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A METHOD FOR STUDYING THE OPTIMALITY OF CONTROLS IN DISCRETE SYSTEMS

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Abstract. Taking into account the specific character of the discrete system, we introduce the notion of zero variation of the quality functional, suggest a new approach and get more general results for the first and second variation of the quality functional. Some applications of these formulas are given.

1. Introduction

Along with continuous optimal control problems a great attention is paid to investigation of discrete optimal control problems with wide applications in economics, engineering, operations research, etc. This is connected with increasing complexity of controlled systems and with the fact that information on the state of this or other systems is received in discrete time. Historically, theory of discrete systems has followed the L.S. Pontryagin maximum principle for continuous optimal control problems and this led to attempts to formulate a discrete analogue of the maximum principle. In 1963 A.G. Butkovsky showed that direct transference of the L. S. Pontryagin maximum principle on discrete systems, in general case is impossible [4]. After the appearance of the paper [4], the researchers began to make efforts to prove the maximum principle in the weakened form (local maximum, stationary state). A number of important results were obtained in this direction. One can find a detailed review of corresponding results in [1-3,5-7,9-13, 15, 17-19].

It should be noted that unlike the continuous one in the discrete case the analogue of the Euler equation is not a consequence of the discrete maximum principle [7,13]. In this connection, we can say that in discrete control systems the Euler equation as a necessary condition has an independent value. Therefore, naturally there arises theoretical and practical interest for obtaining more general and stronger optimality condition as the Euler equation type and second order optimality condition based on the second variation of the quality functional. This paper is devoted to investigation of optimality of controls in such a statement.

By its content the present paper is closest adjacent to variational calculus. Taking into account the specific character of the discrete system, we introduce

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the notion of zero variation of the quality functional, suggest a new approach and get more general conditions for the first and second variations of the quality functional. Some applications of these formulas are given. Note that the obtained results are strengthening and generalization of the earlier known corresponding optimality conditions from [3,7,13].

2. Problem Statement

We consider a problem on the minimum of the functional

$$\mathcal{S}(u) = \Phi(x(t_1)) \tag{2.1}$$

at the constraints

$$u(t) \in U(t) \subset E^r, \quad t \in T = \{t_0,\ t_0+1,\ \ldots,\ \ t_1-1\}, \tag{2.2}$$

$$x(t+1) = f(x(t),\ u(t),\ t\ ),\ t \in T\ ,\ x(t_0) = x^*, \tag{2.3}$$

where $x = (x_1,\ \ldots, x_n)'$ is a state vector ($'$(prime) is the transposition operation), $u = (u_1, \ldots, u_r)'$ is a control vector, t is time (discrete), x^* is an initial vector, E^r is $r-$dimensional Euclidean space; $f(x,u,t)$, $(x,u,t) \in E^n \times E^r \times [t_0,\ t_1]$ is a vector-function continuous in the aggregate of variables together with partial derivatives with respect to $x,\ u$ to second order inclusively, $\Phi(x)$, $x \in E^n$ is a twice differentiable function; the sets $U(t)$, $t \in T \backslash \{t_1{-}1\}$ are open sets, $U(t_1{-}1)$ is an arbitrary set (not necessarily open).

The controls satisfying condition (2.2) will be called admissible. The admissible control $u(t)$, $t \in T$ minimizing functional (2.1) at the constraint (2.3) we call an optimal control, and the corresponding trajectory $x(t)$, $t \in T \cup \{t_1\}$ of the system (2.3) an optimal trajectory. Herewith we call the pair $(u(t), x(t))$ an optimal process.

