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TWO-WEIGHTED INEQUALITY FOR p-ADMISSIBLE $B_{k,n}$–SINGULAR OPERATORS IN WEIGHTED LEBESGUE SPACES

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Abstract. In this paper, we study the boundedness of p-admissible singular operators, associated with the Laplace-Bessel differential operator $B_{k,n} = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum\limits_{j=1}^{k} \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}$ (p-admissible $B_{k,n}$–singular operators) on a weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ including their weak versions. These conditions are satisfied by most of the operators in harmonic analysis, such as the $B_{k,n}$–maximal operator, $B_{k,n}$–singular integral operators and so on. Sufficient conditions on weighted functions ω and ω_1 are given so that p-admissible $B_{k,n}$–singular operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$ and weak p-admissible $B_{k,n}$–singular operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le p < \infty$.

1. Introduction

The singular integral operators considered by S. Mihlin [26] and A. Calderon and A. Zygmund [7] are playing an important role in the theory of Harmonic Analysis and in particular, in the theory of partial differential equations. M. Klyuchantsev [25] and I. Kipriyanov and M. Klyuchantsev [24] have firstly introduced and investigated the boundedness in L_p-spaces of multidimensional singular integrals, generated by the $B_{1,n}$–Laplace-Bessel differential operator ($B_{1,n}$–singular integrals), where

$$B_{1,n} = B_1 + \sum_{j=2}^{n} \frac{\partial^2}{\partial x_j^2}, \; B_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\gamma}{x_1} \frac{\partial}{\partial x_1}, \; \gamma > 0.$$

I.A. Aliev and A.D. Gadjiev [5], A.D. Gadjiev and E.V. Guliyev [11] and E.V. Guliyev [13] have studied the boundedness of $B_{1,n}$ singular integrals in weighted L_p-spaces with radial and general weights consequently. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator $B_{k,n}$–which is known as an important differential operator in analysis and its applications, have been the research areas of many mathematicans

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such as I. Kipriyanov and M. Klyuchantsev [24, 25], L. Lyakhov [29, 30], A.D. Gadjiev and I.A. Aliev [4, 5], I.A. Aliev and S. Bayrakci [2, 3], V.S. Guliyev [15, 16, 17] and others.

In the paper, we shall prove the boundedness of p-admissible singular operators, associated with the Laplace-Bessel differential operator $B_{k,n} = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum\limits_{j=1}^{k} \frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$ (p-admissible $B_{k,n}$–singular operators) on a weighted L_p spaces. Sufficient conditions on weighted functions ω and ω_1 are given so that p-admissible $B_{k,n}$–singular operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$ and weak p-admissible $B_{k,n}$–singular operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le p < \infty$. Note that, our results in the case $k = 1$ were proved in [13], which is some generalization of the paper by I. A. Aliev, A. D. Gadjiev [5].

We point out that the p-admissible $B_{k,n}$–singular operators (see Theorem 2.1). These conditions are satisfied by many interesting operators in harmonic analysis, such as the $B_{k,n}$–Riesz transforms (see [9, 10]), $B_{k,n}$–singular integral operators (for example, for $k = 1$ see [5, 11, 13, 24, 25]), $B_{k,n}$–Hardy–Littlewood maximal operators ([18], for $n = k = 1$ see [32], for $k = 1$ see [17] and for $k = n$ see [15]) and so on.

2. Notations and Background

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$ are vectors in $\mathbb{R}^n$, $(x, \xi) = x_1\xi_1 + \dots + x_n\xi_n$, $|x| = \sqrt{(x, x)}$, $x = (x', x'')$, $x' = (x_1 \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_n)$. Let $\mathbb{R}^k_{++} = \{x \in \mathbb{R}^k : \ x_1 > 0 \dots, x_k > 0\}$, $\mathbb{R}^n_{k,+} = \{x = (x_1, \dots, x_n) : \ x_1, x_2, \dots, x_k > 0\}$, $1 \le k \le n$, $S_{k,+} = \{x \in \mathbb{R}^n_{k,+} : \ |x| = 1\}$.

For $x \in \mathbb{R}^n_{k,+}$ and $r > 0$, we denote by $E(x, r) = \{y \in \mathbb{R}^n_{k,+} : \ |x - y| < r\}$ the open ball centered at x of radius r, and by ${}^{\complement}E(x, r) = \mathbb{R}^n_{k,+} \setminus E(x, r)$ denote its complement, $E'(x', r) = \{y' \in \mathbb{R}^k_{++} : \ |x' - y'| < r\}$, ${}^{\complement}E'(x', r) = \mathbb{R}^k_{++} \setminus E'(x', r)$.

For measurable set $E \subset \mathbb{R}^n_{k,+}$ denote $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E(0, r)|_\gamma = \omega(n, \gamma) r^{n+|\gamma|}$, where $\gamma = (\gamma_1 \dots, \gamma_k)$, $(x')^\gamma = x_1^{\gamma_1} \dots x_k^{\gamma_k}$ and $\omega(n, \gamma) = |E(0, 1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega : \ \mathbb{R}^n_{k,+} \to \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ the set of all measurable functions f on $\mathbb{R}^n_{k,+}$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})} \equiv \|f\|_{p,\omega,\gamma;\mathbb{R}^n_{k,+}} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) (x')^\gamma dx \right)^{1/p}, \qquad 1 \le p < \infty$$

is finite. For $\omega = 1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_{k,+})$, and the norm $\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})}$ by $\|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})}$.

The operator of generalized shift ($B_{k,n}$–shift operator) is defined by the following way (see [18], [30]):

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right) d\nu(\beta),$$

where
$C_{\gamma,k} = \pi^{-\frac{k}{2}}\Gamma^{-1}\left(\frac{|\gamma|}{2}\right)\prod_{i=1}^{k}\Gamma(\frac{\nu_i+1}{2})$, $(x',y')_\beta = ((x_1,y_1)_{\beta_1}...(x_k,y_k)_{\beta_k}), (x_i,y_i)_{\beta_i} = (x_i^2-2x_iy_i\cos\beta_i+y_i^2)^{1/2}$, $1\le i\le k$, $d\nu(\beta)=\prod_{i=1}^{k}\sin^{\gamma_i-1}\beta_i\,d\beta_1\dots d\beta_k$.

Note that this shift operator is closely connected with $B_{k,n}$–Laplace-Bessel singular differential operators (see [18], [30]).

The translation operator T^y generated the corresponding $B_{k,n}$-convolution

$$(f\otimes g)(x)=\int_{\mathbb{R}^n_{k,+}} f(y)[T^y g(x)](y')^\gamma dy,$$

for which the Young inequality

$$\|f\otimes g\|_{L_{r,\gamma}}\le\|f\|_{L_{p,\gamma}}\|g\|_{L_{q,\gamma}}, \quad 1\le p,q,r\le\infty, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$$

holds.

Lemma 2.1. [28] *Let $1\le p\le\infty$. Then for all $y\in\mathbb{R}^n_{k,+}$, T^yf belongs $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and*

$$\|T^yf(\cdot)\|_{L_{p,\gamma}}\le\|f\|_{L_{p,\gamma}}. \tag{2.1}$$

Definition 2.1. A function K defined on $\mathbb{R}^n_{k,+}$, is said to be $B_{k,n}$–singular kernel in the space $\mathbb{R}^n_{k,+}$ if

i) $K\in C^\infty(\mathbb{R}^n_{k,+})$;

ii) $K(rx)=r^{-n-|\gamma|}K(x)$ for each $r>0$, $x\in\mathbb{R}^n_{k,+}$;

iii) $\int_{S_{k,+}}K(x)x^\gamma d\sigma(x)=0$, where $d\sigma$ is the element of area of the $S_{k,+}$.

