partial t}\left(\frac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)\Bigg\}dzdt. \tag{3.4}$$

Since $ch\,(x-t)\le z\le ch\,(x+t)$, then

$$\begin{cases} ch\,(x-r)\le z\le ch\,x \\ x-arcchz\le t\le r \end{cases} \text{ and } \begin{cases} ch\,x\le z\le ch\,(x+r) \\ arcchz-x\le t\le r. \end{cases}$$

And that's why by changing the order of integration in (2.4) we get

$$I(x,r)=\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})}\left(\int\limits_{ch\,(x-r)}^{ch\,x}dz\int\limits_{x-arcchz}^{r}dt+\int\limits_{ch\,x}^{ch\,(x+r)}dz\int\limits_{arcchz-x}^{r}dt\right). \tag{3.5}$$

Consider the integral

$$A(x,z,r)\equiv A(x,r)$$

$$=\int\limits_{x-arcchz}^{r}\left[1-\left(\frac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)^2\right]^{\lambda-1}\frac{\partial}{\partial t}\left(\frac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)dt.$$

Putting here $u=\dfrac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}$, we get

$$A(x,z,r)\equiv A(x,r)=\int\limits_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}(1-u^2)^{\lambda-1}du. \tag{3.6}$$

Since $ch\,t$ is an even function,

$$B(x,r)=\int\limits_{arcchz-x}^{r}\left[1-\left(\frac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)^2\right]^{\lambda-1}\frac{\partial}{\partial t}\left(\frac{ch\,t-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)dt$$

$$=\int\limits_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}(1-u^2)^{\lambda-1}du. \tag{3.7}$$

Taking into account (2.6) and (2.7) in (2.5) we have

$$I(x,r)=\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)}\int\limits_{ch\,(x-r)}^{ch\,(x+r)}|f(z)|\,(z^2-1)^{\lambda-\frac{1}{2}}\int\limits_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}(1-u^2)^{\lambda-1}dudz. \tag{3.8}$$

As $ch\,(x-r)\le z\le ch\,(x+r)$, then

$$\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\ge\frac{ch\,r-ch\,x\cdot ch\,(x+r)}{sh\,x\cdot sh(x+r)}=\frac{2ch\,r-2ch\,x\cdot ch\,(x+r)}{2sh\,x\cdot sh(x+r)}$$

$$=\frac{2ch\,r-ch(2x+r)-ch\,r}{ch(2x+r)-ch\,r}=\frac{ch\,r-ch(2x+r)}{ch(2x+r)-ch\,r}=-1 \tag{3.9}$$

On the other hand for $ch\,(x-r)\le z\le ch\,(x+r)$,

$$\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\le\frac{ch\,r-ch\,x\cdot ch\,(x-r)}{sh\,x\,|sh(x-r)|}=\frac{2ch\,r-2ch\,x\cdot ch\,(x-r)}{2sh\,x\cdot sh(r-x)}$$

$$=\frac{2ch\,r-ch(2x-r)-ch\,r}{ch\,r-ch(2x-r)}=\frac{ch\,r-ch(2x-r)}{ch\,r-ch(2x-r)}=1. \qquad (3.10)$$

From (2.9) and (2.10) it follows that for $ch\,(x-r)\le z\le ch\,(x+r)$, and $0<x\le r$

$$-1\le\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\le 1. \qquad (3.11)$$

From (2.11) it follows that for $0<x\le r\le c$

$$A(x,r)=\int\limits_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}(1-u^2)^{\lambda-1}du\le\int\limits_{-1}^{1}(1-u^2)^{\lambda-1}du=\frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}. \qquad (3.12)$$

But then taking into account (2.12) and (2.8) we obtain that for $0<x\le r\le c$

$$I(x,r)\le\int\limits_{ch\,(x-r)}^{ch\,(x+r)}|f(z)|\,(z^2-1)^{\lambda-\frac{1}{2}}dz=\int\limits_{x-r}^{x+r}|f(ch\,t)|\,sh^{2\lambda}\,tdt. \qquad (3.13)$$

Now let $c<r<x<\infty$ and $ch\,(x-r)\le z\le ch\,(x+r)$
$(c<r<x<\infty)$.
Then we have

$$\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\le\frac{ch\,x-z\cdot ch\,x}{\sqrt{z^2-1}sh\,x}=\frac{(1-z)ch\,x}{\sqrt{z^2-1}sh\,x}=-\frac{\sqrt{z-1}ch\,x}{\sqrt{z+1}sh\,x}\le 0. \qquad (3.14)$$