3. Variation of the quality functional

Let $(u^0(t), x^0(t))$ be some process in problem (2.1)-(2.3). Introduce the sets [16]:

$$U[x^0(\cdot)](t) = \{\hat{u} \in U(t) : \triangle_{\hat{u}} f\left(x^0(t), u^0(t), t\right) =$$

$$= f\left(x^0(t), \hat{u}, t\right) - f\left(x^0(t), u^0(t), t\right) = 0\ \},\quad t \in \mathrm{T}. \tag{3.1}$$

Note that the sets $U[x^0(\cdot)](t)$, $t \in T$ are not empty and even if one set $U[x^0(\cdot)](t)$ consists of at least two elements, it allows to receive additional information on optimality of the control $u^0(t)$, $t \in T$ [14]. We also underline that in most cases it is not difficult to find the elements of the set $U[x^0(\cdot)](\theta)$, $\theta \in T$. For example, in problem (2.1)-(2.3), if $f(x,u,t) = g(x(t),t) + A(x(t),t)u(t)$, $t \in T$, then the finding of the elements $U[x^0(\cdot)](\theta)$ is reduced to solving the linear algebraic system of equations.

Lemma 3.1. *If the control $u^0(t)$, $t \in T$ is optimal, then any control $\hat{u}(t) \in$ $\in U[x^0(\cdot)](t)$, $t \in T$ is optimal, and the pair $(\hat{u}(t), x^0(t))$ is an optimal process.*

The proof of the lemma easily follows from definition of the set (3.1).

Along with $\hat{u}\,(t)\,,\ \ t\in T$ we consider another admissible control $u\,(t;\varepsilon)\,,\ \ t\in \mathrm{T}$ of the following form

$$u\,(t;\varepsilon)=\left\{\begin{array}{l} v, \quad t=t_1-1 \\ \hat{u}(t)+\varepsilon\delta u\,(t)\,,\quad t\in T\backslash\{t_1-1\}, \end{array}\right. \tag{3.2}$$

where $\hat{u}(t)\in U\left[x^0\,(\cdot)\right](t)$, $t\in \mathrm{T}$, $v\in U(t_1-1)$, $\delta u(t)\in E^r, t\in T\backslash\{t_1-1\}$, ε is a number rather small in absolute value, and $u(t;\varepsilon)\in U(t)$, $t\in T$.

Note that the variation of the control $\hat{u}\,(t)\,,\ \ t\in T$ in the form (3.2) is new. This is the basis of the investigation scheme of the problem under consideration (2.1)-(2.3).

Denote by $\triangle^* x\,(t)\,,\ t\in T\cup\{t_1\}$ a special increment in the trajectory $x^0\,(t)\,,$ $t\in T\cup\{t_1\}$, responding to the increment $\triangle^* u\,(t)=u\,(t;\varepsilon)-\hat{u}\,(t)\,,\ t\in T$ of the control $\hat{u}\,(t)\,,\ t\in T$. It is clear that the increment $\triangle^* x\,(t)$, $t\in T\cup\{t_1\}$ is the solution of the system

$$\left\{\begin{array}{l} \triangle^* x\,(t+1)=f\left(x^0\,(t)+\triangle^* x\,(t)\,,\ u\,(t,\varepsilon)\,,t\right)-f\left(x^0\,(t)\,,\hat{u}\,(t)\,,t\right), \\ \triangle x\,(t_0)=0. \end{array}\right. \tag{3.3}$$

Taking into account (3.2) and using the Taylor formula from (3.3) by the steps methods it is easy to show that

$$\|\triangle^* x\,(t)\|\leq K_1\,|\varepsilon|\,,\quad t\in T, \tag{3.4}$$

where $K_1=\mathit{const}>0$, $\|\triangle^* x\,(t)\|$ is the Euclidean norm of the vector $\triangle^* x\,(t)$.