The operator T is called sublinear, if for all $\lambda,\mu>0$ and for all f and g in the domain of T

$$|T(\lambda f+\mu g)(x)|\le\lambda|Tf(x)|+\mu|Tg(x)|.$$

Definition 2.2. (p-admissible $B_{k,n}$–singular operator). Let $1<p<\infty$. A sublinear operator T will be called p-admissible $B_{k,n}$–singular operator, if:

1) T satisfies the size condition of the form

$$\chi_{E(x,r)}(z)\left|T\Big(f\chi_{\mathbb{R}^n_{k,+}\setminus E(x,2r)}\Big)(z)\right| \le C\chi_{E(x,r)}(z)\int_{\mathbb{R}^n_{k,+}\setminus E(x,2r)}T^y|x|^{-n-|\gamma|}\,|f(y)|\,(y')^\gamma dy \tag{2.2}$$

for $x\in\mathbb{R}^n_{k,+}$ and $r>0$;

2) T is bounded in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$.

Definition 2.3. (weak p-admissible $B_{k,n}$–singular operator). Let $1\le p<\infty$. A sublinear operator T will be called the weak p-admissible $B_{k,n}$–singular operator, if:

1) T satisfies the size condition (2.2).

2) T is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to the weak $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$.

Remark 2.1. Note that p-admissible singular operators were introduced and their boundedness on vanishing generalized Morrey spaces was studied in [31]. Also Φ-admissible singular operators and weak Φ-admissible singular operators were introduced and their boundedness on generalized Orlicz-Morrey spaces was studied in [19, 21].

First, we establish the boundedness in weighted $L_{p,\gamma}$ spaces for a large class of p-admissible $B_{k,n}$–singular operator.

Theorem 2.1. *Let $p \in (1,\infty)$ and T be a p-admissible $B_{k,n}$–singular operators.*
Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and the following three conditions are satisfied:
(a) there exist $b>0$ such that

$$\sup_{|x|/8<|y|\le 8|x|} \omega_1(y) \le b\,\omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n_{k,+},$$

$$(b)\quad \mathcal{A} \equiv \sup_{r>0}\left(\int_{\complement E(0,2r)} \omega_1(x)|x|^{-(n+|\gamma|)p}(x')^\gamma dx\right)\left(\int_{E(0,r)} \omega^{1-p'}(x)(x')^\gamma dx\right)^{p-1} <\infty,$$

$$(c)\quad \mathcal{B} \equiv \sup_{r>0}\left(\int_{E(0,r)} \omega_1(x)(x')^\gamma dx\right)\left(\int_{\complement E(0,2r)} \omega^{1-p'}(x)|x|^{-(n+|\gamma|)p'}(x')^\gamma dx\right)^{p-1} <\infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(x)(x')^\gamma dx \le c \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx. \tag{2.3}$$

Moreover, condition (a) can be replaced by the condition
(a') there exist $b>0$ such that

$$\omega_1(x)\left(\sup_{|x|/8<|y|\le 8|x|} \frac{1}{\omega(y)}\right) \le b \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. For $l \in Z$ we define $E_l = \{x \in \mathbb{R}^n_{k,+} : 2^l < |x| \le 2^{l+1}\}$, $E_{l,1} = \{x \in \mathbb{R}^n_{k,+} : |x| \le 2^{l-1}\}$, $E_{l,2} = \{x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x| \le 2^{l+2}\}$, $E_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x| > 2^{l+2}\}$. Then $E_{l,2} = E_{l-1} \cup E_l \cup E_{l+1}$ and the multiplicity of the covering $\{E_{l,2}\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$\begin{aligned} |Tf(x)| &= \sum_{l\in Z} |Tf(x)|\chi_{E_l}(x) \le \sum_{l\in Z} |Tf_{l,1}(x)|\,\chi_{E_l}(x) \\ &+ \sum_{l\in Z} |Tf_{l,2}(x)|\,\chi_{E_l}(x) + \sum_{l\in Z} |Tf_{l,3}(x)|\,\chi_{E_l}(x) \\ &\equiv T_1 f(x) + T_2 f(x) + T_3 f(x), \end{aligned}$$

where χ_{E_l} is the characteristic function of the set E_l, $f_{l,i} = f\chi_{E_{l,i}}$, $i = 1,2,3$.

First we shall estimate $\|T_1 f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in E_l$, $y \in E_{k,1}$ we have $|y| \le 2^{l-1} \le |x|/2$. Moreover, $E_l \cap supp f_{l,1} = \varnothing$ and $|x-y| \ge |x|/2$. Hence by (2.2)

$$\begin{aligned} T_1 f(x) &\le c_0 \sum_{l \in Z} \left(\int_{\mathbb{R}^n_{k,+}} T^y |x|^{-n-|\gamma|} \, |f_{l,1}(y)| (y')^\gamma dy \right) \chi_{E_l} \\ &\le c_0 \int_{E(0,|x|/2)} |x-y|^{-n-|\gamma|} |f(y)| \, (y')^\gamma dy \\ &\le 2^{n+|\gamma|} c_0 |x|^{-n-|\gamma|} \int_{E(0,|x|/2)} |f(y)| \, (y')^\gamma dy \end{aligned}$$

for any $x \in E_l$. Hence we have

$$\begin{aligned} &\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x) \, (x')^\gamma dx \\ &\le \left(2^{n+|\gamma|} c_0\right)^p \int_{\mathbb{R}^n_{k,+}} \left(\int_{E(0,|x|/2)} |f(y)| \, (y')^\gamma dy \right)^p |x|^{-(n+|\gamma|)p} \omega_1(x) \, (x')^\gamma dx. \end{aligned}$$

Since $\mathcal{A} < \infty$, the Hardy inequality

$$\begin{aligned} &\int_{\mathbb{R}^n_{k,+}} \omega_1(x) |x|^{-(n+|\gamma|)p} \left(\int_{E(0,|x|/2)} |f(y)| \, (y')^\gamma dy \right)^p (x')^\gamma dx \\ &\le C \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) \, (x')^\gamma dx \end{aligned}$$

holds and $C \le c' \mathcal{A}$, where c' depends only on n and p. In fact the condition $\mathcal{A} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [8]). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x) \, (x')^\gamma dx \le c_1 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) \, (x')^\gamma dx. \tag{2.4}$$

where c_1 is independent of f.

Next we estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As is easy to verify, for $x \in E_l$, $y \in E_{l,3}$ we have $|y| > 2|x|$ and $|x-y| \ge |y|/2$. Since $E_l \cap supp f_{l,3} = \varnothing$, for $x \in E_l$ by (2.2) we obtain

$$\begin{aligned} T_3 f(x) &\le c_0 \int_{\complement_{E(0,2|x|)}} T^y |x|^{-n-|\gamma|} |f(y)| \, (y')^\gamma dy \\ &\le 2^{n+|\gamma|} c_0 \int_{\complement_{E(0,2|x|)}} |f(y)| |x-y|^{-n-|\gamma|} \, (y')^\gamma dy \\ &\le 2^{n+|\gamma|} c_0 \int_{\complement_{E(0,2|x|)}} |f(y)| |y|^{-n-|\gamma|} \, (y')^\gamma dy. \end{aligned}$$

Hence we have

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x)\ (x')^\gamma dx$$
$$\leq \left(2^{n+|\gamma|} c_0\right)^p \int_{\mathbb{R}^n_{k,+}} \left(\int_{\complement_{E(0,2|x|)}} |f(y)||y|^{-n-|\gamma|}\ (y')^\gamma dy \right)^p \omega_1(x)\ (x')^\gamma dx.$$

Since $\mathcal{B} < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n_{k,+}} \omega_1(x) \left(\int_{\complement_{E(0,2|x|)}} |f(y)||y|^{-n-|\gamma|}\ (y')^\gamma dy \right)^p (x')^\gamma dx$$
$$\leq C \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)\ (x')^\gamma dx$$

holds and $C \leq c'\mathcal{B}$, where c' depends only on n and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [8]). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x)\ (x')^\gamma dx \leq c_2 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)\ (x')^\gamma dx, \tag{2.5}$$

where c_2 is independent of f.