From (2.9) it follows, that

$$\max(1-u)^{\lambda-1}\le\max_{-1\le u\le 0}(1-u)^{\lambda-1}=\max(2^{\lambda-1},1)=1,$$

$$-1\le u\le\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}.$$

Taking into account this circumstance for the integral $A(x,r)$ we obtain of (2.6)

$$A(x,r)=\int\limits_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}\left(1-u^2\right)^{\lambda-1}du$$

$$\le\int\limits_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}(1+u)^{\lambda-1}\,du=\frac{1}{\lambda}\,(1+u)^{\lambda}\Big|_{-1}^{\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}}=\frac{1}{\lambda}\left(1+\frac{ch\,r-z\cdot ch\,x}{\sqrt{z^2-1}\cdot sh\,x}\right)^{\lambda}$$

$$=\frac{1}{\lambda}\left(1-\frac{z\cdot ch\,x-ch\,r}{\sqrt{z^2-1}\cdot sh\,x}\right)^{\lambda}\le\frac{1}{\lambda}\left[1-\left(\frac{z\cdot ch\,x-ch\,r}{\sqrt{z^2-1}\cdot sh\,x}\right)^2\right]^{\lambda}. \qquad (3.15)$$

We find extremum of the function

$$f(z) = 1 - \left(\frac{z \cdot ch\, x - ch\, r}{\sqrt{z^2-1} \cdot sh\, x} \right)^2 .$$

$$f'(z) = -2 \left(\frac{z \cdot ch\, x - ch\, r}{\sqrt{z^2-1} \cdot sh\, x} \right)$$

$$\times \frac{(z^2-1) sh\, x \cdot ch\, x - z^2 sh\, x \cdot ch\, x + z \cdot ch\, r \cdot sh\, x}{(z^2-1)^{\frac{3}{2}} sh^2\, x}$$

$$= -2 \left(\frac{z \cdot ch\, x - ch\, r}{\sqrt{z^2-1} \cdot sh\, x} \right) \frac{z \cdot ch\, r \cdot sh\, x - ch\, x \cdot sh\, x}{(z^2-1)^{\frac{3}{2}} sh^2\, x}$$

$$= \frac{2(z \cdot ch\, x - ch\, r)(ch\, x - z \cdot ch\, r)}{(z^2-1)^2 sh^2\, x}.$$

Since $ch\,(x-r) \leq z \leq ch\,(x+r)$, then the function $f(z)$ at the point $z = ch\, x / ch\, r$ has maximum equal to

$$f_{\max} \left(\frac{ch\, x}{ch\, r} \right) = 1 - \left(\frac{ch^2\, x - ch^2\, r}{\sqrt{ch^2\, x - ch^2\, r} \cdot sh\, x} \right)^2 =$$

$$= 1 - \frac{ch^2\, x - ch^2\, r}{sh^2\, x} = \frac{ch^2\, r - 1}{sh^2\, x} = \left(\frac{sh\, r}{sh\, x} \right)^2 .$$

From the last equation and (2.15) we have

$$A(x,r) \leq \frac{1}{\lambda} \left(\frac{sh\, r}{sh\, x} \right)^{2\lambda} . \tag{3.16}$$

Let $0 < r \leq c$, then taking into account Lemmas 1.1(a) and 1.2(a) for (2.13) with $0 \leq x \leq r \leq c$, we have

$$M_G f(ch\, x) = \sup_{0<r\leq 1} \frac{1}{\mu H(0,r)} \int_0^r A_{ch\, t}^{\lambda} \left| f(ch\, x) \right| d\mu(t)$$

$$= \sup_{0<r\leq 1} \frac{\mu H(x,r)}{\mu H(0,r)} \cdot \frac{1}{\mu H(x,r)} \int_{x-r}^{x+r} \left| f(ch\, t) \right| sh^{2\lambda}\, t dt$$

$$\leq c_\lambda \sup_{0<r\leq 1} \frac{1}{\mu H(x,r)} \int_{H(x,r)} \left| f(ch\, t) \right| d\mu(t) = c_\lambda M_\mu f(ch\, x), \tag{3.17}$$

and for $c < r < x < \infty$ from (2.16) and (2.8) we obtain

$$M_G f(ch\, x) \leq \sup_{0<r\leq 1} \frac{A(x,r) \mu H(x,r)}{\mu H(0,r) \mu H(x,r)} \int_{x-r}^{x+r} \left| f(ch\, t) \right| sh^{2\lambda}\, t dt$$

