TWO-WEIGHTED INEQUALITY FOR p-ADMISSIBLE
B_{k,n}–SINGULAR OPERATORS IN WEIGHTED LEBESGUE
SPACES

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Abstract. In this paper, we study the boundedness of p-admissible singular operators, associated with the Laplace-Bessel differential operator

\[ B_{k,n} = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{n} \gamma_j \frac{\partial}{\partial x_j} \] (p-admissible \( B_{k,n} \)-singular operators) on a weighted Lebesgue spaces \( L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \) including their weak versions. These conditions are satisfied by most of the operators in harmonic analysis, such as the \( B_{k,n} \)-maximal operator, \( B_{k,n} \)-singular integral operators and so on. Sufficient conditions on weighted functions \( \omega \) and \( \omega_1 \) are given so that p-admissible \( B_{k,n} \)-singular operators are bounded from \( L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \) to \( L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+}) \) for \( 1 \leq p < \infty \) and weak p-admissible \( B_{k,n} \)-singular operators are bounded from \( L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \) to \( L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+}) \) for \( 1 \leq p < \infty \).

1. Introduction

The singular integral operators considered by S. Mihlin [26] and A. Calderon and A. Zygmund [7] are playing an important role in the theory of Harmonic Analysis and in particular, in the theory of partial differential equations. M. Klyuchantsev [25] and I. Kipriyanov and M. Klyuchantsev [24] have firstly introduced and investigated the boundedness in \( L_p \)-spaces of multidimensional singular integrals, generated by the \( B_{1,n} \)-Laplace-Bessel differential operator (\( B_{1,n} \)-singular integrals), where

\[ B_{1,n} = B_1 + \sum_{j=2}^{n} \frac{\partial^2}{\partial x_j^2}, \quad B_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\gamma}{x_1} \frac{\partial}{\partial x_1}, \quad \gamma > 0. \]

I.A. Aliev and A.D. Gadjiev [5], A.D. Gadjiev and E.V. Guliyev [11] and E.V. Guliyev [13] have studied the boundedness of \( B_{1,n} \) singular integrals in weighted \( L_p \)-spaces with radial and general weights consequently. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator \( B_{k,n} \)–which is known as an important differential operator in analysis and its applications, have been the research areas of many mathematicians.

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such as I. Kipriyanov and M. Klyuchantsev [24, 25], L. Lyakhov [29, 30], A.D. Gadjiev and I.A. Aliev [4, 5], I.A. Aliev and S. Bayrakci [2, 3], V.S. Gulyiev [15, 16, 17] and others.

In the paper, we shall prove the boundedness of $p$-admissible singular operators, associated with the Laplace-Bessel differential operator $B_{k,n} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{k} \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j}$ ($p$-admissible $B_{k,n}$–singular operators) on a weighted $L_p$ spaces. Sufficient conditions on weighted functions $\omega$ and $\omega_1$ are given so that $p$-admissible $B_{k,n}$–singular operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)$ to $L_{p,\omega_1,\gamma}(\mathbb{R}_{k,+}^n)$ for $1 < p < \infty$ and weak $p$-admissible $B_{k,n}$–singular operators are bounded from $L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)$ to $L_{p,\omega_1,\gamma}(\mathbb{R}_{k,+}^n)$ for $1 \leq p < \infty$. Note that, our results in the case $k = 1$ were proved in [13], which is some generalization of the paper by I. A. Aliev, A. D. Gadjiev [5].

We point out that the $p$-admissible $B_{k,n}$–singular operators (see Theorem 2.1). These conditions are satisfied by many interesting operators in harmonic analysis, such as the $B_{k,n}$–Riesz transforms (see [9, 10]), $B_{k,n}$–singular integral operators (for example, for $k = 1$ see [5, 11, 13, 24, 25]), $B_{k,n}$–Hardy–Littlewood maximal operators ([18], for $n = k = 1$ see [32], for $k = 1$ see [17] and for $k = n$ see [15]) and so on.

2. Notations and Background

Suppose that $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ are vectors in $\mathbb{R}^n$, $(x, \xi) = x_1\xi_1 + \cdots + x_n\xi_n$, $|x| = \sqrt{(x, x)}$, $x = (x', x'') = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$. Let $\mathbb{R}_{k,+}^n = \{x \in \mathbb{R}^k : x_1 > 0, \ldots, x_k > 0\}$, $\mathbb{R}_{k,+}^k = \{x = (x_1, \ldots, x_n) : x_1, x_2, \ldots, x_k > 0\}$, $1 \leq k \leq n$, $S_{k,+} = \{x \in \mathbb{R}_{k,+}^n : |x| = 1\}$.

For $x \in \mathbb{R}_{k,+}^n$ and $r > 0$, we denote by $E(x, r) = \{y \in \mathbb{R}_{k,+}^n : |x - y| < r\}$ the open ball centered at $x$ of radius $r$, and by $E'(x, r) = \mathbb{R}_{k,+}^n \setminus E(x, r)$ denote its complement, $E'(x', r) = \{y' \in \mathbb{R}_{k,+}^k : |x' - y'| < r\}$, $E'(x', r) = \mathbb{R}_{k,+}^k \setminus E'(x', r)$.

For measurable set $E \subset \mathbb{R}_{k,+}^n$ denote $|E| = \int_E \gamma^\gamma d\gamma$, then $|E(0,r)|_\gamma = \omega(n, \gamma) r^{n+|\gamma|}$, where $\gamma = (\gamma_1, \ldots, \gamma_k)$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$ and $\omega(n, \gamma) = |E(0,1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega : \mathbb{R}_{k,+}^n \rightarrow \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)$ the set of all measurable functions $f$ on $\mathbb{R}_{k,+}^n$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)} \equiv \|f\|_{p,\omega,\gamma;\mathbb{R}_{k,+}^n} = \left( \int_{\mathbb{R}_{k,+}^n} |f(x)|^p \omega(x)(x')^\gamma d\gamma \right)^{1/p}, \quad 1 \leq p < \infty$$

is finite. For $\omega = 1$ the space $L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)$ is denoted by $L_{p,\gamma}(\mathbb{R}_{k,+}^n)$, and the norm $\|f\|_{L_{p,\omega,\gamma}(\mathbb{R}_{k,+}^n)}$ by $\|f\|_{L_{p,\gamma}(\mathbb{R}_{k,+}^n)}$.

The operator of generalized shift ($B_{k,n}$–shift operator) is defined by the following way (see [18], [30]):

$$T^\beta f(x) = C_{\gamma,k} \int_0^\pi \cdots \int_0^\pi f((x', y'), \beta, x'' - y'') d\nu(\beta),$$
where
\[ C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma^{-1}\left(\frac{\gamma+1}{2}\right) \prod_{i=1}^{k} \Gamma\left(\frac{\gamma+1}{2} - \frac{i}{2}\right), \quad (x', y')_\beta = ((x_1, y_1)_\beta, \ldots, (x_k, y_k)_\beta), \quad (x_i, y_i)_\beta = (x_i^2 - 2x_iy_i \cos \beta_i + y_i^2)^{1/2}, 1 \leq i \leq k. \]

Note that this shift operator is closely connected with \(B_{k,n}\)-Laplace-Bessel singular differential operators (see [18], [30]).

The translation operator \(T_y\) generated the corresponding \(B_{k,n}\)-convolution
\[(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)[T_y g(x)](y')^\gamma dy,\]
for which the Young inequality
\[\|f \otimes g\|_{L_{r,\gamma}} \leq \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \leq p, q, r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1\]
holds.

**Lemma 2.1.** [28] Let \(1 \leq p \leq \infty\). Then for all \(y \in \mathbb{R}^n_{k,+}\), \(T_y f\) belongs \(L_{p,\gamma}(\mathbb{R}^n_{k,+})\) and
\[\|T_y f(\cdot)\|_{L_{p,\gamma}} \leq \|f\|_{L_{p,\gamma}}. \quad (2.1)\]

**Definition 2.1.** A function \(K\) defined on \(\mathbb{R}^n_{k,+}\), is said to be \(B_{k,n}\)-singular kernel in the space \(\mathbb{R}^n_{k,+}\) if
i) \(K \in C^\infty(\mathbb{R}^n_{k,+})\); ii) \(K(rx) = r^{-n-\gamma} K(x)\) for each \(r > 0, x \in \mathbb{R}^n_{k,+}\); iii) \(\int_{S_{k,+}} K(x)x^\gamma d\sigma(x) = 0\), where \(d\sigma\) is the element of area of the \(S_{k,+}\).

The operator \(T\) is called sublinear, if for all \(\lambda, \mu > 0\) and for all \(f\) and \(g\) in the domain of \(T\)
\[|T(\lambda f + \mu g)(x)| \leq \lambda |Tf(x)| + \mu |Tg(x)|.\]

**Definition 2.2.** (\(p\)-admissible \(B_{k,n}\)-singular operator). Let \(1 < p < \infty\). A sublinear operator \(T\) will be called \(p\)-admissible \(B_{k,n}\)-singular operator, if:
1) \(T\) satisfies the size condition of the form
\[\chi_{E(x,r)}(z) \left| T\left( f_{x_{E(x,r)}} \right)(z) \right| \leq C\chi_{E(x,r)}(z) \int_{\mathbb{R}^n_{k,+} \setminus E(x,2r)} T_y |x|^{-n-\gamma} |y| (y')^\gamma dy \quad (2.2)\]
for \(x \in \mathbb{R}^n_{k,+}\) and \(r > 0\); 2) \(T\) is bounded in \(L_{p,\gamma}(\mathbb{R}^n_{k,+})\).