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of T and condition (a) we have

$$\begin{aligned}
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x)\ (x')^\gamma dx &= \int_{\mathbb{R}^n_{k,+}} \left(\sum_{l\in Z} |Tf_{l,2}(x)|\, \chi_{E_l}(x) \right)^p \omega_1(x)\ (x')^\gamma dx \\
&= \int_{\mathbb{R}^n_{k,+}} \left(\sum_{l\in Z} |Tf_{l,2}(x)|^p\, \chi_{E_l}(x) \right) \omega_1(x)\ (x')^\gamma dx \\
&= \sum_{l\in Z} \int_{E_l} |Tf_{l,2}(x)|^p\, \omega_1(x)\ (x')^\gamma dx \\
&\leq \sum_{l\in Z} \sup_{y\in E_l} \omega_1(y) \int_{\mathbb{R}^n_{k,+}} |Tf_{l,2}(x)|^p\ (x')^\gamma dx \\
&\leq \|T\|^p \sum_{l\in Z} \sup_{y\in E_l} \omega_1(y) \int_{\mathbb{R}^n_{k,+}} |f_{l,2}(x)|^p\ (x')^\gamma dx \\
&= \|T\|^p \sum_{l\in Z} \sup_{y\in E_l} \omega_1(y) \int_{E_{l,2}} |f(x)|^p\ (x')^\gamma dx,
\end{aligned}$$

where $\|T\| \equiv \|T\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{p,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in E_{l,2}$, $2^{l-1} < |x| \leq 2^{l+2}$, we have by condition (a)

$$\sup_{y\in E_l} \omega_1(y) = \sup_{2^{l-1}<|y|\leq 2^{l+2}} \omega_1(y) \leq \sup_{|x|/8<|y|\leq 8|x|} \omega_1(y) \leq b\,\omega(x)$$

for almost all $x \in E_{l,2}$. Therefore

$$\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x)(x')^\gamma dx \le \|T\|^p b \sum_{l\in Z} \int_{E_{l,2}} |f(x)|^p \omega(x)(x')^\gamma dx$$
$$\le c_3 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx, \qquad (2.6)$$

where $c_3 = 3\|T\|^p b$, since the multiplicity of covering $\{E_{l,2}\}_{l\in Z}$ is equal to 3.

Inequalities (2.4), (2.5), (2.6) imply (2.3) which completes the proof. □

Similarly we prove the following weak variant of Theorem 2.1.

Theorem 2.2. *Let $p \in [1,\infty)$ and let T be a p-admissible $B_{k,n}$–singular operators. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied.*

Then there exists a constant c, independent of f, such that

$$\int_{\{x\in\mathbb{R}^n_{k,+}:|Tf(x)|>\lambda\}} \omega_1(x)(x')^\gamma dx \le \frac{c}{\lambda^p} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx \qquad (2.7)$$

for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$.

Let k is a $B_{k,n}$–singular kernel and K be the $B_{k,n}$–singular integral operator

$$Kf(x) = p.v. \int_{\mathbb{R}^n_{k,+}} T^y k(x) f(y)(y')^\gamma dy.$$

Then K is a p-admissible $B_{k,n}$–singular operator for $1 < p < \infty$ and weak p-admissible $B_{k,n}$–singular operators for $1 \le p < \infty$. Thus, we have

Corollary 2.1. *Let $p \in (1,\infty)$, K be a $B_{k,n}$–singular operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied. Then the operator K is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

Corollary 2.2. *Let $p \in [1,\infty)$, K be a $B_{k,n}$–singular operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied. Then the operator K is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.*

Remark 2.2. Note that, the conditions p-admissible $B_{k,n}$–singular operators are satisfied by many interesting operators in harmonic analysis, such as the $B_{k,n}$–maximal operator, $B_{k,n}$–singular integral operators, $B_{k,n}$–Riesz transforms and so on .

Theorem 2.3. *Let $p \in (1,\infty)$ and T be a p-admissible $B_{k,n}$–singular operators.*

Moreover, let $\omega(x')$, $\omega_1(x')$ be a weight functions on $\mathbb{R}^k_{++}$ and the following three conditions be satisfied

(a_1) *there exists a constant $b > 0$ such that*

$$\sup_{|x'|/8<|y'|<8|x'|} \omega_1(y') \le b\,\omega(x') \quad \textit{for a.e. } x' \in \mathbb{R}^k_{++},$$

(b_1) $$\mathcal{A}_1 \equiv \sup_{r>0} \left(\int_{\complement_{E'(0,2r)}} \omega_1(x')|x'|^{-(k+|\gamma|)p}(x')^\gamma dx' \right) \times \left(\int_{E'(0,r)} \omega^{1-p'}(x')(x')^\gamma dx' \right)^{p-1} < \infty,$$

(c_1) $$\mathcal{B}_1 \equiv \sup_{r>0} \left(\int_{E'(0,r)} \omega_1(x')(x')^\gamma dx' \right) \times \left(\int_{\complement_{E'(0,2r)}} \omega^{1-p'}(x')|x'|^{-(k+|\gamma|)p'}(x')^\gamma dx' \right)^{p-1} < \infty.$$

Then there exists a constant c, independent of f, such that for all $f \in L_{p,\omega}(\mathbb{R}^n_{k,+})$

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(x')(x')^\gamma dx \le c \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx. \tag{2.8}$$

Moreover, condition (a) can be replaced by the condition

$(a_1{}')$ *there exists a constant $b > 0$ such that*

$$\omega_1(x') \left(\sup_{|x'|/8<|y'|<8|x'|} \frac{1}{\omega(y')} \right) \le b \quad \textit{for a.e. } x' \in \mathbb{R}^k_{++}.$$

Proof. For $l \in Z$ we define $\widetilde{E}_l = \{x \in \mathbb{R}^n_{k,+} : 2^l < |x'| \le 2^{l+1}\}$, $\widetilde{E}_{l,1} = \{x \in \mathbb{R}^n_{k,+} : |x'| \le 2^{l-1}\}$, $\widetilde{E}_{l,2} = \{x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x'| \le 2^{l+2}\}$, $\widetilde{E}_{l,3} = \{x \in \mathbb{R}^n_{k,+} : |x'| > 2^{l+2}\}$. Then $\widetilde{E}_{l,2} = \widetilde{E}_{l-1} \cup \widetilde{E}_l \cup \widetilde{E}_{l+1}$ and the multiplicity of the covering $\left\{\widetilde{E}_{l,2}\right\}_{l\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, we write

$$\begin{aligned} |Tf(x)| &= \sum_{l\in Z} |Tf(x)|\chi_{\widetilde{E}_l}(x) \le \sum_{l\in Z} |Tf_{l,1}(x)|\,\chi_{\widetilde{E}_l}(x) \\ &+ \sum_{l\in Z} |Tf_{l,2}(x)|\,\chi_{\widetilde{E}_l}(x) + \sum_{l\in Z} |Tf_{l,3}(x)|\,\chi_{\widetilde{E}_l}(x) \\ &\equiv T_1f(x) + T_2f(x) + T_3f(x), \end{aligned} \tag{2.9}$$

where $\chi_{\widetilde{E}_l}$ is the characteristic function of the set $\widetilde{E}_l$, $f_{l,i} = f\chi_{\widetilde{E}_{l,i}}$, $i = 1,2,3$. We shall estimate $\|T_1f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,1}$ we have $|y'| \le 2^{l-1} \le |x'|/2$. Moreover, $\widetilde{E}_l \cap supp f_{l,1} = \varnothing$ and $|x' - y'| \ge |x'|/2$. Hence by (2.2)

$$\begin{aligned} T_1f(x) &\le c_4 \sum_{l\in Z} \left(\int_{\mathbb{R}^n_{k,+}} |f_{l,1}(y)| T^y|x|^{-n-|\gamma|} dy \right) \chi_{\widetilde{E}_l} \\ &\le c_4 \int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} T^y|x|^{-n-|\gamma|} |f(y)|(y')^\gamma dy \\ &\le c_5 \int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} \left(|x'| + |x'' - y''|\right)^{-n-|\gamma|} |f(y)|(y')^\gamma dy' dy'' \end{aligned}$$

for any $x \in E_l$. Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x')(x')^\gamma dx$$
$$\leq c_5^p \int_{\mathbb{R}^n_{k,+}} \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} \big(|x'| + |x'' - y''|\big)^{-n-|\gamma|} |f(y)|(y')^\gamma dy' dy'' \right)^p \times \omega_1(x')(x')^\gamma dx.$$