$$\leq c_\lambda \sup_{0<r\leq 1} \frac{r \cdot ch^{2\lambda}\, x \cdot sh^{2\lambda}\, r}{\mu H(x,r) \left(sh\, \frac{r}{2} \right)^{2\lambda+1} sh^{2\lambda}\, x} \int_{x-r}^{x+r} \left| f(ch\, t) \right| d\mu(t)$$

$$\leq c_\lambda \left(\frac{ch\,x}{sh\,x}\right)^{2\lambda} \sup_{0<r\leq 1} ch^{2\lambda}\frac{r}{2}\cdot\frac{1}{\mu H(x,r)}\int\limits_{x-r}^{x+r}|f(ch\,t)|\,d\mu(t)$$

$$\leq c_\lambda \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^{2\lambda} ch^{2\lambda}\frac{1}{2}\sup_{0<r\leq 1}\frac{1}{\mu H(x,r)}\int\limits_{H(x,r)}|f(ch\,t)|\,d\mu(t)$$

$$\leq c_\lambda\cdot 4^\lambda\cdot e\cdot M_\mu f(ch\,x), \tag{3.18}$$

as $\dfrac{e^{2x}+1}{e^{2x}-1}\leq 2 \Leftrightarrow e^{2x}+1\leq 2e^{2x}-2 \Leftrightarrow e^{2x}\geq 3$ at $x\geq 1$.

From (2.17) and (2.18) it follows that

$$M_G f(ch\,x)\leq c_\lambda M_\mu f(ch\,x), 0<r\leq c, \quad 0\leq x<\infty. \tag{3.19}$$

Now we consider case when $c<r<\infty$.

Note that for $ch\,(x-r)\leq z\leq ch\,(x+r)$ and $x\geq 2r$ the function $f(z)=\dfrac{ch\,r-zch\,x}{\sqrt{z^2-1}sh\,x}$ has maximum equal $-\dfrac{\sqrt{ch^2\,x-ch^2\,r}}{sh\,x}$.

Indeed,

$$f'(z)=-\frac{\sqrt{z^2-1}sh\,xch\,x+\frac{z}{\sqrt{z^2-1}}sh\,x(ch\,r-zch\,x)}{(z^2-1)sh^2\,x}$$

$$=-\frac{(z^2-1)sh\,xch\,x+zsh\,xch\,r-z^2sh\,xch\,x}{(z^2-1)^{\frac{3}{2}}sh^2\,x}=\frac{ch\,x-zch\,r}{(z^2-1)^{\frac{3}{2}}sh\,x}=0\Leftrightarrow z=\frac{ch\,x}{ch\,r}.$$

At this point the function $f(z)$ has maximum equal to

$$f_{\max}(z)=f\left(\frac{ch\,x}{ch\,r}\right)=\frac{ch^2\,r-ch^2\,x}{\sqrt{ch^2\,x-ch^2\,r}\cdot sh\,x}=-\frac{\sqrt{ch^2\,x-ch^2\,r}}{sh\,x}$$

$$=-\frac{ch\,x}{sh\,x}\sqrt{1-\left(\frac{ch\,r}{ch\,x}\right)^2}\sim-\frac{sh\,x}{ch\,x}, \tag{3.20}$$

as

$$\lim_{x\to\infty}\frac{sh\,x}{ch\,x}=\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=1.$$

From (2.15) and (2.20) we obtain

$$A(x,r)\leq\int\limits_{-1}^{\frac{ch\,r-zch\,x}{\sqrt{z^2-1}\;sh\,x}}(1+u)^{\lambda-1}du\leq\int\limits_{-1}^{-\frac{\sqrt{ch^2\,x-ch^2 z}}{sh\,x}}(1+u)^{\lambda-1}du$$

$$\sim\int\limits_{-1}^{-\frac{sh\,x}{ch\,x}}(1+u)^{\lambda-1}du=\frac{1}{\lambda}\left(1-\frac{sh\,x}{ch\,x}\right)^\lambda\leq\frac{1}{\lambda}\left(1-\frac{sh^2\,x}{ch^2\,x}\right)^\lambda$$

$$=\frac{1}{\lambda}\,(ch\,x)^{-2\lambda},\; x\to\infty. \tag{3.21}$$

Now, taking into account Lemmas 1.1(b) and 1.2(b) and also inequalities (2.12) and (2.21), we get

$$\frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \le c_\lambda \begin{cases} \dfrac{ch^{2\lambda} r}{ch^{4\lambda}\dfrac{r}{2}} \\ \dfrac{ch^{2\lambda} x ch^{2\lambda} r}{ch^{2\lambda} x ch^{4\lambda}\dfrac{r}{2}} \end{cases} \le c_\lambda, \quad c < r < \infty. \tag{3.22}$$

