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TWO-WEIGHTED INEQUALITY FOR CERTAIN SUBLINEAR OPERATOR IN WEIGHTED MUSIELAK-ORLICZ SPACES

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Abstract. In this paper we prove a sufficient conditions on general weights ensuring the validity of the two-weight strong type inequalities for certain sublinear operator acting boundedly in weighted Musielak-Orlicz spaces.

1. Introduction

The notion of Musielak-Orlicz spaces was introduced in [38]. The Musielak-Orlicz spaces are closely connected with modular spaces. It is well known that the first systematic study of modular spaces is due to Nakano [37]. Basing on the modular theory Musielak and Orlicz founded in 1959 a theory of Musielak-Orlicz spaces [36]. These spaces have been studied for almost sixty years and there are known a large set of applications of such spaces in various fields of analysis. They were also generalized in many directions, for example, many authors have been considered their generalizations to the vector-valued functions and spaces generated by families of Musielak-Orlicz modulars. Later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see [35]). In this paper we present certain general theorem which imply the boundedness on weighted Musielak-Orlicz spaces for sublinear operator. Now we formulate the historical statement of the considered problem.

Let R^n be the n-dimension standard Euclidean space. A well-known result in [41] states that if

$$Af(x) = p.v. \int_{R^n} K(x,y) f(y)\, dy \tag{1.1}$$

is bounded on $L_p(R^n)$, $1 < p < \infty$, and $K(x,y)$ satisfies the condition

$$|K(x,y)| \leq \frac{C}{|x-y|^n} \text{ for any } x, y \in R^n \text{ and } x \neq y,$$

then A is also bounded on the weighted Lebesgue spaces $L_{p,|x|^\alpha}(R^n)$ for the range $-n < \alpha < n(p-1)$. It is well known that this range of α gives the necessary and sufficient condition on the power weight $|x|^\alpha$ in order to be in the Muckenhoupt

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class A_p. In [40] several generalizations of this result were reduced to the case of sublinear operator satisfying the condition

$$|Bf(x)| \le C \int\limits_{R^n} \frac{|f(y)|}{|x-y|^n}\, dy \tag{1.2}$$

for any $f \in L_1(R^n)$ with compact support and $x \notin supp\ f$, where $C > 0$ is independent of f and x. Similarly, assume that $\widetilde{B}$ represents sublinear operator satisfying that for any $f \in L_1(R^n)$ with compact support and $x \notin supp\ f$

$$|\widetilde{B}f(x)| \le C \int\limits_{R^n} \frac{|f(y)|}{|x-y|^{n-s}}\, dy, \quad 0 < s < n, \tag{1.3}$$

where $C > 0$ is as in (1.2) (see [18]). It is obvious that the class of sublinear operators satisfying (1.2) is immediately generalization of the operators represented by equality (1.1). Besides, many interesting operators in harmonic analysis, such as Hardy-Littlewood maximal operators, Carleson's maximal operators, Fefferman's singular multipliers, Feffer-man's singular integrals, Ricci-Stein's oscillatory singular integrals, Bochner-Riesz means and so on satisfy the conditions (1.2) and (1.3) (see also [40]).

The main goal of this paper is to prove the boundedness of sublinear operator satisfying condition (1.3) and acting in weighted Musielak-Orlicz spaces.

2. Preliminaries

Let Ω be a Lebesgue measurable subset in R^n and $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. The Lebesgue measure of a set Ω will be denoted by $|\Omega|$. It is well known that $|B(0,1)| = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$, where $B(0,1) = \{x : x \in R^n;\ |x| < 1\}$.

Definition 2.1. [35] Let $\Omega \subset R^n$ be a Lebesgue measurable set. A real function $\varphi : \Omega \times [0,\infty) \mapsto [0,\infty)$ is called a generalized φ-function if it satisfies:

a) $\varphi(x,\cdot)$ is a φ-function for all $x \in \Omega$, i.e., $\varphi(x,\cdot) : [0,\infty) \mapsto [0,\infty)$ is convex, left-continuous and satisfies $\varphi(x,0) = 0$, $\lim\limits_{t\to+0} \varphi(x,t) = 0$, $\lim\limits_{t\to\infty} \varphi(x,t) = \infty$;

b) $\psi : x \mapsto \varphi(x,t)$ is measurable for all $t \ge 0$.

If φ is a generalized φ-function on Ω, we shortly write $\varphi \in \Phi$. By $L_0(\Omega)$ we denote the set of all Lebesgue measurable and everywhere finite real function on Ω.