However, the solution of the system (3.3) at the point $t=t_1$ is finite with respect to $\varepsilon:\ \|\triangle^* x\,(t_1)\|\sim\varepsilon^0$. Distinguish the principal part of the increment $\triangle^* x\,(t_1)$. From (3.3) we have

$$\begin{aligned} \triangle^* x\,(t_1)=&\triangle_v f\left(x^0\,(t_1-1)\,,\hat{u}\,(t_1-1)\,,\ t_1-1\right)+ \\ &+\triangle_{\tilde{x}} f\left(x^0\,(t_1-1)\,,v,\ t_1-1\right),\quad v\in U\,(t_1-1)\,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\triangle_{\tilde{x}} f\left(x^0\,(t_1-1)\,,v,\ t_1-1\right)= \\ &=f\left(x^0\,(t_1-1)+\triangle^* x\,(t_1-1)\,,v,t_1-1\right)-f\left(x^0\,(t_1-1)\,,v,\ t_1-1\right). \end{aligned} \tag{3.6}$$

Taking into account (3.4), (3.6) it is easy to see that the second term in (3.5) has order ε, i.e.

$$\left\|\triangle_{\tilde{x}} f\left(x^0\,(t_1-1)\,,v,\ t_1-1\right)\right\|\leq K_2\,|\varepsilon|\,,K_2=\mathit{const}>0. \tag{3.7}$$

Now derive the second order increment in the quality functional. Using (3.4)-(3.7) and the Taylor formula the increment $\triangle^* S\,(\hat{u})=S\,(u\,(t;\varepsilon))-S(\hat{u}\,(t))$ of the functional (2.1) caused by (3.2) may be written in the form

$$\begin{aligned} \triangle^* S\,(\hat{u})=&\triangle_v \Phi(f(x^0\,(t_1-1)\,,\hat{u}\,(t_1-1)\,,t_1-1))+ \\ &+\triangle_1^* S\,(\hat{u})+\frac{1}{2}\triangle_2^* S\,(\hat{u})+o(\varepsilon^2), \end{aligned} \tag{3.8}$$

where $o(\varepsilon^2)/\varepsilon^2\to 0$, as $\varepsilon\to 0$;

$$\begin{aligned} \triangle_v \Phi(f(x^0\,(t_1-1)\,,\hat{u}\,(t_1-1)\,,t_1-1))=&\Phi(f(x^0\,(t_1-1)\,,v,t_1-1))- \\ &-\Phi(f(x^0\,(t_1-1)\,,\hat{u}\,(t_1-1)\,,t_1-1)), \end{aligned} \tag{3.9}$$

$$\triangle_1^* S\,(\hat{u})=\Phi'_x(f(x^0\,(t_1-1)\,,v,t_1-1))\triangle_{\tilde{x}} f\left(x^0\,(t_1-1)\,,v,t_1-1\right), \tag{3.10}$$

$$\triangle_2^* S(\hat{u}) = \triangle_{\tilde{x}} f'\left(x^0(t_1-1), v, t_1-1\right) \times$$
$$\times \Phi_{xx}(f(x^0(t_1-1), v, t_1-1)) \triangle_{\tilde{x}} f\left(x^0(t_1-1), v, t_1-1\right). \quad (3.11)$$

Let's consider the auxiliary vector-function $\widehat{\psi}(t;v)$, $t \in T$ as the solution of the linear discrete system:

$$\begin{cases} \widehat{\psi}(t-1;v) = f_x'\left(x^0(t), \hat{u}(t), t\right) \widehat{\psi}(t;v), \\ \widehat{\psi}(t_1-2;v) = f_x'\left(x^0(t_1-1), v, t_1-1\right) \widehat{\psi}(t_1-1;v), \\ \widehat{\psi}(t_1-1;v) = -\Phi_x(f\left(x^0(t_1-1), v, t_1-1\right)). \end{cases} \quad (3.12)$$

In what follows, assume

$$\Psi^0(t_1-2;v) = f_x'\left(x^0(t_1-1), v, t_1-1\right) \Psi^0(t_1-1;v) f_x\left(x^0(t_1-1), v, t_1-1\right) +$$
$$+ H_{xx}(\widehat{\psi}(t_1-1;v), x^0(t_1-1), v, t_1-1),$$
$$\Psi^0(t_1-1;v) = -\Phi_{xx}\left(f(x^0(t_1-1), v, t_1-1)\right), \quad (3.13)$$

where the function $H(\psi, x, u, t) = \psi\prime f(x,u,t)$ is called the Hamilton-Pontryagin function for problem (2.1)-(2.3).