Applying (2.22) we easy obtain

$$M_G f(ch\, x) = \sup_{r>c} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\, t} |f(ch\, x)|\, d\mu(t)$$

$$= \sup_{r>c} \frac{|H(x,r)|_\lambda}{|H(0,r)|_\lambda} \cdot \frac{1}{|H(x,r)|_\lambda} \int_{|x-r|}^{x+r} |f(ch\, t)|\, sh^{2\lambda} t dt$$

$$\le c_\lambda \frac{1}{|H(x,r)|_\lambda} \int_{H(x,r)} |f(ch\, t)|\, d\mu(t) = c_\lambda M_\mu f(ch\, x). \tag{3.23}$$

Combining (2.19) and (2.23) we get

$$M_G f(ch\, x) = \sup_{0<r<\infty} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\, t} |f(ch\, x)|\, d\mu(t)$$

$$\le \sup_{0<r\le c} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\, t} |f(ch\, x)|\, d\mu(t)$$

$$+ \sup_{r>c} \frac{1}{|H(0,r)|_\lambda} \int_0^r A^\lambda_{ch\, t} |f(ch\, x)|\, d\mu(t) \le c_\lambda M_\mu f(ch\, x).$$

□

Theorem 3.2. *a) If $f \in L_{1,\lambda}[0,\infty)$, then for all $\alpha > 0$*

$$|\{x : M_G f(ch\, x) > \alpha\}|_\lambda \le \frac{c_\lambda}{\alpha} \int_0^\infty |f(ch\, t)|\, sh^{2\lambda} t dt = \frac{c_\lambda}{\alpha} \|f\|_{L_{1,\lambda}[0,\infty)},$$

where $c_\lambda > 0$ and depends only on λ.

b) If $f \in L_{p,\lambda}[0,\infty)$, $1 < p < \infty$, then $M_G f(ch\, x) \in L_{p,\lambda}[0,\infty)$ and $\|M_G f\|_{L_{p,\lambda}[0,\infty)} \le c_\lambda \|f\|_{L_{p,\lambda}[0,\infty)}$.

Corollary 3.1. *If $f \in L_{p,\lambda}[0,\infty)$, $1 \le p \le \infty$, then*

$$\lim_{r\to 0} \frac{1}{|H(0,r)|_\lambda} \int_{H(0,r)} A^\lambda_{ch\, t} f(ch\, x) sh^{2\lambda} t dt = f(ch\, x),$$

for a. e. $x \in [0,\infty)$.

Proof. We need to introduce one maximal function defined on a space of homogeneous type. We mean a topological space X equipped with a continuous preudometric ρ and a positive measure μ, satisfying the doubling condition

$$\mu\left(E(x,2r)\right) \le C\mu\left(E(x,r)\right), \tag{3.24}$$

with a constant $C-$ independent of x and $r > 0$.

Here $E(x,r) = \{y \in X : \rho(x,r) = |x-y| < r\}$.

Let (X, ρ, μ) is a space of homogeneous type. Let us define

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu E(x,r)} \int\limits_{E(x,r)} |f(t)| d\mu(t).$$

It is well known that the maximal function M_μ is weak $(1,1)$ and is bounded on $L_p\left(X, d\mu\right)$ for $1 < p < \infty$ (see [7]). The measure of maximal function $M_\mu f(ch\, x)$ which was introduced at the beginning of Section 1

$$\mu H(x,r) = |H(x,r)|_\lambda = \int\limits_{H(x,r)} sh^{2\lambda}\, t dt,$$

where

$$H(x,r) = \begin{cases} (x-r, x+r), x-r > 0; \\ \\ (0, x+r), \quad x - r < 0, \end{cases}$$

obviously satisfies the condition (2.24), but then the confirmation of Theorem 2.2 follows from Theorem 2.1. □

The proof of Corollary 2.1. At first let us show that for any function $f \in L_{p,\lambda}[0,\infty), 1 \le p \le \infty$, representation $ch\, t \mapsto A^\lambda_{ch\, t} f$ from $\mathbb{R}$ into $L_{p,\lambda}$ continuous, that is

$$\|A^\lambda_{ch\, t} f - f\|_{L_{p,\lambda}} \to 0 \text{ as } t \to 0. \tag{3.25}$$