Definition 2.2. [35] Let $\varphi \in \Phi$ and ρ_φ be defined by the expression

$$\rho_\varphi(f) := \int\limits_\Omega \varphi(y, |f(y)|)\, dy \quad \text{for all} \quad f \in L_0(\Omega).$$

We put $L_\varphi = \{f \in L_0(\Omega) : \ \rho_\varphi(\lambda_0 f) < \infty \quad \text{for some} \quad \lambda_0 > 0\}$ and

$$\|f\|_{L_\varphi} = \inf\left\{\lambda > 0 : \ \rho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{2.1}$$

The space L_φ is called Musielak-Orlicz space.

Note that the space L_φ is a Banach function space with respect to the norm (2.1) (more detail see [16]). In particular, the Musielak-Orlicz spaces include the classical Lebesgue spaces for $\varphi(x,\, t) = t^p$ $(1 \le p < \infty)$, the Orlicz spaces for $\varphi(x,\, t) = \varphi(t)$, and the Lebesgue spaces with variable exponent for $\varphi(x,\, t) = t^{p(x)}$, where $p : \Omega \mapsto [1, \infty]$ is a Lebesgue measurable functions.

Let ω be a weight function on Ω, i.e., ω is a non-negative, almost everywhere positive function on Ω. In this paper we considered the weighted Musielak-Orlicz spaces. We denote

$$L_{\varphi,\,\omega} = \{f \in L_0(\Omega) : \;\; f\omega \in L_\varphi\}\,.$$

It is obvious that the norm in this spaces is given by

$$\|f\|_{L_{\varphi,\,\omega}} = \|f\omega\|_{L_\varphi}.$$

By $L_{p,\omega}(R^n)$ $(1 \le p < \infty)$ we denote the spaces of measurable functions f on R^n such that

$$\|f\|_{L_{p,\,\omega}(R^n)} = \|f\|_{L_{p,\,\omega}} = \left(\int\limits_{R^n} |f(x)\,\omega(x)|^p\,dx\right)^{1/p}.$$

Lemma 2.1. *[6] Let $\Omega_1 \subset R^n$ and $\Omega_2 \subset R^m$. Let $(x,t) \in \Omega_1 \times [0,\infty)$, and $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \le p < \infty$. Suppose $f : \Omega_1 \times \Omega_2 \mapsto R$. Then the inequality*

$$\left\|\|f(x,\cdot)\|_{L_p(\Omega_2)}\right\|_{L_\varphi} \le 2^{1/p}\,\left\|\|f(\cdot,y)\|_{L_\varphi}\right\|_{L_p(\Omega_2)}$$

is valid.

In particular, Lemma 2.1 for $\varphi(x,t) = t^{q(x)}$ and $1 \le p \le q(x) \le \overline{q} < \infty$ was proved in [1].

Definition 2.3. [35] We say that $\varphi \in \Phi$ satisfies the Δ_2-condition if there exists $K \ge 2$ such that

$$\varphi(y, 2t) \le K\,\varphi(y,t)$$

for all $y \in \Omega$ and all $t > 0$. The smallest such K is called the Δ_2-constant of φ.

Lemma 2.2. *[6] Let $\varphi \in \Phi$ and $1 < s \le q(y) \le \overline{q} < \infty$. Suppose for all $C > 0$ the condition*

$$\varphi(y, Ct) \le C^{q(y)}\,\varphi(y,t) \tag{2.2}$$

holds, where $y \in \Omega$ and $t \ge 0$.

Then a function φ satisfies the Δ_2-condition, with constant $K = 2^{\overline{q}}$.

It is clear that if $\varphi(x,t) = t^{q(x)}$, then condition (2.2) satisfies automatically.

Theorem 2.1. *[35] Let $\psi \in \Phi$ and $\delta \ge 1$. Then $L_\psi(R^n) \hookrightarrow L_\delta(R^n)$ if and only if there exists $C > 0$ and $h \in L_1(R^n)$ with $\|h\|_{L_1(R^n)} \le 1$ such that*

$$\left(\frac{t}{C}\right)^\delta \le \psi(x,t) + h(x) \tag{2.3}$$

for almost all $x \in \Omega$ and all $t \ge 0$.

Note that in [3] the embeddings theorems between different variable Lebesgue spaces with measures were proved.