**Definition 2.3.** (weak \(p\)-admissible \(B_{k,n}\)-singular operator). Let \(1 \leq p < \infty\). A sublinear operator \(T\) will be called the weak \(p\)-admissible \(B_{k,n}\)-singular operator, if:
1) \(T\) satisfies the size condition (2.2). 2) \(T\) is bounded from \(L_{p,\gamma}(\mathbb{R}^n_{k,+})\) to the weak \(WL_{p,\gamma}(\mathbb{R}^n_{k,+})\).
Remark 2.1. Note that $p$-admissible singular operators were introduced and their boundedness on vanishing generalized Morrey spaces was studied in [31]. Also $\Phi$-admissible singular operators and weak $\Phi$-admissible singular operators were introduced and their boundedness on generalized Orlicz-Morrey spaces was studied in [19, 21].

First, we establish the boundedness in weighted $L_{p,\gamma}$ spaces for a large class of $p$-admissible $B_{k,n}$-singular operator.

**Theorem 2.1.** Let $p \in (1, \infty)$ and $T$ be a $p$-admissible $B_{k,n}$-singular operators.

Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and the following three conditions are satisfied:
(a) there exist $b > 0$ such that
$$\sup_{|x|/8 <|y| \leq 8|x|} \omega_1(y) \leq b \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n_{k,+},$$

(b) $A \equiv \sup_{r>0} \left( \int_{E(0,2r)} \omega_1(x) |x|^{-(n+|\gamma|)p(x')} \gamma dx \right) \left( \int_{E(0,r)} \omega^{1-p'}(x)(x')^\gamma dx \right)^{p-1} < \infty,$

(c) $B \equiv \sup_{r>0} \left( \int_{E(0,r)} \omega_1(x)(x')^\gamma dx \right) \left( \int_{E(0,2r)} \omega^{1-p'}(x) |x|^{-(n+|\gamma|)p(x')} \gamma dx \right)^{p-1} < \infty.$

Then there exists a constant $c$, independent of $f$, such that for all $f \in L_{p,\omega_1}(\mathbb{R}^n_{k,+})$
$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(x)(x')^\gamma dx \leq c \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx.$$ \hspace{1cm} (2.3)

Moreover, condition (a) can be replaced by the condition
\hspace{1cm} (a') \therefore there exist $b > 0$ such that
$$\omega_1(x) \left( \sup_{|x|/8 <|y| \leq 8|x|} \frac{1}{\omega(y)} \right) \leq b \quad \text{for a.e. } x \in \mathbb{R}^n.$$

**Proof.** For $l \in \mathbb{Z}$ we define $E_l = \{ x \in \mathbb{R}^n_{k,+} : 2^l < |x| \leq 2^{l+1} \}$, $E_{l,1} = \{ x \in \mathbb{R}^n_{k,+} : |x| \leq 2^{l-1} \}$, $E_{l,2} = \{ x \in \mathbb{R}^n_{k,+} : 2^{l-1} < |x| \leq 2^{l+2} \}$, $E_{l,3} = \{ x \in \mathbb{R}^n_{k,+} : |x| > 2^{l+2} \}$. Then $E_{l,2} = E_{l-1} \cup E_l \cup E_{l+1}$ and the multiplicity of the covering $\{E_{l,2}\}_{l \in \mathbb{Z}}$ is equal to 3.

Given $f \in L_{p,\omega_1}(\mathbb{R}^n_{k,+})$, we write
$$|Tf(x)| = \sum_{l \in \mathbb{Z}} |Tf(x)| \chi_{E_l}(x) \leq \sum_{l \in \mathbb{Z}} |Tf_{l,1}(x)| \chi_{E_l}(x)$$
$$+ \sum_{l \in \mathbb{Z}} |Tf_{l,2}(x)| \chi_{E_l}(x) + \sum_{l \in \mathbb{Z}} |Tf_{l,3}(x)| \chi_{E_l}(x)$$
$$\equiv T_1 f(x) + T_2 f(x) + T_3 f(x),$$

where $\chi_{E_l}$ is the characteristic function of the set $E_l$, $f_{l,i} = f \chi_{E_l,i}$, $i = 1, 2, 3$. 

First we shall estimate $\|T_1f\|_{L^{p,\omega_1,\gamma}}$. Note that for $x \in E_t$, $y \in E_{k,1}$ we have $|y| \leq 2^{l-1} \leq |x|/2$. Moreover, $E_t \cap supp f_{k,1} = \emptyset$ and $|x - y| \geq |x|/2$. Hence by (2.2)

$$T_1f(x) \leq c_0 \sum_{t \in \mathbb{Z}} \left( \int_{\mathbb{R}^n_{k,+}} T^y|x|^{-n-|\gamma|} |f_{t,1}(y)| |y|\gamma dy \right) \chi_{E_t}$$

$$\leq c_0 \int_{E(0,|x|/2)} |x - y|^{-n-|\gamma|} |f(y)| (y')\gamma dy$$

$$\leq 2^{n+|\gamma|} c_0 |x|^{-n-|\gamma|} \int_{E(0,|x|/2)} |f(y)| (y')\gamma dy$$

for any $x \in E_t$. Hence we have

$$\int_{\mathbb{R}^n_{k,+}} |T_1f(x)|^p \omega_1(x) (x')\gamma dx$$

$$\leq \left( 2^{n+|\gamma|} c_0 \right)^p \int_{\mathbb{R}^n_{k,+}} \left( \int_{E(0,|x|/2)} |f(y)| (y')\gamma dy \right)^p |x|^{-(n+|\gamma|)} \omega_1(x) (x')\gamma dx.$$  

Since $A < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n_{k,+}} \omega_1(x)|x|^{-(n+|\gamma|)} p \left( \int_{E(0,|x|/2)} |f(y)| (y')\gamma dy \right)^p (x')\gamma dx$$

$$\leq C \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) (x')\gamma dx$$

holds and $C \leq c' A$, where $c'$ depends only on $n$ and $p$. In fact the condition $A < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [8]). Hence, we obtain

$$\int_{\mathbb{R}^n_{k,+}} |T_1f(x)|^p \omega_1(x) (x')\gamma dx \leq c_1 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) (x')\gamma dx. \quad (2.4)$$

where $c_1$ is independent of $f$.

Next we estimate $\|T_3f\|_{L^{p,\omega_1,\gamma}}$. As is easy to verify, for $x \in E_t$, $y \in E_{k,3}$ we have $|y| > 2|x|$ and $|x - y| \geq |y|/2$. Since $E_t \cap supp f_{k,3} = \emptyset$, for $x \in E_t$ by (2.2) we obtain

$$T_3f(x) \leq c_0 \int_{E(0,|x|)} T^y|x|^{-n-|\gamma|} |f(y)| (y')\gamma dy$$

$$\leq 2^{n+|\gamma|} c_0 \int_{E(0,|x|)} |f(y)||x - y|^{-n-|\gamma|} (y')\gamma dy$$

$$\leq 2^{n+|\gamma|} c_0 \int_{E(0,|x|)} |f(y)||y|^{-n-|\gamma|} (y')\gamma dy.$$
Hence we have
\[
\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x) \, (x')^\gamma \, dx \\
\leq \left(2^{n+\gamma}c_0\right)^p \int_{\mathbb{R}^n_{k,+}} \left( \int_{E(0,2|x|)} |f(y)||y|^{-n-\gamma} \, (y')^\gamma \, dy \right)^p \omega_1(x) \, (x')^\gamma \, dx.
\]
Since \( B < \infty \), the Hardy inequality
\[
\int_{\mathbb{R}^n_{k,+}} \omega_1(x) \left( \int_{E(0,2|x|)} |f(y)||y|^{-n-\gamma} \, (y')^\gamma \, dy \right)^p \, (x')^\gamma \, dx \\
\leq C \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) \, (x')^\gamma \, dx
\]
holds and \( C \leq c'B \), where \( c' \) depends only on \( n \) and \( p \). In fact the condition \( B < \infty \) is necessary and sufficient for the validity of this inequality (see [1], [8]). Hence, we obtain
\[
\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x) \, (x')^\gamma \, dx \leq c_2 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x) \, (x')^\gamma \, dx, \tag{2.5}
\]
where \( c_2 \) is independent of \( f \).