Let $f(x)$ is a continuous function defined for $[a,b] \subset [0,\infty)$. Consider the function

$$y(t,x,\varphi) = ch\, t ch\, x - sh\, t sh\, x cos\, \varphi.$$

Hence we have

$$|y(t,x,\varphi) - y(0,x,\varphi)| = |ch\, t ch\, x - sh\, t sh\, x cos\, \varphi - ch\, x|$$

$$= |(ch\, t - 1) ch\, x - sh\, t sh\, x cos\, \varphi - ch\, x| \le 2sh^2 \frac{t}{2} ch\, x + 2sh\, \frac{t}{2} ch\, \frac{t}{2} sh\, x$$

$$\le 2sh\, \frac{t}{2} \left(sh\, \frac{t}{2} ch\, x + ch\, \frac{t}{2} sh\, x \right) = 2sh\, \frac{t}{2} sh(\frac{t}{2} + x)$$

$$\le 2sh\, \frac{t}{2} sh \left(\frac{t}{2} + b \right) \to 0 \quad t \to 0. \tag{3.26}$$

On the strength of uniform continuity of the function $f(x)$ on segment $[a,b]$ for any $\varepsilon > 0$ one may choose the number $\delta > 0$ such that

$$|f[y(t,x,\varphi)] - f[y(0,x,\varphi)]| < \varepsilon, \text{ if}$$

$$|y(t,x,\varphi) - y(0,x,\varphi)| < \delta, \text{ (that follows from (3.26))}.$$

Then we have

$$\left| A^\lambda_{ch\, t} f(ch\, x) - f(ch\, x) \right|$$

$$\leq \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda\right)}\int_0^\pi |f[y(t,x,\varphi)]-f[y(0,x,\varphi)]|(sin\,\varphi)^{2\lambda-1}d\varphi<\varepsilon.$$

It follows that

$$\|A^\lambda_{ch\,t}f-f\|_{\infty,\lambda}=\sup_{x\in[a,b]}|A^\lambda_{ch\,t}f(ch\,x)-f(ch\,x)|<\varepsilon.$$

And for $1\leq p<\infty$

$$\|A^\lambda_{ch\,t}f-f\|_{L_{p,\lambda}[a,b]}=\left(\int_a^b|A^\lambda_{ch\,t}f(ch\,x)-f(ch\,x)|^p sh^{2\lambda}xdx\right)^{\frac{1}{p}}$$

$$<\varepsilon\left(\int_a^b sh^{2\lambda}\,xdx\right)^{\frac{1}{p}}<c_{p,\lambda}\varepsilon.$$

Thus for any continuous function definited by segment $[a,b]\subset[0,\infty)$ and for any number $\varepsilon>0$ the inequality is valid:

$$\|A^\lambda_{ch\,t}f-f\|_{L_{p,\lambda}[a,b]}<\varepsilon\quad 1\leq p\leq\infty. \tag{3.27}$$

It is known the set of all continuous functions with compact support in $[0,\infty)$ is dense in $L_{p,\lambda}[0,\infty)$. Therefore for any number $\varepsilon>0$ there exists a continuous function with compact support in $[0,\infty)$, such that

$$\|f-f_\varepsilon\|_{L_{p,\lambda}[0,\infty)}<\varepsilon. \tag{3.28}$$

We denote $g_\varepsilon=f-f_\varepsilon$. Then $g_\varepsilon\in L_{p,\lambda}[0,\infty)$ and

$$\|g_\varepsilon\|_{L_{p,\lambda}[0,\infty)}<\varepsilon. \tag{3.29}$$

Thus, if $f\in L_{p,\lambda}[0,\infty)$, then for any number $\varepsilon>0$ exists a continuous function f_ε with the compact support and function $g_\varepsilon\in L_{p,\lambda}[0,\infty)$ with condition $\|g_\varepsilon\|_{L_{p,\lambda}[0,\infty)}<\varepsilon$, such that $f=f_\varepsilon+g_\varepsilon$.

Hence we have $A^\lambda_{ch\,t}f(ch\,x)=A^\lambda_{ch\,t}f_\varepsilon(ch\,x)+A^\lambda_{ch\,t}g_\varepsilon(ch\,x)-f(ch\,x)+f_\varepsilon(ch\,x)-f_\varepsilon(ch\,x)$, from which it follows that

$$\|A^\lambda_{ch\,t}f-f\|_{L_{p,\lambda}[0,\infty)}\leq\|A^\lambda_{ch\,t}f_\varepsilon-f_\varepsilon\|_{L_{p,\lambda}[0,\infty)}$$

$$+\|f-f_\varepsilon\|_{L_{p,\lambda}[0,\infty)}+\|A^\lambda_{ch\,t}g_\varepsilon\|_{L_{p,\lambda}[0,\infty)}.$$

Now, taking into account that (see [26], Lemma 2)

$$\|A^\lambda_{ch\,t}f_\varepsilon\|_{L_{p,\lambda}[0,\infty)}\leq\|f\|_{L_{p,\lambda}[0,\infty)},t\in[0,\infty),1\leq p\leq\infty$$

and also the inequality (2.27), (2.28) (2.29), we get

$$\|A^\lambda_{ch\,t}f_\varepsilon-f\|_{L_{p,\lambda}[0,\infty)}\leq 3\varepsilon,$$

from which follows (2.25).