According to (3.4), (3.6), (3.10), (3.12) for $\triangle_1^* S(\hat{u})$ by the Taylor formula it holds the representation

$$\triangle_1^* S(\hat{u}) = -H_x'(\widehat{\psi}(t_1-1;v), x^0(t_1-1), v, t_1-1) \triangle^* x(t_1-1) -$$
$$-\frac{1}{2} \triangle^* x'(t_1-1) H_{xx}(\widehat{\psi}(t_1-1;v),$$
$$x^0(t_1-1), v, t_1-1) \triangle^* x(t_1-1) + o(\varepsilon^2). \quad (3.14)$$

Similar to (3.14), allowing for (3.4), (3.6), (3.13) from (3.11) we get

$$\triangle_2^* S(\hat{u}) = -\triangle^* x'(t_1-1) f_x'(x^0(t_1-1), v, t_1-1) \times$$
$$\times \Psi^0(t_1-1;v) f_x(x^0(t_1-1), v, t_1-1) \triangle^* x(t_1-1) + o(\varepsilon^2). \quad (3.15)$$

Substitute (3.14), (3.15) in (3.8), and taking into account (3.12), (3.13) we have

$$\triangle^* S(\hat{u}) = \triangle_v \Phi\left(f(x^0(t_1-1), \hat{u}(t_1-1), t_1-1)\right) - \widehat{\psi}'(t_1-2;v) \triangle^* x(t_1-1) -$$
$$-\frac{1}{2} \triangle^* x'(t_1-1) \Psi^0(t_1-2;v) \triangle^* x(t_1-1) + o\left(\varepsilon^2\right). \quad (3.16)$$

Here and in the sequel, $o(\varepsilon^2)$ means the total residual term.

Now calculate the second term in (3.16) for admissible control (3.2). Using the identity

$$\widehat{\psi}\prime(t_1-2;v) \triangle^* x(t_1-1) =$$
$$= \sum_{t=t_0}^{t_1-2} \widehat{\psi}\prime(t;v) \triangle^* x(t+1) - \sum_{t=t_0+1}^{t_1-2} \widehat{\psi}\prime(t-1;v) \triangle^* x(t)$$

and taking into account (3.2)-(3.4), (3.12), from the Taylor formula we get

$$\widehat{\psi}\prime(t_1-2;v) \triangle^* x(t_1-1) = \varepsilon \sum_{t=t_0}^{t_1-2} H_u'(\widehat{\psi}(t;v), x^0(t), \hat{u}(t), t) \delta u(t) +$$

$$+\frac{1}{2} \sum_{t=t_0}^{t_1-2} [\triangle^* x'(t) H_{xx}(a(t;\hat{u}(\cdot);v)) \triangle^* x(t) + 2\varepsilon \triangle^* x'(t) H_{xu}(a(t;\hat{u}(\cdot);v)) \delta u(t) +$$

$$+\varepsilon^2\delta u'(t)\, H_{uu}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta u\left(t\right)], \tag{3.17}$$

where

$$a\left(t;\hat{u}\left(\cdot\right);v\right)=\left(\widehat{\psi}\left(t;v\right),x^0\left(t\right),\hat{u}\left(t\right),t\right). \tag{3.18}$$

To achieve the goal it remains to distinguish in $\triangle^* x\left(t\right),\ t\in T$ the principal term with respect to ε. Solving the system (3.3) by the steps method (sequentially with respect to t: $t=t_0,\ldots,t_1-2$) and using the Taylor formula for $\triangle^* x\left(t\right)$, $t\in T$, it is easy to get the expansion of the form

$$\triangle^* x\left(t\right)=\varepsilon\delta x\left(t\right)+o\left(\varepsilon;t\right),\quad t\in T, \tag{3.19}$$

where $o\left(\varepsilon;t_0\right)=0,\ o\left(\varepsilon;t\right)/\varepsilon\to o$ as $\varepsilon\to o,\ t\in T$; and $\delta x\left(t\right),\ \ t\in T$ is the solution of the linear discrete system

$$\begin{cases}\delta x\left(t+1\right)=f_x\left(x^0\left(t\right),\hat{u}\left(t\right),t\right)\delta x\left(t\right)+f_u\left(x^0\left(t\right),\hat{u}\left(t\right),t\right)\delta u\left(t\right),\\ \delta x\left(t_0\right)=0.\end{cases} \tag{3.20}$$