Finally, we estimate \( ||T_2 f||_{L_{p,\omega_1,\gamma}} \). By the \( L_{p,\gamma}(\mathbb{R}^n_{k,+}) \) boundedness of \( T \) and condition (a) we have
\[
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x) \, (x')^\gamma \, dx = \int_{\mathbb{R}^n_{k,+}} \left( \sum_{l \in \mathbb{Z}} |T f_{l,2}(x)| \chi_{E_l}(x) \right)^p \omega_1(x) \, (x')^\gamma \, dx \\
= \int_{\mathbb{R}^n_{k,+}} \left( \sum_{l \in \mathbb{Z}} |T f_{l,2}(x)|^p \chi_{E_l}(x) \right) \omega_1(x) \, (x')^\gamma \, dx \\
= \sum_{l \in \mathbb{Z}} \int_{E_l} |T f_{l,2}(x)|^p \omega_1(x) \, (x')^\gamma \, dx \\
\leq \sum_{l \in \mathbb{Z}} \sup_{y \in E_l} \omega_1(y) \int_{\mathbb{R}^n_{k,+}} |T f_{l,2}(x)|^p \, (x')^\gamma \, dx \\
\leq ||T||^p \sum_{l \in \mathbb{Z}} \sup_{y \in E_l} \omega_1(y) \int_{E_{l,2}} |f_{l,2}(x)|^p \, (x')^\gamma \, dx \\
= ||T||^p \sum_{l \in \mathbb{Z}} \sup_{y \in E_l} \omega_1(y) \int_{E_{l,2}} |f(x)|^p \, (x')^\gamma \, dx,
\]
where \( ||T|| \equiv ||T||_{L_{p,\gamma}(\mathbb{R}^n_{k,+}) \rightarrow L_{p,\gamma}(\mathbb{R}^n_{k,+})} \). Since, for \( x \in E_{l,2}, 2^{l-1} < |x| \leq 2^{l+2} \), we have by condition (a)
\[
\sup_{y \in E_l} \omega_1(y) = \sup_{2^{l-1} < |y| \leq 2^{l+2}} \omega_1(y) \leq \sup_{|x|/8 < |y| \leq 8|x|} \omega_1(y) \leq b \omega(x)
\]

Corollary 2.1. Let $K$ Then the operator $\omega_p(x)(x')^\gamma dx \leq \|T\|_{p,b} \sum_{l \in Z} \int_{E_{l,2}} |f(x)|^p \omega(x)(x')^\gamma dx 
\leq c_3 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx, \quad (2.6)$

where $c_3 = 3\|T\|_{p,b}$, since the multiplicity of covering $\{E_{l,2}\}_{l \in Z}$ is equal to 3. Inequalities (2.4), (2.5), (2.6) imply (2.3) which completes the proof.  

Similarly we prove the following weak variant of Theorem 2.1.

Theorem 2.2. Let $p \in [1, \infty)$ and let $T$ be a $p$-admissible $B_{k,n}$-singular operators. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied.

Then there exists a constant $c$, independent of $f$, such that

$$\int_{\{x \in \mathbb{R}^n_{k,+} : |Tf(x)| > \lambda\}} \omega_1(x)(x')^\gamma dx \leq \frac{c}{\lambda^p} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x)(x')^\gamma dx \quad (2.7)$$

for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$.

Let $k$ is a $B_{k,n}$-singular kernel and $K$ be the $B_{k,n}$-singular integral operator

$$Kf(x) = \text{p.v.} \int_{\mathbb{R}^n_{k,+}} T^y k(x)f(y')(y')^\gamma dy.$$ 

Then $K$ is a $p$-admissible $B_{k,n}$-singular operator for $1 < p \leq \infty$ and weak $p$-admissible $B_{k,n}$-singular operators for $1 \leq p < \infty$. Thus, we have

Corollary 2.1. Let $p \in (1, \infty)$, $K$ be a $B_{k,n}$-singular operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied. Then the operator $K$ is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

Corollary 2.2. Let $p \in [1, \infty)$, $K$ be a $B_{k,n}$-singular operator. Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_{k,+}$ and conditions (a), (b), (c) be satisfied. Then the operator $K$ is bounded from $L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{p,\omega_1,\gamma}(\mathbb{R}^n_{k,+})$.

Remark 2.2. Note that, the conditions $p$-admissible $B_{k,n}$-singular operators are satisfied by many interesting operators in harmonic analysis, such as the $B_{k,n}$-maximal operator, $B_{k,n}$-singular integral operators, $B_{k,n}$-Riesz transforms and so on.

Theorem 2.3. Let $p \in (1, \infty)$ and $T$ be a $p$-admissible $B_{k,n}$-singular operators. Moreover, let $\omega(x')$, $\omega_1(x')$ be a weight functions on $\mathbb{R}^k_{k,+}$ and the following three conditions be satisfied

(a) there exists a constant $b > 0$ such that

$$\sup_{|x'|/8 < |y'| < 8|x'|} \omega_1(y') \leq b \omega(x') \quad \text{for a.e. } x' \in \mathbb{R}^k_{k,+},$$
Moreover, condition (R) covering \( \|x\| \) shall estimate \( \chi_{E} \) for a.e. \( x \) such that for all \( f \in L_{p,\omega}(\mathbb{R}^{n}_{k,+}) \)

\[
\int_{\mathbb{R}^{n}_{k,+}} |Tf(x)|^{p} \omega_{1}(x') dx' \leq c \int_{\mathbb{R}^{n}_{k,+}} |f(x)|^{p} \omega_{1}(x') dx' .
\]  

(2.8)

Moreover, condition (a) can be replaced by the condition (a') there exists a constant \( b > 0 \) such that

\[
\omega_{1}(x') \left( \sup_{|x'|/8 < |y'| < 8|x'|} \frac{1}{\omega(y')} \right) \leq b \quad \text{for a.e. } x' \in \mathbb{R}^{k}_{+}. 
\]

Proof. For \( l \in Z \) we define \( \bar{E}_{l} = \{ x \in \mathbb{R}^{n}_{k,+} : 2^{l} < |x'| \leq 2^{l+1} \} \), \( \bar{E}_{l,1} = \{ x \in \mathbb{R}^{n}_{k,+} : |x'| \leq 2^{l-1} \} \), \( \bar{E}_{l,2} = \{ x \in \mathbb{R}^{n}_{k,+} : 2^{l-1} < |x'| \leq 2^{l+1} \} \), \( \bar{E}_{l,3} = \{ x \in \mathbb{R}^{n}_{k,+} : |x'| > 2^{l+2} \} \). Then \( \bar{E}_{l,2} = \bar{E}_{l-1} \cup \bar{E}_{l} \cup \bar{E}_{l+1} \) and the multiplicity of the covering \( \{ \bar{E}_{l,2} \}_{l \in Z} \) is equal to 3. Given \( f \in L_{p,\omega,\gamma}(\mathbb{R}^{n}_{k,+}) \), we write

\[
|Tf(x)| = \sum_{l \in Z} |Tf(x)| \chi_{\bar{E}_{l}}(x) \leq \sum_{l \in Z} |Tf_{l,1}(x)| \chi_{\bar{E}_{l}}(x) 
\]

\[
+ \sum_{l \in Z} |Tf_{l,2}(x)| \chi_{\bar{E}_{l}}(x) + \sum_{l \in Z} |Tf_{l,3}(x)| \chi_{\bar{E}_{l}}(x) \tag{2.9}
\]

\[
\equiv T_{1} f(x) + T_{2} f(x) + T_{3} f(x),
\]

where \( \chi_{\bar{E}_{l}} \) is the characteristic function of the set \( \bar{E}_{l} \). \( f_{l,i} = f \chi_{\bar{E}_{l,i}} \), \( i = 1, 2, 3 \). We shall estimate \( \|T_{1} f\|_{L_{p,\omega,\gamma}} \). Note that for \( x \in \bar{E}_{l}, y \in \bar{E}_{l,1} \) we have \( |y'| \leq 2^{l-1} \leq |x'|/2 \). Moreover, \( \bar{E}_{l} \cap \text{supp} f_{l,1} = \emptyset \) and \( |x' - y'| \geq |x'|/2 \). Hence by (2.2)

\[
T_{1} f(x) \leq c_{4} \sum_{l \in Z} \left( \int_{\mathbb{R}^{n}_{k,+}} |f_{l,1}(y)| T^{y} |x|^{-n-|\gamma|} dy \right) \chi_{\bar{E}_{l}} 
\]

\[
\leq c_{4} \int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} T^{y} |x|^{-n-|\gamma|} f(y) |(y')^{|\gamma|} dy dy' 
\]

\[
\leq c_{5} \int_{\mathbb{R}^{n-k}} \int_{E'(0,|x'|/2)} |x'| + |x'' - y''|^{-n-|\gamma|} f(y) |(y')^{|\gamma|} dy' dy''
\]
for any $x \in E_l$. Using this last inequality we have

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x')(x')^\gamma dx$$

$$\leq c_5^p \int_{\mathbb{R}^n_{k,+}} \left( \int_{\mathbb{R}^n_{k,-}} E'(0,|x'|/2) \left( |x'| + |x'' - y''| \right)^{-n-\gamma} |f(y')|^\gamma dy' dy'' \right)^p$$

$$\times \omega_1(x')(x')^\gamma dx.$$  

For $x = (x', x'') \in \mathbb{R}^n$ let

$$I(x')$$

$$= \int_{\mathbb{R}^n_{k,-}} \left( \int_{\mathbb{R}^n_{k,-}} E'(0,|x'|/2) \left( |x'| + |x'' - y''| \right)^{-n-\gamma} |f(y', y'')|^\gamma dy' dy'' \right)^p dx''$$

$$= \int_{\mathbb{R}^n_{k,-}} \left( \int_{E'(0,|x'|/2)} \left( \int_{\mathbb{R}^n_{k,-}} \left( |x'| + |x'' - y''| \right)^{-n-\gamma} |f(y', y'')|^\gamma dy' \right)(x')^\gamma dy'' \right)^p dx.'$$