By the locality of the problem, one can account that $f\in L_{1,\lambda}[0,\infty)$. In general case one can multiply f by characteristic function of interval $H(0,r)=[0,r)$ and obtain the required convergence almost everywhere interior to this interval and by tending r to infinity one could obtain it on the whole interval $[0,\infty)$.

Suppose for any $r>0$ and for any $x\in[0,\infty)$

$$f_r(ch\,x)=\frac{1}{|H(0,r)|_\lambda}\int\limits_{H(0,r)}A^\lambda_{ch\,t}f(ch\,x)sh^{2\lambda}\,tdt.$$

Let $r_0 > 0, H = H(0, r_0)$. According to the Minkowski generalized inequality and (2.25), we obtain

$$\|f_r - f\|_{L_{1,\lambda}(B)} = \|\frac{1}{|H(0,r)|_\lambda} \int\limits_{H(0,r)} \left(A^\lambda_{ch\,t} f(ch\,x) - f(ch\,x)\right) sh^{2\lambda}\,tdt\|_{L_{1,\lambda}(B)}$$

$$\leq \frac{1}{|H(0,r)|_\lambda} \int\limits_{H(0,r)} \|A^\lambda_{ch\,t} f - f\|_{L_{1,\lambda}(B)} sh^{2\lambda}\,tdt$$

$$\leq \sup_{|t|\leq r_0} \|A^\lambda_{ch\,t} f - f\|_{L_{1,\lambda}(B)} \to 0, \text{ at } r_o \to +0.$$

It means that there is such sequence r_k, that $r_k \to +0$, $(k \to \infty)$ and

$$\lim_{k\to\infty} f_{r_k}(ch\,x) = f(ch\,x)$$

almost everywhere in $x \in [0, \infty)$.

Now let's prove that $\lim\limits_{r\to+0} f_r(ch\,x)$ exists almost everywhere. For this purpose for any $x \in [0, \infty)$

$$\Omega_f(ch\,x) = \mid \overline{\lim_{r\to+0}} f_r(ch\,x) - \underline{\lim_{r\to+0}} f_r(ch\,x) \mid$$

the oscillation of f_r at the point x as $r \to +0$.

If g is a continuous function with compact support on $[0, \infty)$, then g_r is convergent to g and consequently $\Omega_g \equiv 0$ is identically equal to zero in this case.

Further, if $g \in L_{1,\lambda}[0, \infty)$, then, according to the statement of Theorem 2.2

$$\mid \{x \in [0, \infty) : M_G g(ch\,x) > \varepsilon\} \mid_\lambda \leq \frac{c}{\varepsilon} \|g\|_{L_{1,\lambda}[0,\infty)},\ g \in L_{1,\lambda}[0, \infty).$$

On the other hand it is obvious that $\Omega g(ch\,x) \leq 2M_G g(ch\,x)$. Thus

$$\mid \{x \in [0, \infty) : \Omega_g(ch\,x) > \varepsilon\} \mid_\lambda \leq \frac{2c}{\varepsilon} \|g\|_{L_{1,\lambda}[0,\infty)},\ g \in L_{1,\lambda}[0, \infty).$$

By the same way as it was proved above, any function $f \in L_{p,\lambda}[0, \infty)$ can be written in form $f = h + g$, where h is continuous function and has compact support on $[0, \infty)$, and $g \in L_{p,\lambda}[0, \infty)$, moreover $\|g\|_{L_{p,\lambda}[0,\infty)} < \varepsilon$, for any $\varepsilon > 0$. But $\Omega \leq \Omega_h + \Omega_g \quad \Omega_h \equiv 0$, however h is continuous. Therefore it follows that

$$\mid \{x \in [0, \infty) : \Omega_g(ch\,x) > \varepsilon\} \mid_\lambda \leq \frac{c}{\varepsilon} \|g\|_{L_{1,\lambda}[0,\infty)}.$$

Taking in inequality $\|g\|_{L_{1,\lambda}[0,\infty)} < \varepsilon$ the number ε arbitrarily small, we get $\Omega f = 0$ almost everywhere on $[0, \infty)$. Consequently, $\lim\limits_{r\to 0} f_r(ch\,x)$ exists almost everywhere on $[0, \infty)$, what was required to prove.