Take into account (3.19) in (3.16) and (3.17), and then substitute (3.17) into (3.16). Thus, we get the following formula for $\triangle^* S\left(\hat{u}\right)$:

$$\begin{aligned}\triangle^* S\left(\hat{u}\right)=\triangle_v\Phi\left(f\left(x^0\left(t_1-1\right),\hat{u}\left(t_1-1\right),t_1-1\right)\right)-\\ -\varepsilon\sum_{t=t_0}^{t_1-2}H_u'\left(\widehat{\psi}\left(t;v\right),x^0\left(t\right),\hat{u}\left(t\right),t\right)\delta u\left(t\right)-\\ -\frac{\varepsilon^2}{2}\{\delta x'(t_1-1)\Psi^0(t_1-2;v)\delta x\left(t_1-1\right)+\\ +\sum_{t=t_0}^{t_1-2}[\delta x'(t)H_{xx}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta x\left(t\right)+\\ +2\delta x'(t)\, H_{xu}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta u\left(t\right)+\\ +\delta u'\left(t\right)H_{uu}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta u\left(t\right)]\}+o\left(\varepsilon^2\right),\end{aligned} \tag{3.21}$$

where $\varepsilon\in\left(-\varepsilon_0,\varepsilon_0\right),\ \ \varepsilon_0=\varepsilon_0\left(\delta u\left(\cdot\right)\right)>0,\ \ \delta u\left(t\right)\in E^r,\ \ t\in T\backslash\{t_1-1\},\ \delta x\left(t\right)$, $t\in T$ is the solution of the system (3.20); $\widehat{\psi}\left(t;v\right),\ t\in T$ is the solution of the system (3.12), and $\triangle_v\Phi\left(f\left(\cdot\right)\right),\ \ \Psi^0\left(t_1-2;v\right)$ are defined from (3.9), (3.13).

Definition 3.1. If the increment in the quality functional $\triangle^* S\left(\hat{u}\right)=S\left(u\left(t;\varepsilon\right)\right)-$ $-S(\hat{u}(t))$ allows the representation

$$\triangle^* S\left(\hat{u}\right)=\varepsilon^0 A_0+\varepsilon A_1+\frac{\varepsilon^2}{2}A_2+o(\varepsilon^2),$$

where $A_i, i=0,1,2$ are independent of $\varepsilon\in(-\infty,+\infty)$, then $A_i, i=0,1,2$ are called zero, first and second variations, respectively, of the quality functional $S\left(u\right)$ at the point $\hat{u}=\hat{u}\left(t\right),\ t\in T$. For them as a rule there exist special symbols

$$A_0=\delta^0 S\left(\hat{u};v\right),\ A_1=\delta^1 S\left(\hat{u};\delta u;v\right),\ A_2=\delta^2 S(\hat{u};\delta u;v).$$

From this definition and formula (3.21) it follows that the variations of the quality functional $S\left(u\right)$ have the form:

$$\delta^0 S\left(\hat{u};v\right)=\triangle_v\Phi(f\left(x^0\left(t_1-1\right),\hat{u}\left(t_1-1\right),t_1-1\right)), \tag{3.22}$$

$$\delta^1 S\left(\hat{u};\delta u;v\right)=-\sum_{t=t_0}^{t_1-2}H_u'\left(\widehat{\psi}\left(t;v\right),x^0\left(t\right),\hat{u}\left(t\right),t\right)\delta u\left(t\right), \tag{3.23}$$