Using the Minkowski and Young inequalities we obtain

$$I(x') \leq \left[ \int_{E'(0,|x'|/2)} \left( \int_{\mathbb{R}^n_{k,-}} |f(y', y'')|^p dy'' \right)^{1/p} \left( \int_{\mathbb{R}^n_{k,-}} \left( |x'| + |x''| \right)^{n+\gamma} (x')^\gamma dy' \right)^{1/p} \right]^p$$

$$= \left( \int_{E'(0,|x'|/2)} \| f(\cdot, y'') \|_{p, \mathbb{R}^n_{k,-}} (y')^\gamma dy'' \right)^p \left( \int_{\mathbb{R}^n_{k,-}} \left( |x'| + |x''| \right)^{n+\gamma} (x')^\gamma dy' \right)^p$$

$$= |x'|^{-\gamma(k+|\gamma|)} \left( \int_{E'(0,|x'|/2)} \| f(\cdot, y'') \|_{p, \mathbb{R}^n_{k,-}} (y')^\gamma dy'' \right)^p \left( \int_{\mathbb{R}^n_{k,-}} \left( |x''| + 1 \right)^{n+\gamma} \right)^p$$

$$= c_6 |x'|^{-\gamma(k+|\gamma|)} \left( \int_{E'(0,|x'|/2)} \| f(\cdot, y'') \|_{p, \mathbb{R}^n_{k,-}} (y')^\gamma dy'' \right)^p.$$  

Integrating in $\mathbb{R}^n_{k,+}$ we get

$$\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x')(x')^\gamma dx$$

$$\leq c_7 \int_{\mathbb{R}^n_{k,+}} \omega_1(x') |x'|^{-(k+|\gamma|)} \left( \int_{E'(0,|x'|/2)} \| f(\cdot, y'') \|_{p, \mathbb{R}^n_{k,-}} (y')^\gamma dy'' \right)^p (x')^\gamma dx'.$$

Since $A_1 < \infty$, the Hardy inequality

$$\int_{\mathbb{R}^n_{k,+}} \omega_1(x') |x'|^{-(k+|\gamma|)} \left( \int_{E'(0,|x'|/2)} \| f(\cdot, y'') \|_{p, \mathbb{R}^n_{k,-}} (y')^\gamma dy'' \right)^p (x')^\gamma dx'$$

$$\leq C \int_{\mathbb{R}^n_{k,+}} \| f(\cdot, x') \|_{p, \mathbb{R}^n_{k,-}} \omega_1(x')(x')^\gamma dx'$$

holds and $C \leq c'A_1$, where $c'$ depends only on $n$ and $p$. In fact the condition $A_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [6], [22]).
Hence, we obtain
\[
\int_{\mathbb{R}^n_{k,+}} |T_1 f(x)|^p \omega_1(x')(x')^\gamma dx \leq c_9 \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx.
\] (2.10)

Let us estimate \( |T_3 f| \) \( \|_{L^p,\omega_1,\gamma} \). As is easy to verify, for \( x \in \tilde{E}_l, \ y \in \tilde{E}_{l,3} \) we have \( |y'| > 2|x'| \) and \( |x' - y'| > |y'|/2 \). Since \( \tilde{E}_l \cap \text{supp} f_{k,3} = \emptyset \), for \( x \in \tilde{E}_l \) by (2.2) we obtain
\[
T_3 f(x) \leq c_5 \int_{\mathbb{R}^{n-k}} \int_{\mathcal{E}'(0,2|x'|)} |f(y)| \left( |y'| + |x'' - y''| \right)^{-n-\gamma} (y')^\gamma \ dy' dy''.
\]

Using this last inequality we have
\[
\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x')(x')^\gamma dx
\]
\[
\leq c_5^p \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^{n-k}} \int_{\mathcal{E}'(0,2|x'|)} |f(y)| \left( |y'| + |x'' - y''| \right)^{-n-\gamma} (y')^\gamma \ dy' dy'' \right)^p \omega_1(x')(x')^\gamma dx.
\]

For \( x = (x', x'') \in \mathbb{R}^n \) let
\[
I_1(x') = \int_{\mathbb{R}^{n-k}} \left( \int_{\mathbb{R}^{n-k}} \int_{\mathcal{E}'(0,2|x'|)} |f(y)| \left( |y'| + |x'' - y''| \right)^{-n-\gamma} (y')^\gamma \ dy' dy'' \right)^p \ (x')^\gamma dx''.
\]

Using the Minkowski and Young inequalities we obtain
\[
I_1(x') \leq \left[ \int_{\mathcal{E}'(0,2|x'|)} \left( \int_{\mathbb{R}^{n-k}} |f(y)|^p dy'' \right)^{1/p} \left( \int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y'| + |x''|)^{n+\gamma}} (y')^\gamma dy' \right)^p \right]^p
\]
\[
= c_6 \left( \int_{\mathcal{E}'(0,2|x'|)} \left| y' \right|^{-k-\gamma} \| f(\cdot, y') \|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p \left( \int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y''| + 1)^{n+\gamma}} \right)^p
\]
\[
= c_7 \left( \int_{\mathcal{E}'(0,2|x'|)} \left| y' \right|^{-k-\gamma} \| f(\cdot, y') \|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p.
\]

Integrating over \( \mathbb{R}^k_{++} \) we get
\[
\int_{\mathbb{R}^k_{++}} |T_3 f(x)|^p \omega_1(x')(x')^\gamma dx
\]
\[
\leq c_8 \int_{\mathbb{R}^k_{++}} \left( \int_{\mathcal{E}'(0,2|x'|)} \left| y' \right|^{-k-\gamma} \| f(\cdot, y') \|_{p,\mathbb{R}^{n-k}} (y')^\gamma dy'' \right)^p \omega_1(x')(x')^\gamma dx''.
\]
Since $B_1 < \infty$, the Hardy inequality
\[
\int_{\mathbb{R}^n_{k,+}} \omega_1(x') \left( \int_{E^{2}(0,2|x|)} |y'|^{-k-|\gamma|} \|f(\cdot, y')\|_{p,\mathbb{R}^{n-1}}(y') \gamma dy' \right)^p (x')^\gamma dx' 
\leq C \int_{\mathbb{R}^n_{k,+}} \|f(\cdot, x')\|^p_{p,\mathbb{R}^{n-k}} |x'|^{-(k+|\gamma|)} \|\omega(x')\| |x'|^{(k+|\gamma|)} dx'
\]
holds and $C \leq c' B_1$, where $c'$ depends only on $n, \gamma$ and $p$. In fact the condition $B_1 < \infty$ is necessary and sufficient for the validity of this inequality (see [6], [22]). Hence, we obtain
\[
\int_{\mathbb{R}^n_{k,+}} |T_3 f(x)|^p \omega_1(x')(x')^\gamma dx \leq c_{10} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx. \tag{2.11}
\]
Finally, we estimate $\|T_2 f\|_{L^p,\omega_{1}\gamma}$ by the $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ boundedness of $T$ and condition (a1) we have
\[
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x_n)(x')^\gamma dx = \int_{\mathbb{R}^n_{k,+}} \left( \sum_{l \in Z} |T_{f_{1,2}}(x)| \chi_{E_l}(x) \right)^p \omega_1(x')(x')^\gamma dx
\]
\[
= \int_{\mathbb{R}^n_{k,+}} \left( \sum_{l \in Z} |T_{f_{1,2}}(x)| \chi_{E_l}(x) \right) \omega_1(x')(x')^\gamma dx = \sum_{l \in Z} \int_{E_l} |T_{f_{1,2}}(x)|^p \omega_1(x')(x')^\gamma dx
\]
\[
\leq \sum_{l \in Z} \sup_{x \in E_l} \omega_1(y') \int_{\mathbb{R}^n} |T_{f_{1,2}}(x)|^p (x')^\gamma dx
\]
\[
\leq \|T\|^p \sum_{l \in Z} \sup_{x \in E_l} \omega_1(y') \int_{\mathbb{R}^n} |f_{1,2}(x)|^p (x')^\gamma dx
\]
\[
= \|T\|^p \sum_{l \in Z} \sup_{x \in E_l} \omega_1(y') \int_{E_{1,2}} |f(x)|^p (x')^\gamma dx,
\]
where $\|T\| = \|T\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} \to L_{p,\gamma}(\mathbb{R}^n_{k,+})$. Since, for $x \in \widetilde{E}_{1,2}, 2^{l-1} < |x'| \leq 2^{l+2}$, we have by condition (a1)
\[
\sup_{y \in E_l} \omega_1(y') = \sup_{2^{l-1} < |y'| \leq 2^{l+2}} \omega_1(y') \leq \sup_{|x'|/8 < |y'| < 8|x'|} \omega_1(y') \leq b \omega(x')
\]
for almost all $x \in \widetilde{E}_{1,2}$. Therefore
\[
\int_{\mathbb{R}^n_{k,+}} |T_2 f(x)|^p \omega_1(x')(x')^\gamma dx
\]
\[
\leq \|T\|^p b \sum_{l \in Z} \int_{\widetilde{E}_{1,2}} |f(x)|^p \omega(x')(x')^\gamma dx \leq c_{11} \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(x')(x')^\gamma dx, \tag{2.12}
\]
where $c_{11} = 3 \|T\|^p b$, since the multiplicity of covering $\left\{ \widetilde{E}_{1,2} \right\}_{l \in Z}$ is equal to 3. Inequalities (2.9), (2.10), (2.11), (2.12) imply (2.8) which completes the proof. \hfill \Box
Similarly we prove the following weak variant of Theorem 2.3.

**Theorem 2.4.** Let \( p \in [1, \infty) \) and let \( T \) be a weak \( p \)-admissible \( B_{k,n} \)-singular operators. Moreover, let \( \omega(x'), \omega_1(x') \) be weight functions on \( \mathbb{R}_+^k \) and conditions \((a_1), (b_1), (c_1)\) be satisfied.