Lemma 2.3. *Let $\psi \in \Phi$, $\gamma \geq 1$ and $1 \leq q(x) \leq \overline{q} < \infty$. Further, let*

$$\min_{s>0} \{s,\ s^{\gamma}\}\, \psi(x,t) \leq \psi(x,st) \leq \max_{s>0} \left\{s,\ s^{q(x)}\right\} \psi(x,t) \tag{2.4}$$

for almost all $x \in \Omega$ and all $t \geq 0$. Then $\rho_\psi \left(\frac{f}{\|f\|_{L_\psi}} \right) = 1$ and

$$\min_{\|f\|_{L_\psi}} \left\{ \|f\|_{L_\psi},\ \|f\|_{L_\psi}^{\gamma} \right\} \leq \rho_\psi(f) \leq \max_{\|f\|_{L_\psi}} \left\{ \|f\|_{L_\psi},\ \|f\|_{L_\psi}^{q(x)} \right\}$$

Proof. Let $0 < \|f\|_{L_\psi} < \infty$ and $\rho_\psi \left(\frac{f}{\|f\|_{L_\psi}} \right) < 1$. We choose a positive number $\lambda \leq \|f\|_{L_\psi}$ such that $\rho_\psi \left(\frac{f}{\lambda} \right) < 1$. Indeed, we put $\lambda = \|f\|_{L_\psi} \cdot \rho_\psi^{1/\overline{q}} \left(\frac{f}{\|f\|_{L_\psi}} \right)$. Then $\lambda < \|f\|_{L_\psi}$ and by virtue of condition (2.4) for $s > 1$, we have

$$\rho_\psi \left(\frac{f}{\lambda} \right) = \int\limits_{R^n} \psi \left(x,\ \frac{|f(x)|}{\|f\|_{L_\psi} \cdot \rho_\psi^{1/\overline{q}} \left(\frac{f}{\|f\|_{L_\psi}} \right)} \right) dx \leq$$

$$\leq \int\limits_{R^n} \rho_\psi^{-q(x)/\overline{q}} \left(\frac{f}{\|f\|_{L_\psi}} \right) \psi \left(x,\ \frac{|f(x)|}{\|f\|_{L_\psi}} \right) dx$$

$$\leq \rho_\psi^{-1} \left(\frac{f}{\|f\|_{L_\psi}} \right) \int\limits_{R^n} \psi \left(x,\ \frac{|f(x)|}{\|f\|_{L_\psi}} \right) dx = \rho_\psi^{-1} \left(\frac{f}{\|f\|_{L_\psi}} \right) \rho_\psi \left(\frac{f}{\|f\|_{L_\psi}} \right) = 1.$$

Lemma 2.3 is proved. □

Example. *Let $1 \leq p(x) \leq q(x) \leq \overline{q} < \infty$ or $2 \leq \theta + 1 \leq q(x) \leq \overline{q} < \infty$, where $x \in R^n$. Suppose $\psi(x,t) = t^{p(x)}$ or $\psi(x,t) = t^{\theta} \ln(1+t)$.*
Then $\rho_\psi \left(\frac{f}{\|f\|_{L_\psi}} \right) = 1$.

Definition 2.4. A mapping S from one Musielak-Orlicz space L_φ to another Musielak-Orlicz space L_ψ is said to be sublinear if for all $f, g \in L_\varphi$ and $\lambda > 0$, we have

(1) $S(\lambda f) = \lambda\, S(f)$;
(2) $S(f+g) \leq S(f) + S(g)$.

The following Lemma characterize bounded, sublinear operators from one Musielak-Orlicz spaces to another.

Lemma 2.4. *[16] Let $\varphi, \psi \in \Phi$ and φ and ψ satisfy the Δ_2 condition. Suppose $S : L_\varphi \mapsto L_\psi$ be sublinear. Then the following conditions are equivalent:*
(a) S is bounded, i.e. there exists $C > 0$ such that $\|Sf\|_{L_\psi} \leq C\, \|f\|_{L_\varphi}$;
(b) there exists $M_1, M_2 > 0$ such that $\|f\|_{L_\varphi} \leq M_1 \implies \rho_{L_\psi}(SF) \leq M_2$.

We consider the multidimensional Hardy type operator and its dual operator

$$Hf(x) = \int\limits_{|y|<|x|} f(y)\, dy \quad \text{and} \quad H^* f(x) = \int\limits_{|y|>|x|} f(y)\, dy,$$

where $f \geq 0$ and $x \in R^n$.

Now we formulate a two-weight criterion for multidimensional Hardy type operator acting from the weighted Musielak-Orlicz spaces to weighted Lebesgue spaces.