For $x = (x', x'') \in \mathbb{R}^n$ let

$$I(x') = \int_{\mathbb{R}^{n-k}} \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} \big(|x'| + |x'' - y''|\big)^{-n-|\gamma|} |f(y', y'')|(y')^\gamma dy' dy'' \right)^p dx''$$
$$= \int_{\mathbb{R}^{n-k}} \left(\int_{E'(0,|x'|/2)} \left(\int_{\mathbb{R}^{n-k}} \big(|x'| + |x'' - y''|\big)^{-n-|\gamma|} |f(y', y'')| dy' \right) (y')^\gamma dy' \right)^p dx''.$$

Using the Minkowski and Young inequalities we obtain

$$I(x') \leq \left[\int_{E'(0,|x'|/2)} \left(\int_{\mathbb{R}^{n-k}} |f(y', y'')|^p dy'' \right)^{1/p} \left(\int_{\mathbb{R}^{n-k}} \frac{dx''}{(|x'| + |x''|)^{n+|\gamma|}} \right) (y')^\gamma dy' \right]^p$$
$$= \left(\int_{E'(0,|x'|/2)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p \left(\int_{\mathbb{R}^{n-k}} \frac{dx''}{(|x'| + |x''|)^{n+|\gamma|}} \right)^p$$
$$= |x'|^{-(k+|\gamma|)p} \left(\int_{E'(0,|x'|/2)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p \left(\int_{\mathbb{R}^{n-k}} \frac{dx''}{(|x''| + 1)^{n+|\gamma|}} \right)^p$$
$$= c_6 |x'|^{-(k+|\gamma|)p} \left(\int_{E'(0,|x'|/2)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p.$$

Integrating in $\mathbb{R}^k_{++}$ we get

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x')(x')^\gamma dx$$
$$\leq c_7 \int_{\mathbb{R}^k_{++}} \omega_1(x')|x'|^{-(k+|\gamma|)p} \left(\int_{E'(0,|x'|/2)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p (x')^\gamma dx'.$$

Since $\mathcal{A}_1 < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^k_{++}} \omega_1(x')|x'|^{-(k+\gamma)p} \left(\int_{E'(0,|x'|/2)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p (x')^\gamma dx'$$
$$\leq C \int_{\mathbb{R}^k_{++}} \|f(\cdot, x')\|^p_{p,\mathbb{R}^{n-k}} \omega(x')(x')^\gamma dx'$$

holds and $C \leq c' \mathcal{A}_1$, where c' depends only on n and p. In fact the condition $\mathcal{A}_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [6], [22]).

Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x')(x')^\gamma dx \le c_9 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx. \tag{2.10}$$

Let us estimate $\|T_3 f\|_{L_{p,\omega_1,\gamma}}$. As is easy to verify, for $x \in \widetilde{E}_l$, $y \in \widetilde{E}_{l,3}$ we have $|y'| > 2|x'|$ and $|x'-y'| \ge |y'|/2$. Since $\widetilde{E}_l \cap supp f_{k,3} = \varnothing$, for $x \in \widetilde{E}_l$ by (2.2) we obtain

$$T_3 f(x) \le c_5 \int_{\mathbb{R}^{n-k}} \int_{\complement E'(0,2|x'|)} |f(y)| \left(|y'| + |x''-y''|\right)^{-n-|\gamma|} (y')^\gamma dy' dy''.$$

Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x')(x')^\gamma dx$$

$$\le c_5^p \int\limits_{\mathbb{R}^n_{k,+}} \left(\int\limits_{\mathbb{R}^{n-k}} \int\limits_{\complement E'(0,2|x'|)} |f(y)| \left(|y'| + |x''-y''|\right)^{-n-|\gamma|} (y')^\gamma dy' dy'' \right)^p \omega_1(x')(x')^\gamma dx.$$

For $x = (x', x'') \in \mathbb{R}^n$ let

$$I_1(x') = \int\limits_{\mathbb{R}^{n-k}} \left(\int\limits_{\complement E'(0,2|x'|)} \int\limits_{\mathbb{R}^{n-k}} |f(y)| \left(|y'| + |x''-y''|\right)^{-n-|\gamma|} (y')^\gamma dy' dy'' \right)^p (x')^\gamma dx''.$$

Using the Minkowski and Young inequalities we obtain

$$\begin{aligned} I_1(x') &\le \left[\int_{\complement E'(0,2|x'|)} \left(\int_{\mathbb{R}^{n-k}} |f(y)|^p dy'' \right)^{1/p} \left(\int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y'| + |y''|)^{n+|\gamma|}} \right) (y')^\gamma dy' \right]^p \\ &= c_6 \left(\int_{\complement E'(0,2|x'|)} |y'|^{-k-|\gamma|} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p \left(\int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y''|+1)^{n+|\gamma|}} \right)^p \\ &= c_7 \left(\int_{\complement E'(0,2|x'|)} |y'|^{-k-|\gamma|} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p . \end{aligned}$$

Integrating over $\mathbb{R}^k_{++}$ we get

$$\begin{aligned} &\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x')(x')^\gamma dx \\ &\le c_8 \int_{\mathbb{R}^k_{++}} \left(\int_{\complement E'(0,2|x'|)} |y'|^{-k-|\gamma|} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'' \right)^p \omega_1(x')(x')^\gamma dx''. \end{aligned}$$

Since $\mathcal{B}_1 < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^k_{++}} \omega_1(x') \left(\int_{\complement_{E'(0,2|x'|)}} |y'|^{-k-|\gamma|} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-1}} (y')^\gamma dy' \right)^p (x')^\gamma dx'$$
$$\le C \int_{\mathbb{R}^k_{++}} \|f(\cdot,x')\|^p_{p,\mathbb{R}^{n-k}} |x'|^{-(k+|\gamma|)p} \omega(x') |x'|^{(k+|\gamma|)p} (x')^\gamma dx'$$
$$= C \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx$$

holds and $C \le c'\mathcal{B}_1$, where c' depends only on n, γ and p. In fact the condition $\mathcal{B}_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [6], [22]). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x')(x')^\gamma dx \le c_{10} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx. \tag{2.11}$$

Finally, we estimate $\|T_2 f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of T and condition (a_1) we have

$$\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x_n)(x')^\gamma dx = \int_{\mathbb{R}^n_{k,+}} \left(\sum_{l\in Z} |Tf_{l,2}(x)| \, \chi_{\widetilde{E}_l}(x) \right)^p \omega_1(x')(x')^\gamma dx$$
$$= \int_{\mathbb{R}^n_{k,+}} \left(\sum_{l\in Z} |Tf_{l,2}(x)|^p \, \chi_{\widetilde{E}_l}(x) \right) \omega_1(x')(x')^\gamma dx = \sum_{l\in Z} \int_{\widetilde{E}_l} |Tf_{l,2}(x)|^p \, \omega_1(x')(x')^\gamma dx$$
$$\le \sum_{l\in Z} \sup_{y\in \widetilde{E}_l} \omega_1(y') \int_{\mathbb{R}^n} |Tf_{l,2}(x)|^p \, (x')^\gamma dx$$
$$\le \|T\|^p \sum_{l\in Z} \sup_{y\in \widetilde{E}_l} \omega_1(y') \int_{\mathbb{R}^n} |f_{l,2}(x)|^p \, (x')^\gamma dx$$
$$= \|T\|^p \sum_{l\in Z} \sup_{y\in \widetilde{E}_l} \omega_1(y') \int_{\widetilde{E}_{l,2}} |f(x)|^p (x')^\gamma dx,$$

where $\|T\| \equiv \|T\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+}) \to L_{p,\gamma}(\mathbb{R}^n_{k,+})}$. Since, for $x \in \widetilde{E}_{l,2}$, $2^{l-1} < |x'| \le 2^{l+2}$, we have by condition (a_1)

$$\sup_{y\in \widetilde{E}_l} \omega_1(y') = \sup_{2^{l-1}<|y'|\le 2^{l+2}} \omega_1(y') \le \sup_{|x'|/8<|y'|<8|x'|} \omega_1(y') \le b\omega(x')$$

for almost all $x \in \widetilde{E}_{l,2}$. Therefore

$$\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x')(x')^\gamma dx$$
$$\le \|T\|^p b \sum_{l\in Z} \int_{\widetilde{E}_{l,2}} |f(x)|^p \omega(x') dx \le c_{11} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx, \tag{2.12}$$

where $c_{11} = 3\|T\|^p b$, since the multiplicity of covering $\left\{ \widetilde{E}_{l,2} \right\}_{l\in Z}$ is equal to 3.