Then there exists a constant \( c\), independent of \( f\), such that

\[
\int_{\{x \in \mathbb{R}_+^k : |Tf(x)| > \lambda\}} \omega_1(x')(x')^\gamma dx \leq \frac{c}{\lambda^p} \int_{\mathbb{R}_+^k} |f(x)|^p \omega(x')(x')^\gamma dx \quad (2.13)
\]

for all \( f \in L_{p,\omega,\gamma}(\mathbb{R}_+^k)\).

**Corollary 2.3.** Let \( p \in (1, \infty), T \) be the \( p \)-admissible \( B_{k,n} \)-singular operators. Moreover, let \( \omega(x'), \omega_1(x') \) be weight functions on \( \mathbb{R}_+^k \) and conditions \((a_1), (b_1), (c_1)\) be satisfied. Then inequality \((2.8)\) is valid.

**Corollary 2.4.** Let \( p \in [1, \infty) \), \( T \) be the weak \( p \)-admissible \( B_{k,n} \)-singular operators. Moreover, let \( \omega(x'), \omega_1(x') \) be weight functions on \( \mathbb{R}_+^k \) and conditions \((a_1), (b_1), (c_1)\) be satisfied. Then inequality \((2.13)\) is valid.

**Remark 2.3.** Note that, if instead of \( \omega(x'), \omega_1(x') \) respectively put \( \omega(x'), \omega_1(\infty) \), then from conditions \((a), (b), (c)\) will not follows conditions \((a_1), (b_1), (c_1)\) respectively.

**Theorem 2.5.** Let \( p \in (1, \infty) \) and \( T \) be a \( p \)-admissible \( B_{k,n} \)-singular operators. Moreover, let \( \omega(t) \) be a weight function on \( (0, \infty) \), \( \omega_1(t) \) be a positive increasing function on \( (0, \infty) \) and the weighted pair \((\omega(|x|), \omega_1(|x|))\) satisfies conditions \((a), (b)\). Then there exists a constant \( c > 0 \), such that for all \( f \in L_{p,\omega,\gamma}(\mathbb{R}_+^k)\)

\[
\int_{\mathbb{R}_+^k} |Tf(x)|^p \omega_1(|x|)(x')^\gamma dx \leq c \int_{\mathbb{R}_+^k} |f(x)|^p \omega(|x|)(x')^\gamma dx. \quad (2.14)
\]

**Proof.** Suppose that \( f \in L_{p,\omega,\gamma}(\mathbb{R}_+^k) \) and \( \omega_1 \) are positive increasing functions on \((0, \infty)\) and \( \omega, \omega_1 \) satisfied the conditions \((a), (b)\).

Without loss of generality we can suppose that \( \omega_1 \) may be represented by

\[
\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda)d\lambda,
\]

where \( \omega_1(0+) = \lim_{t \to 0+} \omega_1(t) \) and \( \omega_1(t) \geq 0 \) on \((0, \infty)\). In fact there exists a sequece of increasing absolutely continuous fuctions \( \varpi_1 \), such that \( \varpi_1(t) \leq \omega_1(t) \) and \( \lim_{n \to \infty} \varpi_1(t) = \omega_1(t) \) for any \( t \in (0, \infty) \) (see [12], [14] for details).

We have

\[
\int_{\mathbb{R}_+^k} |Tf(x)|^p \omega_1(|x|)(x')^\gamma dx = \omega_1(0+) \int_{\mathbb{R}_+^k} |Tf(x)|^p (x')^\gamma dx + \int_{\mathbb{R}_+^k} |Tf(x)|^p \left( \int_0^{|x|} \psi(\lambda)d\lambda \right) (x')^\gamma dx = J_1 + J_2.
\]

If \( \omega_1(0+) = 0 \), then \( J_1 = 0 \). If \( \omega_1(0+) \neq 0 \) by the boundedness of \( T \) in \( L_{p,\gamma}(\mathbb{R}_+^k) \) thanks to \((a)\)

\[
J_1 \leq \|T\|^p \omega_1(0+) \int_{\mathbb{R}_+^k} |f(x)|^p (x')^\gamma dx
\]
\[ \leq ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x|)(x')^\gamma \, dx \leq b \, ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^\gamma \, dx. \]

After changing the order of integration in \( J_2 \) we have
\[
J_2 = \int_0^\infty \psi(\lambda) \left( \int_{E(0, \lambda)} |Tf(x)|^p (x')^\gamma \, dx \right) d\lambda \\
\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left( \int_{E(0, \lambda)} |T(\chi_{E(0, \lambda/2)})(x)|^p (x')^\gamma \, dx \right) d\lambda \\
+ \int_{E(0, \lambda)} |T(\chi_{E(0, \lambda/2)})(x)|^p (x')^\gamma \, dx \, d\lambda = J_{21} + J_{22}.
\]

Using the boundedness of \( T \) in \( L_{p, \gamma}(\mathbb{R}^n_{k,+}) \) and condition (a) we have
\[
J_{21} \leq ||T||^p \int_0^\infty \psi(t) \left( \int_{E(0, \lambda/2)} |f(y)|^p (y')^\gamma \, dy \right) dt \\
= ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \left( \int_0^{2|y|} \psi(\lambda) d\lambda \right) (y')^\gamma \, dy \\
\leq ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega_1(2|y|)(y')^\gamma \, dy \\
\leq b \, ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma \, dy.
\]

Let us estimate \( J_{22} \). For \( |x| > \lambda \) and \( |y| \leq \lambda/2 \) we have
\[
|x|/2 \leq |x - y| \leq 3|x|/2,
\]
and so
\[
J_{22} \leq c_4 \int_0^\infty \psi(\lambda) \left( \int_{E(0, \lambda)} \left( \int_{E(0, 2\lambda)} T^y |x|^{-n+|\gamma|}|f(y)|(y')^\gamma \, dy \right)^p (x')^\gamma \, dx \right) d\lambda \\
\leq c_5 \int_0^\infty \psi(\lambda) \left( \int_{E(0, \lambda)} \left( \int_{E(0, 2\lambda)} |f(y)|(y')^\gamma \, dy \right)^p |x|^{-(n+|\gamma|)}(x')^\gamma \, dx \right) d\lambda \\
= c_6 \int_0^\infty \psi(\lambda) \lambda^{-(n+|\gamma|)(p-1)} \left( \int_{E(0, \lambda/2)} |f(y)|(y')^\gamma \, dy \right)^p d\lambda.
\]

The Hardy inequality
\[
\int_0^\infty \psi(\lambda) \lambda^{-(n+|\gamma|)(p-1)} \left( \int_{E(0, \lambda/2)} |f(y)|(y')^\gamma \, dy \right)^p d\lambda \\
\leq C \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma \, dy
\]
ie valid, for \( p \in (1, \infty) \) is valid by the condition \( C \leq c' A' \) (see \([6], [22]\)), where
\[
A' \equiv \sup_{\tau > 0} \left( \int_{2\tau}^{\infty} \psi(t) t^{-(n+|\gamma|)(p-1)} d\tau \right) \left( \int_{E(0, \tau)} \omega^{-p'}(|y|)(y')^{\gamma} dy \right)^{p^{-1}} < \infty.
\]
Note that
\[
\int_{2t}^{\infty} \psi(\tau) \tau^{-(n+|\gamma|)(p-1)} d\tau \\
= (n + |\gamma|)(p-1) \int_{2t}^{\infty} \psi(\tau) d\tau \int_{\tau}^{\infty} \lambda^{-k-(n+|\gamma|)(p-1)} d\lambda \\
= (n + |\gamma|)(p-1) \int_{2t}^{\infty} \lambda^{-k-(n+|\gamma|)(p-1)} d\lambda \int_{2t}^{\infty} \psi(\tau) d\tau \\
\leq (n + |\gamma|)(p-1) \int_{2t}^{\infty} \lambda^{-k-(n+|\gamma|)(p-1)} \omega_1(\lambda) d\lambda \\
= \frac{(p-1)}{\omega(n, |\gamma|)} \int_{E(0, 2t)} \omega_1(|y|) |y|^{-(n+|\gamma|)p} (y')^{\gamma} dy.
\]
Condition (b) of the theorem guarantees that \( A' \leq \frac{(n+|\gamma|)(p-1)}{\omega(n, |\gamma|)} A < \infty \). Hence, applying the Hardy inequality, we obtain
\[
J_{22} \leq c_7 \int_{\mathbb{R}_+^n} |f(x)|^p \omega(|x|)(x')^\gamma dx.
\]
Combining the estimates of \( J_1 \) and \( J_2 \), we get (2.14) for \( \omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau) d\tau \). By Fatou’s theorem on passing to the limit under the Lebesgue integral sign, this implies (2.14). The theorem is proved.

**Corollary 2.5.** Let \( p \in (1, \infty) \), \( k \) be a \( B_{k,n} \)-singular kernel and \( K \) be the corresponding operator. Moreover, let \( \omega(t) \) be a weight function on \((0, \infty)\), \( \omega_1(t) \) be a positive increasing function on \((0, \infty)\) and the weighted pair \( (\omega(|x|), \omega_1(|x|)) \) satisfies conditions (a), (b). Then for the operator \( K \) the inequality (2.14) is valid.