$$\delta^2 S\left(\hat{u};\delta u;v\right)=-\delta x^{'}\left(t_1-1\right)\Psi^0\left(t_1-2;v\right),\delta x\left(t_1-1\right)-$$
$$-\sum_{t=t_0}^{t_1-2}[\delta x^{'}\left(t\right)H_{xx}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta x\left(t\right)+$$
$$+2\delta x^{'}\left(t\right)H_{xu}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta u\left(t\right)+\delta u'\left(t\right)H_{uu}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta u\left(t\right)], \qquad (3.24)$$

where $\Delta_v\Phi(\cdot)$ and $a\left(t;\hat{u}\left(\cdot\right);v\right)$ are determined according to (3.9) and (3.18).

4. Necessary optimality conditions

Introduce the sets:

$$U_0\left[x^0\left(\cdot\right)\right]\left(t_1-1\right)=$$
$$=\{v\in U\left(t_1-1\right):\triangle_v\Phi\left(f\left(x^0\left(t_1-1\right),\hat{u}\left(t_1-1\right),t_1-1\right)\right)=0\}. \qquad (4.1)$$

Obviously $U[x^0(\cdot)]\left(t_1-1\right)\subset U_0[x^0(\cdot)]\left(t_1-1\right)$.

Let $\left(u^0\left(t\right),x^0\left(t\right)\right)$ be an optimal process. Then, taking into account (3.22)-(4.1) and the optimality condition $\triangle^* S\left(\hat{u}\right)\geq 0$, from (3.21) we arrive at the following conclusion.

Theorem 4.1. *For the optimality of the process* $\left(u^0\left(t\right),x^0\left(t\right)\right)$ *it is necessary that for any* $\hat{u}\left(t\right)\in U\left[x^0\left(\cdot\right)\right]\left(t\right),\ t\in T$ *the following relations be fulfilled:*

$$\delta^0 S\left(\hat{u};v\right)\geq 0, \quad \forall v\in U\left(t_1-1\right); \qquad (4.2)$$
$$\delta^1 S\left(\hat{u};\delta u;v\right)=0,\ \forall\delta u\left(t\right)\in E^r,t\in T\backslash\{t_1-1\},\ \forall v\in U_0\left[x^0\left(\cdot\right)\right]\left(t_1-1\right); \quad (4.3)$$
$$\delta^2 S\left(\hat{u};\delta u;v\right)\geq 0,\ \forall\delta u\left(t\right)\in E^r,t\in T\backslash\left\{t_1-1\right\},\ \forall v\in U_0\left[x^0\left(\cdot\right)\right]\left(t_1-1\right), \quad (4.4)$$

where $\delta^i S\left(\cdot\right),\ i=0,1,2$ *are determined from* (3.22)-(3.24), *and the set* $U_0\left[x^0\left(\cdot\right)\right]\left(t_1-1\right)$ *from* (4.1).

As is seen, the conditions (4.3), (4.4) are effectively inverifiable necessary optimality conditions. But constructively verifiable necessary optimality conditions may be obtained from these conditions.

Now deal with this issue. Assume that $\left(u^0\left(t\right),x^0\left(t\right)\right)$ is an optimal process. Using the arbitrariness of $\delta u\left(t\right)\in E^r$, $t\in T\backslash\{t_1-1\}$, determine it in the following way:

$$\delta u\left(t\right)=\begin{cases}\tilde{v},\ t=\theta\in T_1\equiv\{t_0,\ldots,t_1-2\}, \\ 0,t\in T_1\backslash\{\theta\},\end{cases} \qquad (4.5)$$

where $\tilde{v}\in E^r$.