Theorem 2.2. *Let $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \leq p < \infty$, $x \in R^n$. Suppose that $v(x)$ and $w(x)$ are weights on R^n. Then the inequality*

$$\|Hf\|_{L_{\varphi,\,w}} \leq C\ \|f\|_{L_{p,\,v}}$$

holds, for every $f \geq 0$ if and only if there exists $\alpha \in (0,1)$ such that

$$A(\alpha) = \sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-p'}\, dy \right)^{\frac{\alpha}{p'}} \left\| \chi_{\{|z|>t\}}(\cdot) \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\alpha}{p'}} \right\|_{L_{\varphi,\,w}} < \infty.$$

Moreover, we have

$$\sup_{0<\alpha<1} \frac{p'\, A(\alpha)}{(1-\alpha)\ \left[\left(\frac{p'}{1-\alpha} \right)^p + \frac{1}{\alpha(p-1)} \right]^{1/p}} \leq \|H\|_{L_{p,\,v} \mapsto L_{\varphi,\,w}} \leq 2^{1/p} \inf_{0<\alpha<1} \frac{A(\alpha)}{(1-\alpha)^{1/p'}}.$$

For the dual operator, the below stated theorem is proved analogously.

Theorem 2.3. *Let $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 \leq p < \infty$, $x \in R^n$. Suppose that $v(x)$ and $w(x)$ are weights on R^n. Then the inequality*

$$\|H^* f\|_{L_{\varphi,\,w}} \leq C\ \|f\|_{L_{p,\,v}}$$

holds, for every $f \geq 0$ if and only if there exists $\gamma \in (0,1)$ such that

$$B(\gamma) = \sup_{t>0} \left(\int\limits_{|y|>t} [v(y)]^{-p'}\, dy \right)^{\frac{\gamma}{p'}} \left\| \chi_{\{|z|<t\}}(\cdot) \left(\int\limits_{|y|>|\cdot|} [v(y)]^{-p'}\, dy \right)^{\frac{1-\gamma}{p'}} \right\|_{L_{\varphi,\,w}} < \infty.$$

Moreover, we have

$$\sup_{0<\gamma<1} \frac{p'\, B(\gamma)}{(1-\gamma)\ \left[\left(\frac{p'}{1-\gamma} \right)^p + \frac{1}{\gamma(p-1)} \right]^{1/p}} \leq \|H\|_{L_{p,\,v} \mapsto L_{\varphi,\,w}} \leq 2^{1/p} \inf_{0<\gamma<1} \frac{B(\gamma)}{(1-\gamma)^{1/p'}}.$$

Corollary 2.1. *Note that Theorem 2.2 and Theorem 2.3 was proved in [6]. In the case $L_{\varphi,w} = L_{q,w}$, $1 < p \leq q < \infty$, for $x \in (0,\infty)$, $\alpha = \dfrac{s-1}{p-1}$ and $s \in (1,\, p)$ Theorem 2.2 and Theorem 2.3 was proved in [43]. For $x \in R^n$ in the case $L_{\varphi,w} = L_{q(x),\,w}$ and $1 < p \leq q(x) \leq ess \sup\limits_{x \in R^n} q(x) < \infty$ Theorem 2.2 and Theorem 2.3 was proved in [2]. In [5] the Hardy inequality was connected with a certain nonlinear differential equation. The boundedness of Hardy type operators in weighted Musielak-Orlicz spaces was considered in [39], [9], [11] and etc.*

Remark 2.1. In the case $n = 1$, $L_{\varphi,w} = L_{q,\,w}$, $1 < p \leq q \leq \infty$, at $x \in (0,\infty)$, for classical Lebesgue spaces the various variants of Theorem 2.2 and Theorem 2.3 were proved in [10], [22], [25], [26], [33], [34], [42] and etc. In the case of Lebesgue spaces with variable exponent the boundedness of Hardy type operator was proved in [12], [14], [15], [17], [19], [23], [24], [27]-[32] and etc. For $L_{\varphi,w} = L_{q(x),\,w}$, two-weighted criterion for multidimensional Hardy type operator in the case $L_{\varphi,w} = L_{q(x),\,w}$, $1 < \underline{p} \le p(x) \le q(x) \le ess \sup\limits_{x\in R^n} q(x) < \infty$ and $x \in R^n$ was proved in [27], [28], [32] and etc (see also [24], [17], [6]). In the papers [30] and [31], it was obtained a criterion on exponent $p(x)$ for boundedness of Hardy type operator in weighted variable Lebesgue spaces. Also, for details we refer to [15]. Recently, in one dimensional case, the boundedness of Hardy operator in weighted $L_{p(x),v}(0,\infty)$ for $0 < \underline{p} \le p(x) < 1$ was proved in [7].