**Example 2.1.** Let
\[
\omega(t) = \begin{cases} 
 t^{(n+|\gamma|)(p-1)} \ln^p \frac{1}{t^\gamma}, & \text{for } t \in \left(0, \frac{1}{2}\right) \\
 2^{p-1} \ln^p t^\beta, & \text{for } t \in \left[\frac{1}{2}, \infty\right),
\end{cases}
\]
\[
\omega_1(t) = \begin{cases} 
 t^{(n+|\gamma|)(p-1)} \nu^\alpha, & \text{for } t \in \left(0, \frac{1}{2}\right) \\
 2^{p-1} \nu^\alpha, & \text{for } t \in \left[\frac{1}{2}, \infty\right),
\end{cases}
\]
where \( 0 < \alpha \leq \beta < (n + |\gamma|)(p-1) \). Then the weighted pair \( (\omega(|x|), \omega_1(|x|)) \) satisfies the condition of Theorem 2.5.

**Theorem 2.6.** Let \( p \in (1, \infty) \) and \( T \) be a \( p \)-admissible \( B_{k,n} \)-singular operators. Moreover, let \( \omega(t) \) be a weight function on \((0, \infty)\), \( \omega_1(t) \) be a positive decreasing function on \((0, \infty)\) and the weighted pair \( (\omega(|x|), \omega_1(|x|)) \) satisfies conditions (a), (c). Then inequality (2.14) is valid.

**Proof.** Without loss of generality we can suppose that \( \omega_1 \) may be represented by
\[
\omega_1(t) = \omega_1(+\infty) + \int_t^{\infty} \psi(\tau) d\tau,
\]
where $\omega_1(+\infty) = \lim_{t \to \infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a sequence of decreasing absolutely continuous functions $\omega_n$ such that $\omega_n(t) \leq \omega_1(t)$ and $\lim_{n \to \infty} \omega_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [12], [14] for details).

We have

$$\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x|)(x')^\gamma dx = \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p (x')^\gamma dx$$

$$+ \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \left( \int_{|x|}^\infty \psi(\tau)d\tau \right) (x')^\gamma dx$$

$$= I_1 + I_2.$$  

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, by the boundedness of $T$ in $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and condition (a) we have

$$J_1 \leq ||T|| \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx$$

$$\leq ||T|| \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x|)(x')^\gamma dx$$

$$\leq b ||T|| \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x|)(x')^\gamma dx.$$  

After changing the order of integration in $J_2$ we have

$$J_2 = \int_0^\infty \psi(\lambda) \left( \int_{E(0,\lambda)} |Tf(x)|^p (x')^\gamma dx \right) d\lambda$$

$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left( \int_{E(0,\lambda)} |T(f\chi_{E(0,2\lambda)})(x)|^p (x')^\gamma dx \right)$$

$$+ \int_{E(0,\lambda)} |T(f\chi_{E(0,2\lambda)})(x)|^p (x')^\gamma dx \right) d\lambda$$

$$= J_{21} + J_{22}.$$  

Using the boundedness of $T$ in $L_p(\mathbb{R}^n_{k,+})$ and condition (a) we obtain

$$J_{21} \leq ||T|| \int_0^\infty \psi(t) \left( \int_{|y|<2\lambda} |f(y)|^p (y')^\gamma dy \right) dt$$

$$= ||T|| \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \left( \int_{|y|/2}^\infty \psi(\lambda)d\lambda \right) (y')^\gamma dy$$

$$\leq ||T|| \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega_1(|y|/2)(y')^\gamma dy$$

$$\leq b ||T|| \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y|)(y')^\gamma dy.$$
Let us estimate $J_{22}$. For $|x| < \lambda$ and $|y| \geq 2\lambda$ we have $|y|/2 \leq |x - y| \leq 3|y|/2$, and so
\[
J_{22} \leq c_8 \int_0^\infty \psi(\lambda) \left( \int_{E(0,\lambda)} \left( \int_{E(0,2\lambda)} T^y |x|^{-n-|\gamma|} |f(y)(y')^\gamma dy \right)^p (x')^\gamma dx \right) d\lambda
\]
\[
\leq 2^n c_8 \int_0^\infty \psi(\lambda) \left( \int_{E(0,\lambda)} \left( \int_{E(0,2\lambda)} |y|^{-n-|\gamma|} |f(y)(y')^\gamma dy \right)^p (x')^\gamma dx \right) d\lambda
\]
\[
= c_9 \int_0^\infty \psi(\lambda) \lambda^{n+|\gamma|} \left( \int_{E(0,2\lambda)} |y|^{-n-|\gamma|} |f(y)(y')^\gamma dy \right)^p d\lambda.
\]

The Hardy inequality
\[
\int_0^\infty \psi(\lambda) \lambda^{n+|\gamma|} \left( \int_{E(0,2\lambda)} |y|^{-n-|\gamma|} |f(y)(y')^\gamma dy \right)^p d\lambda
\]
\[
\leq C \int_{\mathbb{R}_{k,+}^n} |f(y)|^p |y|^{-(n+|\gamma|)p} |y|^{(n+|\gamma|)p} \omega(|y|)(y')^\gamma dy = C \int_{\mathbb{R}_{k,+}^n} |f(y)|^p \omega(|y|)(y')^\gamma dy
\]
is valid, for $p \in (1, \infty)$ is valid by the condition $C \leq c\mathcal{B}'$ (see [6], [22]), where
\[
\mathcal{B}' = \sup_{\tau > 0} \left( \int_{E(0,\tau)} \omega^{1-p'}(|y|)|y|^{-(n+|\gamma|)p'}(y')^\gamma dy \right)^{p-1} < \infty.
\]

Note that
\[
\int_0^\tau \psi(t)t^{n+|\gamma|} dt = (n + |\gamma|) \int_0^\tau \psi(t) dt \int_0^t \lambda^{n+|\gamma|-1} d\lambda
\]
\[
= (n + |\gamma|) \int_0^\tau \lambda^{n+|\gamma|-1} d\lambda \int_0^\tau \psi(t) d\tau
\]
\[
\leq (n + |\gamma|) \int_0^\tau \lambda^{n+|\gamma|-1} \omega_1(\lambda) d\lambda
\]
\[
= \frac{n + |\gamma|}{\omega(n, |\gamma|)} \int_{E(0,\tau)} \omega_1(|x|)(x')^\gamma dx.
\]

Condition (c) of the theorem guarantees that $\mathcal{B}' \leq \frac{n+|\gamma|}{\omega(n,|\gamma|)} \mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain
\[
J_{22} \leq c_{10} \int_{\mathbb{R}_{k,+}^n} |f(x)|^p \omega(|x|)(x')^\gamma dx.
\]

Combining the estimates of $J_1$ and $J_2$, we get (2.14) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(t) d\tau$. By Fatou’s theorem on passing to the limit under the Lebesgue integral sign, this implies (2.14). The theorem is proved.

**Corollary 2.6.** Let $p \in (1, \infty)$, $k$ be a $B_{k,n}$–singular kernel and $K$ be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive decreasing function on $(0, \infty)$ and the weighted pair $(\omega(|x|), \omega_1(|x|))$ satisfies conditions (a), (c). Then for the operator $K$ the inequality (2.14) is valid.
Example 2.2. Let
\[
\omega(t) = \begin{cases} 
\frac{1}{(n+|\gamma|+\alpha)} \ln ^\frac{\nu}{\alpha} \frac{1}{t}, & \text{for } t < d \\
\ (d-n-|\gamma|) t^\alpha, & \text{for } t \geq d,
\end{cases}
\]

\[
\omega_1(t) = \begin{cases} 
\frac{1}{(n+|\gamma|+\alpha)} \ln ^\frac{\beta}{\alpha} \frac{1}{t}, & \text{for } t < d \\
\ (d-n-|\gamma|-\lambda) t^\lambda, & \text{for } t \geq d
\end{cases}
\]

where \( \beta < \nu \leq 0, -n - |\gamma| < \lambda < \alpha < 0, d = e^{\frac{\beta}{n+|\gamma|}} \).

Then the weighted pair \((\omega(|x|), \omega_1(|x|))\) satisfies the condition of Theorem 2.6.

Theorem 2.7. Let \( p \in (1, \infty) \) and \( T \) be a \( p \)-admissible \( B_{k,n} \)-singular operators. Moreover, let \( \omega(t) \) be a weight function on \((0, \infty)\), \( \omega_1(t) \) be a positive increasing function on \((0, \infty)\) and \( \omega(|x'|), \omega_1(|x'|) \) be satisfied the conditions (a1), (b1).

Then there exists a constant \( c > 0 \), such that for all \( f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \)
\[
\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x'|)(x')^\gamma dx \leq c \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x'|)(x')^\gamma dx.
\] (2.15)

Proof. Suppose that \( f \in L_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \), \( \omega_1 \) are positive increasing functions on \((0, \infty)\) and \( \omega(t), \omega_1(t) \) satisfied the conditions (a1), (b1).

Without loss of generality we can suppose that \( \omega_1 \) may be represented by
\[
\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\lambda) d\lambda,
\]
where \( \omega_1(0+) = \lim_{t \to 0} \omega_1(t) \) and \( \omega_1(t) \geq 0 \) on \((0, \infty)\). In fact there exists a sequence of increasing absolutely continuous functions \( \omega_n \) such that \( \omega_n(t) \leq \omega_1(t) \) and \( \lim_{n \to \infty} \omega_n(t) = \omega_1(t) \) for any \( t \in (0, \infty) \) (see [12], [14] for details).