Inequalities (2.9), (2.10), (2.11), (2.12) imply (2.8) which completes the proof. □

Similarly we prove the following weak variant of Theorem 2.3.

Theorem 2.4. *Let $p \in [1, \infty)$ and let T be a weak p-admissible $B_{k,n}$–singular operators. Moreover, let $\omega(x')$, $\omega_1(x')$ be weight functions on $\mathbb{R}^k_{++}$ and conditions (a_1), (b_1), (c_1) be satisfied.*

Then there exists a constant c, independent of f, such that

$$\int_{\{x\in\mathbb{R}^n_{k,+}\,:\,|Tf(x)|>\lambda\}} \omega_1(x')(x')^\gamma dx \le \frac{c}{\lambda^p}\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx \tag{2.13}$$

for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$.

Corollary 2.3. *Let $p \in (1, \infty)$, T be the p-admissible $B_{k,n}$–singular operators. Moreover, let $\omega(x')$, $\omega_1(x')$ be weight functions on $\mathbb{R}^k_{++}$ and conditions (a_1), (b_1), (c_1) be satisfied. Then inequality (2.8) is valid.*

Corollary 2.4. *Let $p \in [1, \infty)$, T be the weak p-admissible $B_{k,n}$–singular operators. Moreover, let $\omega(x')$, $\omega_1(x')$ be weight functions on $\mathbb{R}^k_{++}$ and conditions (a_1), (b_1), (c_1) be satisfied. Then inequality (2.13) is valid.*

Remark 2.3. Note that, if instead of $\omega(x)$, $\omega_1(x)$ respectively put $\omega(x')$, $\omega_1(x')$, then from conditions (a), (b), (c) will not follows conditions (a_1), (b_1), (c_1) respectively.

Theorem 2.5. *Let $p \in (1, \infty)$ and T be a p-admissible $B_{k,n}$–singular operators. Moreover, let $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies conditions (a), (b). Then there exists a constant $c > 0$, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$*

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x|)(x')^\gamma dx \le c\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^\gamma dx. \tag{2.14}$$

Proof. Suppose that $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and ω_1 are positive increasing functions on $(0, \infty)$ and ω, ω_1 satisfied the conditions (a), (b).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda)d\lambda,$$

where $\omega_1(0+) = \lim_{t\to 0}\omega_1(t)$ and $\omega_1(t) \ge 0$ on $(0, \infty)$. In fact there exists a seqence of increasing absolutely continuous fuctions ϖ_n, such that $\varpi_n(t) \le \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [12], [14] for details).

We have

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x|)(x')^\gamma dx = \omega_1(0+)\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p (x')^\gamma dx$$

$$+\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \left(\int_0^{|x|}\psi(\lambda)d\lambda\right)(x')^\gamma dx = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \ne 0$ by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ thanks to (a)

$$J_1 \le \|T\|^p \omega_1(0+)\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx$$

$$\leq \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x|)(x')^\gamma dx \leq b\,\|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^\gamma dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{\complement_{E(0,\lambda)}} |Tf(x)|^p (x')^\gamma dx \right) d\lambda$$
$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{\complement_{E(0,\lambda)}} |T(f\chi_{\complement_{E(0,\lambda/2)}})(x)|^p (x')^\gamma dx \right.$$
$$\left. + \int_{\complement_{E(0,\lambda)}} |T(f\chi_{E(0,\lambda/2)})(x)|^p (x')^\gamma dx \right) d\lambda = J_{21} + J_{22}.$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and condition (a) we have

$$J_{21} \leq \|T\|^p \int_0^\infty \psi(t) \left(\int_{\complement_{E(0,\lambda/2)}} |f(y)|^p (y')^\gamma dy \right) dt$$
$$= \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \left(\int_0^{2|y|} \psi(\lambda) d\lambda \right) (y')^\gamma dy$$
$$\leq \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega_1(2|y|)(y')^\gamma dy$$
$$\leq b\,\|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma dy.$$

Let us estimate J_{22}. For $|x| > \lambda$ and $|y| \leq \lambda/2$ we have

$$|x|/2 \leq |x-y| \leq 3|x|/2,$$

and so

$$J_{22} \leq c_4 \int_0^\infty \psi(\lambda) \left(\int_{\complement_{E(0,\lambda)}} \left(\int_{E(0,2\lambda)} T^y |x|^{-n-|\gamma|} |f(y)| (y')^\gamma dy \right)^p (x')^\gamma dx \right) d\lambda$$
$$\leq c_5 \int_0^\infty \psi(\lambda) \left(\int_{\complement_{E(0,\lambda)}} \left(\int_{E(0,2\lambda)} |f(y)| (y')^\gamma dy \right)^p |x|^{-(n+|\gamma|)p} (x')^\gamma dx \right) d\lambda$$
$$= c_6 \int_0^\infty \psi(\lambda) \lambda^{-(n+|\gamma|)(p-1)} \left(\int_{E(0,\lambda/2)} |f(y)| (y')^\gamma dy \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{-(n+|\gamma|)(p-1)} \left(\int_{E(0,\lambda/2)} |f(y)| (y')^\gamma dy \right)^p d\lambda$$
$$\leq C \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma dy$$

ie valid, for $p \in (1, \infty)$ is valid by the condition $C \le c'\mathcal{A}'$ (see [6], [22]), where

$$\mathcal{A}' \equiv \sup_{\tau>0} \left(\int_{2\tau}^{\infty} \psi(t) t^{-(n+|\gamma|)(p-1)} d\tau \right) \left(\int_{E(0,\tau)} \omega^{1-p'}(|y|)(y')^{\gamma} dy \right)^{p-1} < \infty.$$

Note that

$$\begin{aligned}
&\int_{2t}^{\infty} \psi(\tau)\tau^{-(n+|\gamma|)(p-1)} d\tau \\
&= (n+|\gamma|)(p-1) \int_{2t}^{\infty} \psi(\tau) d\tau \int_{\tau}^{\infty} \lambda^{-k-(n+|\gamma|)(p-1)} d\lambda \\
&= (n+|\gamma|)(p-1) \int_{2t}^{\infty} \lambda^{-k-(n+|\gamma|)(p-1)} d\lambda \int_{2t}^{\lambda} \psi(\tau) d\tau \\
&\le (n+|\gamma|)(p-1) \int_{2t}^{\infty} \lambda^{-k-(n+|\gamma|)(p-1)} \omega_1(\lambda) d\lambda \\
&= \frac{(p-1)}{\omega(n,|\gamma|)} \int_{\complement E(0,2t)} \omega_1(|y|)|y|^{-(n+|\gamma|)p} (y')^{\gamma} dy.
\end{aligned}$$

Condition (b) of the theorem guarantees that $\mathcal{A}' \le \frac{(n+|\gamma|)(p-1)}{\omega(n,|\gamma|)} \mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \le c_7 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^{\gamma} dx.$$

Combining the estimates of J_1 and J_2, we get (2.14) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (2.14). The theorem is proved. □

Corollary 2.5. *Let $p \in (1,\infty)$, k be a $B_{k,n}$–singular kernel and K be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies conditions (a), (b). Then for the operator K the inequality* (2.14) *is valid.*

Example 2.1. *Let*

$$\omega(t) = \begin{cases} t^{(n+|\gamma|)(p-1)} \ln^p \frac{1}{t}, & for \quad t \in \left(0, \frac{1}{2}\right) \\ \left(2^{\beta-p+1} \ln^p 2\right) t^{\beta}, & for \quad t \in \left[\frac{1}{2}, \infty\right) \end{cases},$$

$$\omega_1(t) = \begin{cases} t^{(n+|\gamma|)(p-1)}, & for \quad t \in \left(0, \frac{1}{2}\right) \\ 2^{\alpha-p+1} t^{\alpha}, & for \quad t \in \left[\frac{1}{2}, \infty\right) \end{cases},$$

where $0 < \alpha \le \beta < (n+|\gamma|)(p-1)$. Then the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies the condition of Theorem 2.5.