Remark 3.1. Theorem 2.2 was proved earlier by W. C. Connett and A. L. Schwartz [9] for the Jacobi-type hypergroups.

Remark 3.2. If $f \in L_{1,\lambda}[0, \infty)$, then (see [20], Theorem 2.1)

$$\lim_{r\to 0} \frac{1}{\left(sh\,\frac{r}{2}\right)^{2\lambda+1}} \int_0^r \left| A^\lambda_{ch\,t} f(ch\,x) - f(ch\,x)\right| sh^{2\lambda}\,tdt = 0,$$

almost everywhere on $x \in [0, \infty)$.

This implies that for any $\varepsilon > 0$ one can find $\delta > 0$ such that for all $r < \delta$ the following inequality holds:

$$\frac{1}{\left(sh\,\frac{r}{2}\right)^{2\lambda+1}} \int_0^r \left| A^{\lambda}_{ch\,t} f(ch\,x) - f(ch\,x) \right| sh^{2\lambda}\,tdt < \varepsilon.$$

But then from Lemma 1.1 (a) we obtain

$$\left| \frac{1}{|H(0,r)|_{\lambda}} \int\limits_{H(0,r)} \left[A^{\lambda}_{ch\,t} f(ch\,x) - f(ch\,x) \right] sh^{2\lambda}\,tdt \right|$$

$$\leq \frac{1}{\left(sh\,\frac{r}{2}\right)^{2\lambda+1}} \int_0^r \left| A^{\lambda}_{ch\,t} f(ch\,x) - f(ch\,x) \right| sh^{2\lambda}\,tdt < \varepsilon,$$

for all $r < \delta$, which means that Corollary 2.1 is valid under assumption $f \in L_{1,\lambda}[0,\infty)$.

4. Some Morrey embeddings, associated with the Gegenbauer expansion

We shall define function spaces, generated by the Gegenbauer expansion G.

Definition 4.1. [13, 14] Let $1 \leq p < \infty$, $0 \leq \gamma \leq 2\lambda + 1$, $[r]_1 = \min\{1, r\}$. We denote by $L_{p,\lambda,\gamma}([0,\infty), G)$ Morrey-Gegenbauer spaces (G- Morrey spaces) and by $\widetilde{L}_{p,\lambda,\gamma}([0,\infty), G)$ modified G- Morrey spaces which are the sets of functions f locally integrable on $[0,\infty)$ with finite norms

$$\|f\|_{L_{p,\lambda,\gamma}([0,\infty),G)} = \sup_{x,r\in(0,\infty)} \left(\left(sh\,\tfrac{r}{2}\right)^{-\gamma} \int\limits_{H(0,r)} (A^{\lambda}_{ch\,t} |f(ch\,x)|)^p sh^{2\lambda}\,tdt \right)^{\frac{1}{p}}$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}([0,\infty),G)} = \sup_{x,r\in(0,\infty)} \left(\left[sh\,\tfrac{r}{2}\right]_1^{-\gamma} \int\limits_{H(0,r)} A^{\lambda}_{ch\,t} |f(ch\,x)|)^p sh^{2\lambda}\,tdt \right)^{\frac{1}{p}}.$$

Definition 4.2. [13, 14] We denote by BMO $([0,\infty), G)$ the BMO-Gegenbauer spaces (G-BMO space) as the set of functions locally integrable on $[0,\infty)$, with finite norm

$$\|f\|_{*,G} = \sup_{x,r\in[0,\infty)} \frac{1}{|H(0,r)|_{\lambda}} \int\limits_{H(0,r)} \left| A^{\lambda}_{ch\,t} f(ch\,x) - \underset{H(0,r)}{f}(ch\,x) \right| sh^{2\lambda}\,tdt,$$

where

$$f_{H(0,r)}(ch\,x) = \frac{1}{|H(0,r)|_{\lambda}} \int\limits_{H(0,r)} A^{\lambda}_{ch\,t} |f(ch\,x)|\, sh^{2\lambda}\,tdt.$$

Note that

$$\widetilde{L}_{p,\lambda,0}[0,\infty) = L_{p,\lambda}[0,\infty),\ L_{p,\lambda,2\lambda+1}[0,\infty) = L_{\infty,\lambda},$$

$$\widetilde{L}_{p,\lambda,\gamma}[0,\infty) \subseteq L_{p,\lambda}[0,\infty) \text{ and } \|f\|_{L_{p,\lambda}[0,\infty)} \leq \|f\|_{\widetilde{L}_{p,\lambda,\gamma}[0,\infty)}.$$