We have
\[
\int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \omega_1(|x'|)(x')^\gamma dx = \omega_1(0+) \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p(x')^\gamma dx +
\]
\[
+ \int_{\mathbb{R}^n_{k,+}} |Tf(x)|^p \left( \int_0^{x'} \psi(\lambda) d\lambda \right) (x')^\gamma dx = J_1 + J_2.
\]

If \( \omega_1(0+) = 0 \), then \( J_1 = 0 \). If \( \omega_1(0+) \neq 0 \) by the boundedness of \( T \) in \( L_{p,\gamma}(\mathbb{R}^n_{k,+}) \) thanks to (a)
\[
J_1 \leq ||T||^p \omega_1(0+) \int_{\mathbb{R}^n_{k,+}} |f(x)|^p(x')^\gamma dx
\]
\[
\leq ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x'|)(x')^\gamma dx
\]
\[
\leq ||T||^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x'|)(x')^\gamma dx.
\]
After changing the order of integration in $J_2$ we have

$$J_2 = \int_0^\infty \psi(\lambda) \left( \int_{\mathbb{R}^{n-k}_+} \int_{E'(0,\lambda)} |Tf(x)|^p(x') \gamma dx \right) d\lambda$$

$$\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left( \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} |T(f\chi_{|x'|>\lambda/2})(x)|^p(x') \gamma dx \right) d\lambda$$

$$+ \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} |T(f\chi_{|x'|\leq\lambda/2})(x)|^p(x') \gamma dx \right) d\lambda = J_{21} + J_{22}.$$ 

Using the boundedness of $T$ in $L_{p,\gamma}(\mathbb{R}^n)$ we obtain

$$J_{21} \leq \|T\|^p \int_0^\infty \psi(t) \left( \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda/2)} |f(y)|^p(y') \gamma dy \right) dt$$

$$= \|T\|^p \int_0^\infty \psi(t) \left( \int_{E'(0,\lambda/2)} \|f(\cdot, y')\|^p_{p,\mathbb{R}^{n-k}} \gamma dy' \right) dt$$

$$= \|T\|^p \int_{\mathbb{R}^{n-k}_+} \|f(\cdot, y')\|^p_{p,\mathbb{R}^{n-k}} \left( \int_0^2|y'| \psi(\lambda)d\lambda \right) (y') \gamma dy'$$

$$\leq \|T\|^p \int_{\mathbb{R}^{n-k}_+} \|f(\cdot, y')\|^p_{p,\mathbb{R}^{n-k}} \omega_1(2|y'|)(y') \gamma dy'$$

$$\leq b \|T\|^p \int_{\mathbb{R}^{n-k}_+} |f(y)|^p\omega(|y'|)(y') \gamma dy.$$

Let us estimate $J_{22}$. For $|x'| > \lambda$ and $|y'| \leq \lambda/2$ we have $|x'|/2 \leq |x'| - |y'| \leq 3|x'|/2$, and so

$$J_{22} \leq c_9 \int_0^\infty \psi(\lambda) \left( \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} \left( \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda/2)} \frac{|f(y)|}{|x-y|^{n+|\gamma|}} dy \right)^p (x') \gamma dx \right) d\lambda \leq$$

$$c_{10} \int_0^\infty \psi(\lambda) \left( \int_{E'(0,\lambda)} \int_{E'(0,\lambda/2)} \left( \int_{\mathbb{R}^{n-k}} \int_{E'(0,\lambda)} \frac{|f(y)|}{(|x'| + |x'' - y'|)^{n+|\gamma|}} (y') \gamma dy \right)^p (x') \gamma dx \right) d\lambda.$$ 

For $x = (x', x'') \in \mathbb{R}^{n-k}_+$ let

$$J(x', \lambda) = \int_{\mathbb{R}^{n-k}} \left( \int_{E'(0,\lambda/2)} \left( \int_{\mathbb{R}^{n-k}} \frac{|f(y)|}{(|x'| + |x'' - y''|)^{n+|\gamma|}} (y') \gamma dy \right)^p dx''$$
Using the Minkowski and Young inequalities we obtain

\[ J(x', \lambda) \leq \left[ \int_{E'(0, \lambda/2)} \left( \int_{\mathbb{R}^{n-k}} |f(y)|^p \, dy'' \right)^{1/p} \left( \int_{\mathbb{R}^{n-k}} \frac{dy''}{(|y''| + |x'|)^{n+|\gamma|}} \right) (y')^\gamma \, dy' \right]^p \]

\[ \leq \left( \int_{E'(0, \lambda/2)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma \, dy' \right)^p \left( \int_{\mathbb{R}^{n-k}} \frac{dy'}{(|y''| + |x'|)^{n+|\gamma|}} \right)^p \]

\[ = c_3 |x'|^{-(k+|\gamma|)} \left( \int_{E'(0, \lambda/2)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma \, dy' \right)^p \]

\[ \times \left( \int_{\mathbb{R}^{n-k}} \frac{dy'}{(1 + |y'|)^{n+|\gamma|}} \right)^p \]

\[ = c_4 |x'|^{-(k+|\gamma|)} \left( \int_{E'(0, \lambda/2)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma \, dy' \right)^p. \]

Integrating in \((0, \infty) \times E'(0, \lambda)\) we get

\[ J_{22} \leq c_5 \int_0^\infty \psi(\lambda) \]

\[ \times \left( \int_{E'(0, \lambda)} \left( \int_{E(0, \lambda/2)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma \, dy' \right)^p |x'|^{-(k+|\gamma|)} (x')^\gamma \, dx' \right) d\lambda \]

\[ = \frac{2c_5}{p-1} \int_0^\infty \psi(\lambda)|\lambda|^{-(k+|\gamma|)} \left( \int_{E(0, \lambda/2)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma \, dy' \right)^p d\lambda. \]

The Hardy inequality

\[ \int_0^\infty \psi(\lambda)|\lambda|^{-(k+|\gamma|)} \left( \int_{E(0, \lambda/2)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma \, dy' \right)^p d\lambda \]

\[ \leq C \int_{\mathbb{R}^{n-k}_{++}} \|f(\cdot, y')\|^p_{p, \mathbb{R}^{n-k}} \omega(|y'|)(y')^\gamma \, dy' \]

\[ = C \int_{\mathbb{R}^{n-k}_{++}} |f(y)|^p \omega(|y'|)(y')^\gamma \, dy. \]

is valid, for \(p \in (1, \infty)\) is valid by the condition \(C \leq c' A''\), where

\[ A'' \equiv \sup_{\tau > 0} \left( \int_{2\tau}^\infty \psi(t) t^{-(k+|\gamma|)p+|\gamma|+k} \, dt \right) \left( \int_0^\tau \omega^{1-p'} (t) t^{\gamma} \, dt \right)^{p-1} < \infty. \]

Note that

\[ \int_{2\tau}^\infty \psi(\tau) \tau^{-(k+|\gamma|)p+|\gamma|+k} \, d\tau = (k + |\gamma|)(p-1) \int_{2\tau}^\infty \psi(\tau) \, d\tau \int_{\tau}^\infty \lambda^{-(k+|\gamma|)p+\gamma} \, d\lambda \]

\[ = (k + |\gamma|)(p-1) \int_{2\tau}^\infty \lambda^{-(k+|\gamma|)p+|\gamma|} \, d\lambda \int_{2\tau}^\lambda \psi(\tau) \, d\tau \]

\[ \leq (k + |\gamma|)(p-1) \int_{2\tau}^\infty \lambda^{-(k+|\gamma|)p+|\gamma|} \omega_1(\lambda) \, d\lambda. \]
Condition (b1) of the theorem guarantees that $A'' \leq (k + |\gamma|)(p - 1)A_1 < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{11} \int_{\mathbb{R}^n_+} |f(x)|^p \omega(|x'|)(x')^\gamma dx.$$  

Combining the estimates of $J_1$ and $J_2$, we get (2.14) for $\omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau) d\tau$. By Fatou’s theorem on passing to the limit under the Lebesgue integral sign, this implies (2.15). The theorem is proved. \hfill $\Box$

**Example 2.3.** Let

$$\omega(t) = \begin{cases} \frac{t^{p-1}}{(2^{\beta-p+1} + \ln p) t^\beta}, & \text{for } t \in \left(0, \frac{1}{2}\right), \\ \frac{t^{p-1}}{2^{\alpha-p+1} t^\alpha}, & \text{for } t \in \left[\frac{1}{2}, \infty\right). \end{cases}$$

$$\omega_1(t) = \begin{cases} \frac{t^{p-1}}{2^{\alpha-p+1} t^\alpha}, & \text{for } t \in \left(0, \frac{1}{2}\right), \\ \frac{t^{p-1}}{(2^{\beta-p+1} + \ln p) t^\beta}, & \text{for } t \in \left[\frac{1}{2}, \infty\right). \end{cases}$$

where $0 < \alpha < \beta < p - 1$. Then the pair $(\omega(|x'|), \omega_1(|x'|))$ satisfies the condition of Theorem 2.7.

**Corollary 2.7.** Let $p \in (1, \infty)$, $k$ be a $B_{k,n}$-singular kernel and $K$ be the corresponding operator. Moreover, let $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive increasing function on $(0, \infty)$ and $\omega(|x'|), \omega_1(|x'|)$ be satisfied the conditions (a1), (b1). Then for the operator $K$ the inequality (2.15) is valid.

**Theorem 2.8.** Let $p \in (1, \infty)$ and $T$ be a $p$-admissible $B_{k,n}$-singular operators. Moreover, let $\omega(t)$ be a weight function on $(0, \infty)$, $\omega_1(t)$ be a positive decreasing function on $(0, \infty)$ and $\omega(|x'|), \omega_1(|x'|)$ be satisfied the conditions (a1), (c1). Then inequality (2.15) is valid.