Theorem 2.6. *Let $p \in (1,\infty)$ and T be a p-admissible $B_{k,n}$–singular operators. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies conditions (a), (c). Then inequality* (2.14) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^{\infty} \psi(\tau) d\tau,$$

where $\omega_1(+\infty) = \lim_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$(see [12], [14] for details).

We have

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x|)(x')^\gamma dx = \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p (x')^\gamma dx$$
$$+ \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \left(\int_{|x|}^{\infty} \psi(\tau) d\tau \right) (x')^\gamma dx$$
$$= I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and condition (a) we have

$$J_1 \leq \|T\| \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx$$
$$\leq \|T\| \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x|)(x')^\gamma dx$$
$$\leq b\, \|T\| \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^\gamma dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{E(0,\lambda)} |Tf(x)|^p (x')^\gamma dx \right) d\lambda$$
$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{E(0,\lambda)} |T(f\chi_{E(0,2\lambda)})(x)|^p (x')^\gamma dx \right.$$
$$\left. + \int_{E(0,\lambda)} |T(f\chi_{E(0,2\lambda)})(x)|^p (x')^\gamma dx \right) d\lambda$$
$$= J_{21} + J_{22}.$$

Using the boundeedness of T in $L_p(\mathbb{R}^n_{k,+})$ and condition (a) we obtain

$$J_{21} \leq \|T\| \int_0^\infty \psi(t) \left(\int_{|y|<2\lambda} |f(y)|^p (y')^\gamma dy \right) dt$$
$$= \|T\| \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \left(\int_{|y|/2}^\infty \psi(\lambda) d\lambda \right) (y')^\gamma dy$$
$$\leq \|T\| \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega_1(|y|/2)(y')^\gamma dy$$
$$\leq b\, \|T\| \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma dy.$$

Let us estimate J_{22}. For $|x| < \lambda$ and $|y| \geq 2\lambda$ we have $|y|/2 \leq |x-y| \leq 3|y|/2$, and so

$$J_{22} \leq c_8 \int_0^\infty \psi(\lambda) \left(\int_{E(0,\lambda)} \left(\int_{\complement E(0,2\lambda)} T^y |x|^{-n-|\gamma|} |f(y)| (y')^\gamma dy \right)^p (x')^\gamma dx \right) d\lambda$$

$$\leq 2^n c_8 \int_0^\infty \psi(\lambda) \left(\int_{E(0,\lambda)} \left(\int_{\complement E(0,2\lambda)} |y|^{-n-|\gamma|} |f(y)| (y')^\gamma dy \right)^p (x')^\gamma dx \right) d\lambda$$

$$= c_9 \int_0^\infty \psi(\lambda) \lambda^{n+|\gamma|} \left(\int_{\complement E(0,2\lambda)} |y|^{-n-|\gamma|} |f(y)| (y')^\gamma dy \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{n+|\gamma|} \left(\int_{\complement E(0,2\lambda)} |y|^{-n-|\gamma|} |f(y)| (y')^\gamma dy \right)^p d\lambda$$

$$\leq C \int_{\mathbb{R}^n_{k,+}} |f(y)|^p |y|^{-(n+|\gamma|)p} |y|^{(n+|\gamma|)p} \omega(|y|)(y')^\gamma dy = C \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma dy$$

is valid, for $p \in (1,\infty)$ is valid by the condition $C \leq c\mathcal{B}'$ (see [6], [22]), where

$$\mathcal{B}' \equiv \sup_{\tau>0} \left(\int_0^\tau \psi(t) t^{n+|\gamma|} d\tau \right) \left(\int_{\complement E(0,2\tau)} \omega^{1-p'}(|y|) |y|^{-(n+|\gamma|)p'} (y')^\gamma dy \right)^{p-1} < \infty.$$

Note that

$$\begin{aligned}
\int_0^\tau \psi(t) t^{n+|\gamma|} dt &= (n+|\gamma|) \int_0^\tau \psi(t) dt \int_0^t \lambda^{n+|\gamma|-1} d\lambda \\
&= (n+|\gamma|) \int_0^\tau \lambda^{n+|\gamma|-1} d\lambda \int_\lambda^t \psi(\tau) d\tau \\
&\leq (n+|\gamma|) \int_0^\tau \lambda^{n+|\gamma|-1} \omega_1(\lambda) d\lambda \\
&= \frac{n+|\gamma|}{\omega(n,|\gamma|)} \int_{E(0,r)} \omega_1(|x|)(x')^\gamma dx.
\end{aligned}$$

Condition (c) of the theorem guarantees that $\mathcal{B}' \leq \frac{n+|\gamma|}{\omega(n,|\gamma|)} \mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{10} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^\gamma dx.$$

Combining the estimates of J_1 and J_2, we get (2.14) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau) d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (2.14). The theorem is proved. □

Corollary 2.6. *Let $p \in (1,\infty)$, k be a $B_{k,n}$–singular kernel and K be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies conditions (a), (c). Then for the operator K the inequality* (2.14) *is valid.*

Example 2.2. *Let*

$$\omega(t)=\begin{cases} \frac{1}{t^{n+|\gamma|}}\ln^{\nu}\frac{1}{t}, & for \quad t<d \\ \left(d^{-n-|\gamma|-\alpha}\ln^{\nu}\frac{1}{d}\right)t^{\alpha}, & for \quad t\geq d \end{cases},$$

$$\omega_1(t)=\begin{cases} \frac{1}{t^{n+|\gamma|}}\ln^{\beta}\frac{1}{t}, & for \quad t<d \\ \left(d^{-n-|\gamma|-\lambda}\ln^{\beta}\frac{1}{d}\right)t^{\lambda}, & for \quad t\geq d \end{cases},$$

where $\beta<\nu\leq 0$, $-n-|\gamma|<\lambda<\alpha<0$, $d=e^{\frac{\beta}{n+|\gamma|}}$. *Then the weighted pair* $(\omega(|x|),\omega_1(|x|))$ *satisfies the condition of Theorem 2.6.*

Theorem 2.7. *Let* $p\in(1,\infty)$ *and* T *be a* p*-admissible* $B_{k,n}$*–singular operators. Moreover, let* $\omega(t)$ *be a weight function on* $(0,\infty)$, $\omega_1(t)$ *be a positive increasing function on* $(0,\infty)$ *and* $\omega(|x'|)$, $\omega_1(|x'|)$ *be satisfied the conditions* (a_1), (b_1).

Then there exists a constant $c>0$, *such that for all* $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$

$$\int_{\mathbb{R}^n_{k,+}}|Tf(x)|^p\omega_1(|x'|)(x')^{\gamma}dx\leq c\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(|x'|)(x')^{\gamma}dx. \tag{2.15}$$

Proof. Suppose that $f\in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$, ω_1 are positive increasing functions on $(0,\infty)$ and $\omega(t)$, $\omega_1(t)$ satisfied the conditions (a_1), (b_1).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t)=\omega_1(0+)+\int_0^t\psi(\lambda)d\lambda,$$

where $\omega_1(0+)=\lim_{t\to 0}\omega_1(t)$ and $\omega_1(t)\geq 0$ on $(0,\infty)$. In fact there exists a seqence of increasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t)\leq\omega_1(t)$ and $\lim\limits_{n\to\infty}\varpi_n(t)=\omega_1(t)$ for any $t\in(0,\infty)$ (see [12], [14] for details).