Theorem 4.1. *Let* $1 \le p < \infty$, $0 \le \gamma \le 2\lambda+1$ *and* $\alpha p = 2\lambda+1-\gamma$. *Then*

$$L_{p,\lambda,\gamma}[0,\infty) \subset L_{1,\lambda,2\lambda+1-\alpha}[0,\infty) \text{ and } \|f\|_{L_{1,\lambda,2\lambda+1-\alpha}} \le c_{\lambda,p}\|f\|_{L_{p,\lambda,\gamma}}.$$

Proof. Let $f \in L_{p,\lambda,\gamma}[0,\infty)$, $1 \le p < \infty$, $0 \le \gamma \le 2\lambda+1$, $1/p+1/q = 1$ and $\alpha p = 2\lambda+1-\gamma$.

Applying Holder's inequality we have

$$\int\limits_{H(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|\, sh^{2\lambda} t dt$$

$$\le \left(\int\limits_{H(0,r)} (A^{\lambda}_{ch\,t}|f(ch\,x)|)^p sh^{2\lambda} t dt\right)^{\frac{1}{p}} \left(\int\limits_{H(0,r)} sh^{2\lambda} t dt\right)^{\frac{1}{q}}. \tag{4.1}$$

From Lemma 1.1 (b) it follows that for $r > c$

$$|H(0,r)|_\lambda \le c_\lambda ch^{4\lambda}\frac{r}{2} < c_\lambda \left(ch\,\frac{r}{2}\right)^{2\lambda+1} \le c_\lambda\left(3sh\,\frac{r}{2}\right)^{2\lambda+1} = c_\lambda\left(sh\,\frac{r}{2}\right)^{2\lambda+1}. \tag{4.2}$$

From Lemma 1.1 (a) and (3.2) it follows that for any $0 < r < \infty$

$$|H(0,r)|_\lambda \le c_\lambda \left(sh\,\frac{r}{2}\right)^{2\lambda+1}. \tag{4.3}$$

Taking into account (3.3) and (3.1) we obtain

$$\int\limits_{H(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|\, sh^{2\lambda} t dt$$

$$\le c_{\lambda,p}\left(sh\,\frac{r}{2}\right)^{\frac{2\lambda+1}{q}}\left(\int\limits_{H(0,r)} (A^{\lambda}_{ch\,t}|f(ch\,x)|)^p sh^{2\lambda} t dt\right)^{\frac{1}{p}}.$$

Further,

$$\left(sh\,\frac{r}{2}\right)^{\alpha-2\lambda-1}\int\limits_{H(0,r)} A^{\lambda}_{ch\,t}|f(ch\,x)|\, sh^{2\lambda} t dt$$

$$\le c_{\lambda,p}\left(sh\,\frac{r}{2}\right)^{\alpha-2\lambda-1+\frac{2\lambda+1}{q}}\left(\int\limits_{H(0,r)} (A^{\lambda}_{ch\,t}|f(ch\,x)|)^p sh^{2\lambda} t dt\right)^{\frac{1}{p}}$$

$$= c_{\lambda,p}\left(sh\,\tfrac{r}{2}\right)^{\alpha-2\lambda-1+(2\lambda+1)\left(1-\frac{1}{p}\right)}\left(\int\limits_{H(0,r)} (A^{\lambda}_{ch\,t}|f(ch\,x)|)^p sh^{2\lambda} t dt\right)^{\frac{1}{p}}$$

$$= c_{\lambda,p}\left(sh\,\tfrac{r}{2}\right)^{\alpha-\frac{2\lambda+1}{p}}\left(\int\limits_{H(0,r)} (A^{\lambda}_{ch\,t}|f(ch\,x)|)^p sh^{2\lambda} t dt\right)^{\frac{1}{p}}$$

$$= c_{\lambda,p}\left\{\left(sh\,\tfrac{r}{2}\right)^{-\gamma}\int\limits_{H(0,r)} (A^{\lambda}_{ch\,t}|f(ch\,x)|)^p sh^{2\lambda} t dt\right\}^{\frac{1}{p}} = c_{\lambda,p}\|f\|_{L_{p,\lambda,\gamma}}.$$

Thus

$$f \in L_{1,\lambda,2\lambda+1-\alpha}[0,\infty) \text{ and } \|f\|_{L_{1,\lambda,2\lambda+1-\alpha}} \le c_{\lambda,p}\|f\|_{L_{p,\lambda,\gamma}}.$$

□

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