**Proof.** Without loss of generality we can suppose that $\omega_1$ may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int_0^\infty \psi(\tau) d\tau,$$

where $\omega_1(+\infty) = \lim_{t \to \infty} \omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a sequence of decreasing absolutely continuous functions $\omega_n$ such that $\omega_n(t) \leq \omega_1(t)$ and $\lim_{n \to \infty} \omega_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [12], [14] for details). We have

$$\int_{\mathbb{R}^n_+} |Tf(x)|^p \omega_1(|x'|)(x')^\gamma dx = \omega_1(+\infty) \int_{\mathbb{R}^n_+} |Tf(x)|^p (x')^\gamma dx$$

$$+ \int_{\mathbb{R}^n_+} |Tf(x)|^p \left( \int_{|x'|}^\infty \psi(\tau) d\tau \right) (x')^\gamma dx$$

$$= \int_1 + \int_2.$$
If \( \omega_1(+\infty) = 0 \), then \( I_1 = 0 \). If \( \omega_1(+\infty) \neq 0 \) by the boundedness of \( T \) in \( L_{p,\gamma}(\mathbb{R}^n_{k,+}) \)

\[
J_1 \leq \|T\|^p \omega_1(+\infty) \int_{\mathbb{R}^n_{k,+}} |f(x)|^p(x')^\gamma dx
\]
\[
\leq \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega_1(|x'|)(x')^\gamma dx
\]
\[
\leq b \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(x)|^p \omega(|x'|)(x')^\gamma dx.
\]

After changing the order of integration in \( J_2 \) we have

\[
J_2 = \int_0^\infty \psi(\lambda) \left( \int_{\mathbb{R}^n_{k,+}} \int_{E'(0,\lambda)} |Tf(x)|^p(x')^\gamma dx \right) d\lambda
\]
\[
\leq 2^{p-1} \int_0^\infty \psi(\lambda) \left( \int_{\mathbb{R}^n_{k,+}} \int_{E'(0,\lambda)} |Tf(|x'|<2\lambda)|^p(x')^\gamma dx \right.
\]
\[
+ \int_{\mathbb{R}^n_{k,+}} \int_{E'(0,\lambda)} |Tf(|x'|\geq2\lambda)|^p(x')^\gamma dx \right) d\lambda
\]
\[
= J_{21} + J_{22}.
\]

Using the boundedness of \( T \) in \( L_{p,\gamma}(\mathbb{R}^n_{k,+}) \) we obtain

\[
J_{21} \leq \|T\|^p \int_0^\infty \psi(t) \left( \int_{\mathbb{R}^n_{k,+}} \int_{E'(0,2\lambda)} |f(y)|^p(y')^\gamma dy \right) dt
\]
\[
= \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \left( \int_0^\infty \psi(\lambda) d\lambda \right) (y')^\gamma dy
\]
\[
\leq \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega_1(|y'/2|)(y')^\gamma dy
\]
\[
\leq b \|T\|^p \int_{\mathbb{R}^n_{k,+}} |f(y)|^p \omega(|y'|)(y')^\gamma dy.
\]
Let us estimate $J_{22}$. For $|x'| < \lambda$ and $|y'| \geq 2\lambda$ we have $|y'|/2 \leq |x' - y'| \leq 3|y'|/2$, and so

$$J_{22} \leq c_{12} \int_0^\infty \psi(\lambda) \times \left( \int_{\mathbb{R}^{n-k}} \int_{E(0,\lambda)} \int_{\mathbb{R}^{n-k}} \int_{E(0,2\lambda)} \frac{|f(y)|(y')^\gamma dy}{(|x' - y'| + |x'' - y''|)^{n+|\gamma|}} (x')^\gamma dx \right) d\lambda \leq 2^{2n} c_{12} \int_0^\infty \psi(\lambda) \times \left( \int_{\mathbb{R}^{n-k}} \int_{E(0,\lambda)} \int_{\mathbb{R}^{n-k}} \int_{E(0,2\lambda)} \frac{|f(y)|(y')^\gamma dy}{(|x'' - y''| + |y'|)^{n+|\gamma|}} (x')^\gamma dx \right) d\lambda.$$

For $x = (x', x'') \in \mathbb{R}^n_{k, +}$ let

$$J_1(x', \lambda) = \int_{\mathbb{R}^{n-k}} \int_{E(0,2\lambda)} \int_{\mathbb{R}^{n-k}} \int_{E(0,2\lambda)} \frac{|f(y)|(y')^\gamma dy}{(|x'' - y''| + |y'|)^{n+|\gamma|}} (x')^\gamma dx'. $$

Using the Minkowski and Young inequalities we obtain

$$J_1(x', \lambda) \leq \int_{E(0,2\lambda)} \left( \int_{\mathbb{R}^{n-k}} |f(y)|^p dy \right)^{1/p} \left( \int_{\mathbb{R}^{n-k}} \frac{dy'}{|y''| + |y'|)^{n+\gamma}} (y')^\gamma dy' \right)^p \leq \left( \int_{E(0,2\lambda)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} (y')^\gamma dy' \right)^p \left( \int_{\mathbb{R}^{n-k}} \frac{dy''}{(1 + |y'|)^{n+|\gamma|}} \right)^p = c_3 \left( \int_{E(0,2\lambda)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p \left( \int_{\mathbb{R}^{n-k}} \frac{dy''}{(1 + |y'|)^{n+|\gamma|}} \right)^p = c_4 \left( \int_{E(0,2\lambda)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p.$$  

Integrating in $(0, \infty) \times (0, \lambda)$ we get

$$J_{22} \leq c_5 \int_0^\infty \psi(\lambda) \times \left( \int_{E(0,2\lambda)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p (x')^\gamma dx d\lambda = 2c_5 \int_0^\infty \psi(\lambda) \lambda^{k+|\gamma|} \left( \int_{E(0,2\lambda)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{1+|\gamma|} \left( \int_{E(0,2\lambda)} \|f(\cdot, y')\|_{p, \mathbb{R}^{n-k}} |y'|^{-k-|\gamma|} (y')^\gamma dy' \right)^p d\lambda$$
Combining the estimates of the Hardy inequality, we obtain

$$
\leq C \int_{\mathbb{R}^n_{++}} |f(\cdot, x')|_p^p,_{\mathbb{R}^{n-k}} \omega(|x'|)(x')^\gamma dx' = C \int_{\mathbb{R}^n_{++}} |f(y)|_p^p \omega(|y'|)(y')^\gamma dy,
$$

is valid, for \( p \in (1, \infty) \) is valid by the condition \( C \leq c'B'' \), where

$$
B'' \equiv \sup_{\tau > 0} \left( \int_0^\tau \psi(t) t^{k+|\gamma|} d\tau \right) \left( \int_0^\infty \omega^{1-p'}(t) t^{-(k+|\gamma|)p' |\gamma|} dt \right)^{p-1} < \infty.
$$

Note that

$$
\int_0^\tau \psi(t) t^{k+|\gamma|} dt = (k + |\gamma|) \int_0^\tau \psi(t) dt \int_0^t \lambda^{|\gamma|} d\lambda = (k + |\gamma|) \int_0^\tau \lambda^{|\gamma|} d\lambda \int_\lambda^\tau \psi(\tau) d\tau \leq (k + |\gamma|) \int_0^\tau \omega(\lambda) \lambda^{|\gamma|} d\lambda.
$$

Condition \((c_1)\) of the theorem guarantees that \(B'' \leq B_1 < \infty\). Hence, applying the Hardy inequality, we obtain

$$
J_{22} \leq c \int_{\mathbb{R}^n_{++}} |f(x)|_p^p \omega(|x'|)(x')^\gamma dx.
$$

Combining the estimates of \( J_1 \) and \( J_2 \), we get (2.14) for \( \omega_1(t) = \omega_1(+\infty) + \int_0^\infty \psi(\tau) d\tau \). By Fatou’s theorem on passing to the limit under the Lebesgue integral sign, this implies (2.15). The theorem is proved.

**Corollary 2.8.** Let \( p \in (1, \infty) \), \( k \) be a \( B_{k,n} \)-singular kernel and \( K \) be the corresponding operator. Moreover, let \( \omega(t) \) be a weight function on \((0, \infty)\), \( \omega_1(t) \) be a positive decreasing function on \((0, \infty)\) and \( \omega(|x'|), \omega_1(|x'|) \) be satisfied the conditions \((a_1), (c_1)\). Then for the operator \( K \) the inequality (2.15) is valid.

**Example 2.4.** Let

$$
\omega(t) = \begin{cases} 
\frac{1}{2} \ln^\nu \frac{1}{t}, & \text{for } t < d \\
(d^{-1-\alpha} \ln^\alpha \frac{d}{t})^\alpha, & \text{for } t \geq d,
\end{cases}
$$

$$
\omega_1(t) = \begin{cases} 
\frac{1}{2} \ln^\beta \frac{1}{t}, & \text{for } t < d \\
(d^{-1-\lambda} \ln^\beta \frac{d}{t})^\lambda, & \text{for } t \geq d,
\end{cases}
$$

where \( \beta < \nu \leq 0 \), \(-1 < \lambda < \alpha < 0\), \( d = e^\beta\). Then the pair \((\omega(|x'|), \omega_1(|x'|))\) satisfies the condition of Theorem 2.8.

**References**


[24] I.A. Kipriyanov and M.I. Klyuchantsev, On singular integrals generated by the


[28] J. L"ofstrom, J. Peetre, Approximation theorems connected with generalized trans-

[29] L.N. Lyakhov, On a class of spherical functions and singular pseudodifferential op-


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