We have

$$\int_{\mathbb{R}^n_{k,+}}|Tf(x)|^p\omega_1(|x'|)(x')^{\gamma}dx=\omega_1(0+)\int_{\mathbb{R}^n_{k,+}}|Tf(x)|^p(x')^{\gamma}dx+$$

$$+\int_{\mathbb{R}^n_{k,+}}|Tf(x)|^p\left(\int_0^{x'}\psi(\lambda)d\lambda\right)(x')^{\gamma}dx=J_1+J_2.$$

If $\omega_1(0+)=0$, then $J_1=0$. If $\omega_1(0+)\neq 0$ by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ thanks to (a)

$$\begin{aligned} J_1 &\leq \|T\|^p\omega_1(0+)\int_{\mathbb{R}^n_{k,+}}|f(x)|^p(x')^{\gamma}dx \\ &\leq \|T\|^p\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega_1(|x'|)(x')^{\gamma}dx \\ &\leq \|T\|^p\int_{\mathbb{R}^n_{k,+}}|f(x)|^p\omega(|x'|)(x')^{\gamma}dx. \end{aligned}$$

After changing the order of integration in J_2 we have

$$J_2 = \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}_+^{n-k}} \int_{\complement_{E'(0,\lambda)}} |Tf(x)|^p (x')^\gamma dx \right) d\lambda$$
$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-k}} \int_{\complement_{E'(0,\lambda)}} |T(f\chi_{\{|x'|>\lambda/2\}})(x)|^p (x')^\gamma dx \right.$$
$$\left. + \int_{\mathbb{R}^{n-k}} \int_{\complement_{E'(0,\lambda)}} |T(f\chi_{\{|x'|\leq\lambda/2\}})(x)|^p (x')^\gamma dx \right) d\lambda = J_{21} + J_{22}.$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ we obtain

$$J_{21} \leq \|T\|^p \int_0^\infty \psi(t) \left(\int_{\mathbb{R}^{n-k}} \int_{\complement_{E'(0,\lambda/2)}} |f(y)|^p (y')^\gamma dy \right) dt$$
$$= \|T\|^p \int_0^\infty \psi(t) \left(\int_{\complement_{E'(0,\lambda/2)}} \|f(\cdot, y')\|^p_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right) dt$$
$$= \|T\|^p \int_{\mathbb{R}^n_{k,+}} \|f(\cdot, y')\|^p_{p,\mathbb{R}^{n-k}} \left(\int_0^{2|y'|} \psi(\lambda) d\lambda \right) (y')^\gamma dy'$$
$$\leq \|T\|^p \int_{\mathbb{R}^n_{k,+}} \|f(\cdot, y')\|^p_{p,\mathbb{R}^{n-k}} \omega_1(2|y'|)(y')^\gamma dy'$$
$$\leq b\, \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y'|)(y')^\gamma dy.$$

Let us estimate J_{22}. For $|x'| > \lambda$ and $|y'| \leq \lambda/2$ we have $|x'|/2 \leq ||x'| - |y'|| \leq 3|x'|/2$, and so

$$J_{22} \leq c_9 \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\mathbb{R}^{n-k}} \int\limits_{\complement_{E'(0,\lambda)}} \Big(\int\limits_{\mathbb{R}^{n-k}} \int\limits_{E'(0,\lambda/2)} \frac{|f(y)|}{|x-y|^{n+|\gamma|}} dy \Big)^p (x')^\gamma dx \Big) d\lambda \leq$$

$$c_{10} \int\limits_0^\infty \psi(\lambda) \Big(\int\limits_{\complement_{E'(0,\lambda)}} \int\limits_{\mathbb{R}^{n-k}} \Big(\int\limits_{E'(0,\lambda/2)} \int\limits_{\mathbb{R}^{n-k}} \frac{|f(y)|}{(|x'| + |x'' - y''|)^{n+|\gamma|}} (y')^\gamma dy \Big)^p (x')^\gamma dx \Big) d\lambda.$$

For $x = (x', x'') \in \mathbb{R}^n_{k,+}$ let

$$J(x', \lambda) = \int_{\mathbb{R}^{n-k}} \left(\int_{E'(0,\lambda/2)} \int_{\mathbb{R}^{n-k}} \frac{|f(y)|}{(|x'| + |x'' - y''|)^{n+|\gamma|}} (y')^\gamma dy \right)^p dx''$$

Using the Minkowski and Young inequalities we obtain

$$J(x',\lambda) \le \left[\int_{E'(0,\lambda/2)} \left(\int_{\mathbb{R}^{n-k}} |f(y)|^p dy''\right)^{1/p} \left(\int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y''|+|x'|)^{n+|\gamma|}}\right) (y')^\gamma dy'\right]^p$$

$$\le \left(\int_{E'(0,\lambda/2)} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'\right)^p \left(\int_{\mathbb{R}^{n-k}} \frac{dy'}{(|y''|+|x'|)^{n+|\gamma|}}\right)^p$$

$$= c_3 |x'|^{-(k+|\gamma|)p} \left(\int_{E'(0,\lambda/2)} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'\right)^p$$

$$\times \left(\int_{\mathbb{R}^{n-k}} \frac{dy'}{(1+|y'|)^{n+|\gamma|}}\right)^p$$

$$= c_4 |x'|^{-(k+|\gamma|)p} \left(\int_{E'(0,\lambda/2)} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'\right)^p.$$

Integrating in $(0,\infty) \times ({}^{\complement}E'(0,\lambda))$ we get

$$J_{22} \le c_5 \int_0^\infty \psi(\lambda)$$

$$\times \left(\int_{{}^{\complement}E'(0,\lambda)} \left(\int_{E(0,\lambda/2)} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'\right)^p |x'|^{-(k+|\gamma|)p} (x')^\gamma dx\right) d\lambda$$

$$= \frac{2c_5}{p-1} \int_0^\infty \psi(\lambda) \lambda^{-(k+|\gamma|)p+|\gamma|+k} \left(\int_{E(0,\lambda/2)} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'\right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{-(k+|\gamma|)p+|\gamma|+k} \left(\int_{E(0,\lambda/2)} \|f(\cdot,y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy\right)^p d\lambda$$

$$\le C \int_{\mathbb{R}^k_{++}} \|f(\cdot,y')\|^p_{p,\mathbb{R}^{n-k}} \omega(|y'|)(y')^\gamma dy'$$

$$= C \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y'|)(y')^\gamma dy.$$

is valid, for $p \in (1,\infty)$ is valid by the condition $C \le c'\mathcal{A}''$, where

$$\mathcal{A}'' \equiv \sup_{\tau>0} \left(\int_{2\tau}^\infty \psi(t) t^{-(k+|\gamma|)p+|\gamma|+k} d\tau\right) \left(\int_0^\tau \omega^{1-p'}(t) t^{|\gamma|} dt\right)^{p-1} < \infty.$$

Note that

$$\int_{2t}^\infty \psi(\tau)\tau^{-(k+|\gamma|)p+|\gamma|+k} d\tau = (k+|\gamma|)(p-1) \int_{2t}^\infty \psi(\tau) d\tau \int_\tau^\infty \lambda^{-(k+|\gamma|)p+\gamma} d\lambda$$

$$= (k+|\gamma|)(p-1) \int_{2t}^\infty \lambda^{-(k+|\gamma|)p+|\gamma|} d\lambda \int_{2t}^\lambda \psi(\tau) d\tau$$

$$\le (k+|\gamma|)(p-1) \int_{2t}^\infty \lambda^{-(k+|\gamma|)p+|\gamma|} \omega_1(\lambda) d\lambda.$$

Condition (b_1) of the theorem guarantees that $\mathcal{A}'' \leq (k+|\gamma\|)(p-1)\mathcal{A}_1 < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{11} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x'|)(x')^\gamma dx.$$

Combining the estimates of J_1 and J_2, we get (2.14) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this iplies (2.15). The theorem is proved. □

Example 2.3. *Let*

$$\omega(t) = \begin{cases} t^{p-1} \ln^p \frac{1}{t}, & for \quad t \in \left(0, \frac{1}{2}\right) \\ \left(2^{\beta-p+1} \ln^p 2\right) t^\beta, & for \quad t \in \left[\frac{1}{2}, \infty\right) \end{cases},$$

$$\omega_1(t) = \begin{cases} t^{p-1}, & for \quad t \in \left(0, \frac{1}{2}\right) \\ 2^{\alpha-p+1} t^\alpha, & for \quad t \in \left[\frac{1}{2}, \infty\right) \end{cases},$$

where $0 < \alpha \leq \beta < p-1$. *Then the pair* $(\omega(|x'|), \omega_1(|x'|))$ *satisfies the condition of Theorem 2.7.*

Corollary 2.7. *Let* $p \in (1,\infty)$, *k be a* $B_{k,n}$*–singular kernel and K be the corresponding operator. Moreover, let* $\omega(t)$ *be a weight function on* $(0,\infty)$, $\omega_1(t)$ *be a positive increasing function on* $(0,\infty)$ *and* $\omega(|x'|)$, $\omega_1(|x'|)$ *be satisfied the conditions* (a_1), (b_1). *Then for the operator K the inequality* (2.15) *is valid.*