3. Main result

Now we formulate a two-weight strong type inequality for sublinear operator satisfying the condition (1.3).

Theorem 3.1. *Let $\varphi\left(x, t^{1/p}\right) \in \Phi$ for some $1 < p < \infty$ and a function $\psi \in \Phi$ satisfy the conditions (2.3) and (2.4), where $x \in R^n$. Suppose that $v(x)$ and $w(x)$ are weight functions on R^n. Let T be a sublinear operator acting boundedly from $L_\psi(R^n)$ to $L_\varphi(R^n)$ and satisfying (1.3). Let there exists $1 < \theta \le r(x) \le ess \sup\limits_{x\in R^n} r(x) < \infty$ such that, for all $C > 0$ $\varphi(x, Ct) \le C^{r(x)} \varphi(x,t)$. Moreover, let $v(x)$ and $w(x)$ are weight functions on R^n and satisfies the following conditions:*

$$\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-p'}\,dy\right)^{\frac{\alpha}{p'}}\left\|\frac{w(\cdot)}{|\cdot|^{n-s}}\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-p'}\,dy\right)^{\frac{1-\alpha}{p'}}\right\|_{L_\varphi(|\cdot|>t)} < \infty, \quad (3.1)$$

$$\sup_{t>0}\left(\int\limits_{|y|>t}[v(y)|y|^{n-s}]^{-p'}dy\right)^{\frac{\beta}{p'}}\left\|w(\cdot)\left(\int\limits_{|y|>|\cdot|}[v(y)|y|^{n-s}]^{-p'}dy\right)^{\frac{1-\beta}{p'}}\right\|_{L_\varphi(|\cdot|<t)} < \infty, \quad (3.2)$$

There exists $M > 0$ such that

$$\sup_{|x|/2<|y|\le 4\,|x|} w(y) \le M \inf_{|x|/2<|y|\le 4\,|x|} v(x). \quad (3.3)$$

Then there exists a positive constant C, independent of f, such that for all $f \in L_{p,v}(R^n)$

$$\|Tf\|_{L_{\varphi,w}} \le C\|f\|_{L_{\psi,v}(R^n)}.$$

Proof. Let $Z = \{0, \pm1, \pm2, \dots,\}$. For $k \in Z$ we define

$$E_k = \left\{x \in R^n : 2^k < |x| \le 2^{k+1}\right\}, \quad E_{k,1} = \left\{x \in R^n :\ |x| \le 2^{k-1}\right\},$$

$$E_{k,2} = \left\{x \in R^n :\ 2^{k-1} < |x| \le 2^{k+2}\right\}, \quad E_{k,3} = \left\{x \in R^n :\ |x| > 2^{k-1}\right\}.$$

Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,v}(R^n)$, we write

$$|Tf(x)| = \sum_{k\in Z}|Tf(x)|\,\chi_{E_k}(x) \le \sum_{k\in Z}|Tf_{k,1}(x)|\,\chi_{E_k}(x) + \sum_{k\in Z}|Tf_{k,2}(x)|\,\chi_{E_k}(x) +$$

$$+\sum_{k\in Z}|Tf_{k,3}(x)|\ \chi_{E_k}(x)=T_1f(x)+T_2f(x)+T_3f(x),$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i}=f\chi_{E_{k,i}}$, $i=1,2,3$.

First we shall estimate $\|T_1f\|_{L_{p,\omega_2}}$. Note that for $x\in E_k$, $y\in E_{k,1}$ we have $|y|<2^{k-1}\le|x|/2$. Moreover, $E_k\cap supp\, f_{k,1}=\emptyset$ and $|x-y|\ge|x|-|y|\ge|x|-|x|/2=|x|/2$. Hence by (1.3)

$$|T_1f(x)|\le C\sum_{k\in Z}\left(\int\limits_{R^n}\frac{|f_{k,1}(y)|}{|x-y|^{n-s}}\,dy\right)\chi_{E_k}\le C\int\limits_{|y|<|x|/2}\frac{|f(y)|}{|x-y|^{n-s}}\,dy\le$$

$$\le C\int\limits_{|y|<|x|}\frac{|f(y)|}{|x-y|^{n}}\,dy\le 2^{n-s}\,C\,\frac{1}{|x|^{n-s}}\int\limits_{|y|<|x|}|f(y)|\,dy$$

for any $x\in E_k$. Hence we have

$$\|T_1f\|_{L_{\varphi,\,w}}\le 2^{n-s}\,C\left\|\frac{1}{|x|^{n-s}}\int\limits_{|y|<|x|}|f(y)|\,dy\right\|_{L_{\varphi,\,w}}=\left\|\int\limits_{|y|<|x|}|f(y)|\,dy\right\|_{L_{\varphi,\,\frac{w}{|x|^{n-s}}}}.$$

By the condition (3.1), Theorem 2.2 and Theorem 2.1, we obtain

$$\|T_1f\|_{L_{\varphi,\,w}}\le C_1\,\|f\|_{L_{p,v}(R^n)}\le C_2\|f\|_{L_{\psi,v}(R^n)} \tag{3.4}$$

where $C_i>0$ $(i=1,2)$ is independent of f.

Next we estimate $\|T_3f\|_{L_{p,w}(R^n)}$. It is obviously that, for $x\in E_k$, $y\in E_{k,3}$ we have $|y|>2\,|x|$ and $|x-y|\ge|y|-|x|\ge|y|-|y|/2=|y|/2$. Since $E_k\cap supp\, f_{k,3}=\emptyset$, for $x\in E_k$ by (1.3), we have

$$|T_3f(x)|\le C\int\limits_{|y|>2|x|}\frac{|f(y)|}{|x-y|^{n-s}}\,dy\le 2^{n-s}\,C\int\limits_{|y|>2|x|}\frac{|f(y)|}{|y|^{n-s}}\,dy.$$

Hence we obtain

$$\|T_3f\|_{L_{\varphi,\,w}}\le 2^{n-s}\,C\left\|\int\limits_{|y|>2|\cdot|}|f(y)|\,|y|^{s-n}\,dy\right\|_{L_{\varphi,\,w}}\le$$

$$\le 2^{n-s}\,C\left\|\int\limits_{|y|>|x|}|f(y)|\,|y|^{s-n}\,dy\right\|_{L_{\varphi,\,w}}.$$

By the condition (3.2), Theorem 2.3 and Theorem 2.1, we obtain

$$\|T_3f\|_{L_{\varphi,\,w}}\le C_3\,\|f\|_{L_{p,v}(R^n)}\le C_4\|f\|_{L_{\psi,v}(R^n)}, \tag{3.5}$$

where $C_i>0$ $(i=3,4)$ is independent of f.

Let $T:L_p\left(R^n\right)\to L_\varphi\left(R^n\right)$. We have

$$\|Tf_{k,2}\|_{L_{\varphi,\,w}(R^n)}=\left\|\sum_{k\in Z}|Tf_{k,2}|\ \chi_{E_k}\right\|_{L_{\varphi,\,w}(R^n)}.$$

By virtue of Lemma 2.4 it suffices to prove that from $\|f\|_{L_{\psi,\,v}(R^n)} \le 1$ implies $\int\limits_{R^n} \varphi\left(y, w \sum\limits_{k\in Z} |Tf_{k,2}|\, \chi_{E_k}\right) dy \le C$, where $C>0$ is independent on $k \in Z$.

Finally, we estimate $\|T_2 f\|_{L_{\varphi,\,w}}$. By the $L_\psi(R^n) \mapsto L_\varphi(R^n)$ boundedness of T and condition (3.3), we have

$$\int\limits_{R^n} \varphi\left(y, w(y) \sum_{k\in Z} |Tf_{k,2}(y)|\, \chi_{E_k}(y)\right) dy =$$

$$= \sum_{m\in Z} \int\limits_{E_m} \varphi\left(y, w(y) \sum_{k\in Z} |Tf_{k,2}(y)|\, \chi_{E_k}(y)\right) dy =$$

$$\sum_{k\in Z} \int\limits_{E_k} \varphi\left(y, w(y)\, |Tf_{k,2}(y)|\right) dy =$$

$$= \sum_{k\in Z} \int\limits_{E_k} \varphi\left(y, C\, w(y)\, \|f_{k,2}\|_{L_\psi(R^n)}\, \frac{|Tf_{k,2}|}{C\,\|f_{k,2}\|_{L_\psi(R^n)}}\right) dy \le$$

$$\sum_{k\in Z} \int\limits_{E_k} \left(C\, w(y)\, \|f_{k,2}\|_{L_\psi(R^n)}\right)^{r(y)} \varphi\left(y, \frac{|Tf_{k,2}|}{C\,\|f_{k,2}\|_{L_\psi(R^n)}}\right) dy \le$$

$$C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(w(y)\, \|f\|_{L_\psi(E_{k,2})}\right)^{r(y)} \int\limits_{R^n} \varphi\left(y, \frac{|Tf_{k,2}|}{C\,\|f_{k,2}\|_{L_\psi(R^n)}}\right) dy \le$$