Theorem 2.8. *Let* $p \in (1,\infty)$ *and T be a p-admissible* $B_{k,n}$*–singular operators. Moreover, let* $\omega(t)$ *be a weight function on* $(0,\infty)$, $\omega_1(t)$ *be a positive decreasing function on* $(0,\infty)$ *and* $\omega(|x'|)$, $\omega_1(|x'|)$ *be satisfied the conditions* (a_1), (c_1). *Then inequality* (2.15) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [12], [14] for details). We have

$$\begin{aligned}\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x'|)(x')^\gamma dx &= \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p (x')^\gamma dx \\ &+ \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \left(\int_{|x'|}^\infty \psi(\tau)d\tau \right) (x')^\gamma dx \\ &= I_1 + I_2.\end{aligned}$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$ by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$

$$
\begin{aligned}
J_1 &\le \|T\|^p \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \\
&\le \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x'|)(x')^\gamma dx \\
&\le b\,\|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x'|)(x')^\gamma dx.
\end{aligned}
$$

After changing the order of integration in J_2 we have

$$
\begin{aligned}
J_2 &= \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} |Tf(x)|^p (x')^\gamma dx \right) d\lambda \\
&\le 2^{p-1} \int_0^\infty \psi(\lambda) \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} |T(f\chi_{\{|x'|<2\lambda\}})(x)|^p (x')^\gamma dx \right. \\
&+ \left. \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} |T(f\chi_{\{|x'|\ge 2\lambda\}})(x)|^p (x')^\gamma dx \right) d\lambda \\
&= J_{21} + J_{22}.
\end{aligned}
$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ we obtain

$$
\begin{aligned}
J_{21} &\le \|T\|^p \int_0^\infty \psi(t) \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,2\lambda)} |f(y)|^p (y')^\gamma dy \right) dt \\
&= \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \left(\int_{|y'|/2}^\infty \psi(\lambda) d\lambda \right) (y')^\gamma dy \\
&\le \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega_1(|y'/2|)(y')^\gamma dy \\
&\le b\,\|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y'|)(y')^\gamma dy.
\end{aligned}
$$

Let us estimate J_{22}. For $|x'| < \lambda$ and $|y'| \geq 2\lambda$ we have $|y'|/2 \leq |x'-y'| \leq 3|y'|/2$, and so

$$J_{22} \leq c_{12} \int_0^\infty \psi(\lambda)$$
$$\times \left(\int_{\mathbb{R}^{n-k}} \int_{\complement E'(0,\lambda)} \left(\int_{\mathbb{R}^{n-k}} \int_{\complement E'(0,2\lambda)} \frac{|f(y)|(y')^\gamma dy}{(|x'-y'|+|x''-y''|)^{n+|\gamma|}} \right)^p (x')^\gamma dx \right) d\lambda$$
$$\leq 2^n c_{12} \int_0^\infty \psi(\lambda)$$
$$\times \left(\int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} \left(\int_{\mathbb{R}^{n-k}} \int_{\complement E'(0,2\lambda)} \frac{|f(y)|(y')^\gamma dy}{(|x''-y''|+|y'|)^{n+|\gamma|}} \right)^p (x')^\gamma dx \right) d\lambda.$$

For $x = (x', x'') \in \mathbb{R}^n_{k,+}$ let

$$J_1(x', \lambda) = \int_{\mathbb{R}^{n-k}} \left(\int_{\complement E'(0,2\lambda)} \int_{\mathbb{R}^{n-k}} \frac{|f(y)|(y')^\gamma dy}{(|x''-y''|+|y'|)^{n+|\gamma|}} \right)^p dx'.$$

Using the Minkowski and Young inequalities we obtain

$$J_1(x', \lambda) \leq \left[\int_{\complement E'(0,2\lambda)} \left(\int_{\mathbb{R}^{n-k}} |f(y)|^p dy' \right)^{1/p} \left(\int_{\mathbb{R}^{n-k}} \frac{dy'}{(|y''|+|y'|)^{n+\gamma}} \right) (y')^\gamma dy_n \right]^p$$
$$\leq \left(\int_{\complement E'(0,2\lambda)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p \left(\int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y''|+|y'|)^{n+|\gamma|}} \right)^p$$
$$= c_3 \left(\int_{\complement E'(0,2\lambda)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p \left(\int_{\mathbb{R}^{n-k}} \frac{dy''}{(1+|y''|)^{n+|\gamma|}} \right)^p$$
$$= c_4 \left(\int_{\complement E'(0,2\lambda)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p.$$

Integrating in $(0,\infty) \times (0,\lambda)$ we get

$$J_{22} \leq c_5 \int_0^\infty \psi(\lambda)$$
$$\times \left(\int_{E'(0,\lambda)} \left(\int_{\complement E'(0,2\lambda)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p (x')^\gamma dx' \right) d\lambda$$
$$= 2c_5 \int_0^\infty \psi(\lambda) \lambda^{k+|\gamma|} \left(\int_{\complement E'(0,2\lambda)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{1+|\gamma|} \left(\int_{\complement E'(0,2\lambda)} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-1}} |y'|^{-k-|\gamma|} (y')^\gamma dy \right)^p d\lambda$$

$$\leq C\int_{\mathbb{R}^k_{++}} \|f(\cdot,x')\|^p_{p,\mathbb{R}^{n-k}}\omega(|x'|)(x')^\gamma dx' = C\int_{\mathbb{R}^n_{k,+}} |f(y)|^p\omega(|y'|)(y')^\gamma dy,$$

is valid, for $p \in (1,\infty)$ is valid by the condition $C \leq c'\mathcal{B}''$, where

$$\mathcal{B}'' \equiv \sup_{\tau>0}\left(\int_0^\tau \psi(t)t^{k+|\gamma|}d\tau\right)\left(\int_{2\tau}^\infty \omega^{1-p'}(t)t^{-(k+|\gamma|)p'}t^{|\gamma|}dt\right)^{p-1} < \infty.$$

Note that

$$\begin{aligned}\int_0^\tau \psi(t)t^{k+|\gamma|}dt &= (k+|\gamma|)\int_0^\tau \psi(t)dt\int_0^t \lambda^{|\gamma|}d\lambda\\ &= (k+|\gamma|)\int_0^\tau \lambda^{|\gamma|}d\lambda\int_\lambda^t \psi(\tau)d\tau\\ &\leq (k+|\gamma|)\int_0^\tau \omega(\lambda)\lambda^{|\gamma|}d\lambda.\end{aligned}$$

Condition (c_1) of the theorem guarantees that $\mathcal{B}'' \leq \mathcal{B}_1 < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c\int_{\mathbb{R}^n_{k,+}} |f(x)|^p\omega(|x'|)(x')^\gamma dx.$$

Combining the estimates of J_1 and J_2, we get (2.14) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this iplies (2.15). The theorem is proved. □

Corollary 2.8. *Let $p \in (1,\infty)$, k be a $B_{k,n}$–singular kernel and K be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive decreasing function on $(0,\infty)$ and $\omega(|x'|)$, $\omega_1(|x'|)$ be satisfied the conditions (a_1), (c_1). Then for the operator K the inequality (2.15) is valid.*

Example 2.4. *Let*

$$\omega(t) = \begin{cases} \frac{1}{t}\ln^\nu\frac{1}{t}, & for \quad t<d\\ \left(d^{-1-\alpha}\ln^\nu\frac{1}{d}\right)t^\alpha, & for \quad t\geq d\end{cases},$$

$$\omega_1(t) = \begin{cases} \frac{1}{t}\ln^\beta\frac{1}{t}, & for \quad t<d\\ \left(d^{-1-\lambda}\ln^\beta\frac{1}{d}\right)t^\lambda, & for \quad t\geq d\end{cases},$$

where $\beta < \nu \leq 0$, $-1 < \lambda < \alpha < 0$, $d = e^\beta$. Then the pair $(\omega(|x'|),\omega_1(|x'|))$ satisfies the condition of Theorem 2.8.

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