$$C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(w(y)\, \|f\|_{L_\psi(E_{k,2})}\right)^{r(y)} = C_2 \sum_{k\in Z} \sup_{y\in E_k} \left(\|f\, w\|_{L_\psi(E_{k,2})}\right)^{r(y)} \le$$

$$C_3 \sum_{k\in Z} \sup_{y\in E_k} \left(\|f \inf_{y\in E_{k,2}} v(y)\|_{L_\psi(E_{k,2})}\right)^{r(y)} \le C_3 \sum_{k\in Z} \sup_{y\in E_k} \left(\|f\, v\|_{L_\psi(E_{k,2})}\right)^{r(y)} =$$

$$C_3 \sum_{k\in Z} \left(\|f\|_{L_{\psi,\,v}(E_{k,2})}\right)^{\inf\limits_{y\in E_k} r(y)} \le C_3 \sum_{k\in Z} \left(\|f\|_{L_{\psi,\,v}(E_{k,2})}\right)^{\underline{r}} \le$$

$$C_3 \sum_{k\in Z} \left(\int\limits_{E_{k,2}} \psi\left(y, |f(y)| v(y)\right) dy\right)^{\underline{r}/\gamma} \le$$

$$C_3 \left(\sum_{k\in Z} \left[\int\limits_{E_{k-1}} + \int\limits_{E_k} + \int\limits_{E_{k+1}}\right] \psi\left(y, |f(y)| v(y)\right) dy\right)^{\underline{r}/\gamma} =$$

$$C_3 \left(3 \sum_{k\in Z} \int\limits_{E_k} \psi\left(y, |f(y)| v(y)\right) dy\right)^{\underline{r}/\gamma} \le 3^{\underline{r}/\gamma}.$$

Thus

$$\|T_2 f\|_{L_{\varphi,\,w}} \le C_4, \tag{3.6}$$

where $C > 0$ is independent of f and $x \in R^n$.

Combining the inequalities (3.4),(3.5) and (3.6) we obtain the proof of Theorem 3.1. $\square$

In particular, taking $\varphi(x,t) = t^{q(x)}$, $\psi(x,t) = t^{p(x)}$ and $p = \underline{p}$ from Theorem 3.1 we have the following Corollary.

Corollary 3.1. *Let $\underline{p} > 1$, $\overline{p} < n/s$, $\underline{q} \geq \overline{p}$ and there exists $\eta \in (0,1)$ such that $\int\limits_E \eta^{\frac{\underline{p}\, p(x)}{p(x)-\underline{p}}} dx < \infty$, where $E = \left\{x \in R^n : \underline{p} < p(x)\right\}$. Let T be a sublinear operator acting boundedly from $L_{p(x)}(R^n)$ to $L_{q(x)}(R^n)$ and satisfying (1.3). Moreover, let $v(x)$ and $w(x)$ are weight functions on R^n and satisfies the following conditions:*

$$\sup_{t>0} \left(\int\limits_{|y|<t} [v(y)]^{-\overline{p}'} dy \right)^{\frac{\alpha}{\overline{p}'}} \left\| \frac{w(\cdot)}{|\cdot|^{n-s}} \left(\int\limits_{|y|<|\cdot|} [v(y)]^{-\overline{p}'} dy \right)^{\frac{1-\alpha}{\overline{p}'}} \right\|_{L_{q(\cdot)}(|\cdot|>t)} < \infty,$$

$$\sup_{t>0} \left(\int\limits_{|y|>t} [v(y)|y|^{n-s}]^{-\overline{p}'} dy \right)^{\frac{\beta}{\overline{p}'}} \left\| w(\cdot) \left(\int\limits_{|y|>|\cdot|} [v(y)|y|^{n-s}]^{-\overline{p}'} dy \right)^{\frac{1-\beta}{\overline{p}'}} \right\|_{L_{q(\cdot)}(|\cdot|<t)} < \infty,$$

There exists $M > 0$ such that

$$\sup_{|x|/2<|y|\leq 4\,|x|} w(y) \leq M \inf_{|x|/2<|y|\leq 4\,|x|} v(x).$$

Then there exists a positive constant C, independent of f, such that for all $f \in L_{p(x),v}(R^n)$

$$\|Tf\|_{L_{q(\cdot),w}(R^n)} \leq C\|f\|_{L_{p(\cdot),v}(R^n)}.$$

We consider the Riesz potential $\mathcal{R}^s f(x) = \int\limits_{R^n} \frac{f(y)}{|x-y|^{n-s}} dy$ and the fractional maximal operator $M^s f(x) = \sup\limits_{r>0} \frac{1}{|B(x,r)|^{1-\frac{s}{n}}} \int\limits_{B(x,r)} |f(y)|\, dy$ and $0 < s < n$.

Further, we assume that the exponent $p(x)$ satisfies the standard conditions

$$|p(x) - p(y)| \leq \frac{M_1}{-\ln|x-y|}, \quad 0 < |x-y| \leq \frac{1}{2}, \quad x, y \in R^n, \tag{3.7}$$

together with the following conditions at infinity:

$$|p(x) - p(y)| \leq \frac{M_2}{\ln(e+|x|)}, \quad |x| \geq |y|, \quad x, y \in R^n, \tag{3.8}$$

where the positive constants M_1 and M_2 are independent of x and y. Note that, from condition (3.8) implies that there is some number p_∞ such that $p(x) \to p_\infty$ as $|x| \to \infty$, and this limit holds uniformly in all directions. It is known that if $p(x)$ satisfies (3.8), $p_\infty = \underline{p}$ and $\frac{1}{r(x)} = \frac{1}{\underline{p}} - \frac{1}{p(x)}$, then $\frac{1}{r(x)}$ satisfies (3.8), $\lim\limits_{|x|\to\infty} r(x) = \infty$ and $L_{p(x)}(R^n) \hookrightarrow L_{\underline{p}}(R^n)$ (see [16]).

Let $T=\mathcal{R}^s$ or $T=M^s$. It is obvious that T is a sublinear operator and $M^s f(x)\le |B(0,1)|^{1-\frac{s}{n}}\,\mathcal{R}^s|f|(x)$ for a.e. $x\in R^n$. Note that in [13] show that, if the exponent function $p(x)$ or $q(x)$ satisfies the conditions (3.6) and (3.7), then Riesz potential is bounded from $L_{p(x)}(R^n)$ to $L_{q(x)}(R^n)$.

From Theorem 3.1 for Riesz potential and fractional maximal operator we have the following Corollary.

Corollary 3.2. *Let* $\dfrac{1}{p(x)}-\dfrac{1}{q(x)}=\dfrac{s}{n}$, $\underline{p}>1$, $\overline{p}<n/s$, $\underline{q}\ge\overline{p}$ *and* $p(x)$ *satisfy conditions (2.5) and (2.6) with* $p_\infty=\underline{p}$. *Moreover, let* $v(x)$ *and* $w(x)$ *are weight functions on* R^n *and satisfies the following conditions:*

$$\sup_{t>0}\left(\int\limits_{|y|<t}[v(y)]^{-\overline{p}'}\,dy\right)^{\frac{\alpha}{\overline{p}'}}\left\|\frac{w(\cdot)}{|\cdot|^{n-s}}\left(\int\limits_{|y|<|\cdot|}[v(y)]^{-\overline{p}'}\,dy\right)^{\frac{1-\alpha}{\overline{p}'}}\right\|_{L_{q(\cdot)}(|\cdot|>t)}<\infty,$$

$$\sup_{t>0}\left(\int\limits_{|y|>t}[v(y)|y|^{n-s}]^{-\overline{p}'}\,dy\right)^{\frac{\beta}{\overline{p}'}}\left\|w(\cdot)\left(\int\limits_{|y|>|\cdot|}[v(y)|y|^{n-s}]^{-\overline{p}'}\,dy\right)^{\frac{1-\beta}{\overline{p}'}}\right\|_{L_{q(\cdot)}(|\cdot|<t)}<\infty,$$

where $0<\alpha,\beta<1$.

There exists a constant $M>0$ *such that*

$$\sup_{|x|/4<|y|\le 4\,|x|} w(y)\le M\inf_{|x|/4<|y|\le 4\,|x|} v(y)\quad \text{for a.e. } x\in R^n.$$

Then

$$\|Tf\|_{L_{q(\cdot),w}(R^n)}\le C\|f\|_{L_{p(\cdot),v}(R^n)},$$

for any $f\in L_{p(x),v}(R^n)$, *where a positive constant* C *independent of* f.

Remark 3.1. In the case $p(x)=p=const$ for classical Lebesgue spaces the Theorem 3.1 was proved in [44] (see also [20]). The boundedness of sublinear operators and commutators on generalized Morrey spaces was proved in [18], [21] and etc. In particular, the boundedness of certain convolution operator in a weighted Lebesgue space with kernel satisfying the generalized Hörmander's condition was proved in [4], [8] and etc.

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