Herewith, in the first place, taking into account (3.23) and arbitrariness $\tilde{v}\in E^r$ from (4.3), we get

$$H_u\left(\widehat{\psi}\left(t;v\right),x^0\left(\theta\right),\hat{u}\left(\theta\right),\theta\right)=0,\ \forall v\in U_0\left[x^0\left(\cdot\right)\right]\left(t_1-1\right),\ \forall\theta\in T_1, \qquad (4.6)$$

secondly, according to (3.20), (3.24), (4.5) from (4.4) we have:

$$\delta x'(t_1-1)\Psi^0(t_1-2;v)\delta x\left(t_1-1\right)+\tilde{v}'H_{uu}(a(\theta;\hat{u}\left(\cdot\right);v))\tilde{v}+$$
$$+\sum_{t=\theta_1}^{t_1-2}\delta x^{'}\left(t\right)H_{xx}\left(a\left(t;\hat{u}\left(\cdot\right);v\right)\right)\delta x\left(t\right)\leq 0, \qquad (4.7)$$
$$\forall v\in U_0\left[x^0\left(\cdot\right)\right]\left(t_1-1\right),\ \forall\tilde{v}\in E^r,\ \forall\theta\in T_1,$$

where $\Psi^0(t_1-2;v)$ are determined from (3.13), $\delta x\,(t)$ $t\in T$ is the solution of the system

$$\begin{cases} \delta x\,(t+1) = f_x\left(x^0\,(t)\,,\ \hat{u}\,(t)\,,t\right)\delta x\,(t)\,, \quad t\in\{\theta_1,\ldots,t_1-2\}\,, \\ \delta x\,(t)=0, t\in\{t_0,\ldots,\theta\}\,, \delta x\,(\theta_1) = f_u\left(x^0\,(\theta)\,,\hat{u}\,(\theta)\,,\theta\right)\tilde{v}, \theta_1=\theta+1. \end{cases} \tag{4.8}$$

Let's consider symmetric matrix function $\widehat{\Psi}\,(t;v)$, $t\in\{\theta,\ldots,t_1-2\}$ as the solution of the linear discrete system

$$\begin{cases} \widehat{\Psi}\,(t-1;v) = f_x'\left(x^0\,(t)\,,\hat{u}\,(t)\,,t\right)\widehat{\Psi}\,(t;v)\,f_x\left(x^0\,(t)\,,\hat{u}\,(t)\,,t\right)+ \\ +H_{xx}\left(\widehat{\psi}\,(t;v)\,,x^0\,(t)\,,\hat{u}\,(t)\,,t\right), \\ \widehat{\Psi}\,(t_1-2;v) = \Psi^0(t_1-2;v) \end{cases} \tag{4.9}$$

and use the following identity [7]:

$$\delta x'(t_1-1)\widehat{\Psi}\,(t_1-2;v)\,\delta x(t_1-1) =$$

$$=\sum_{t=\theta}^{t_1-2}\delta x'\,(t+1)\widehat{\Psi}\,(t;v)\,\delta x(t+1) - \sum_{t=\theta_1}^{t_1-2}\delta x'\,(t)\widehat{\Psi}\,(t-1;v)\,\delta x\,(t)\,,$$

where $\delta x\,(t)\,,\ t\in T$ is the solution of the system (4.8).

Then without special difficulty, allowing for (3.18), (4.9) inequality (4.7) takes the form:

$$\tilde{v}'[f_u'\left(x^0\,(\theta)\,,\hat{u}\,(\theta)\,,\theta\right)\widehat{\Psi}\,(\theta;v)\,f_u\left(x^0\,(\theta)\,,\hat{u}\,(\theta)\,,\theta\right)+$$

$$+H_{uu}(\widehat{\psi}\,(\theta;v)\,,x^0\,(\theta)\,,\hat{u}\,(\theta)\,,\theta)]\tilde{v}\le 0, \tag{4.10}$$

$$\forall v\in U_0\left[x^0\,(\cdot)\right](t_1-1)\,,\ \forall\tilde{v}\in E^r,\ \forall\theta\in\{t_0,\ldots,t_1-2\},$$

where $\widehat{\Psi}\,(\theta;v)$ is the value of the solution of the system (4.9) at the point $t=\theta$.

Thus, the following theorem is proved.

Theorem 4.2. *For the optimality of the process* $(u^0\,(t)\,,x^0(t))$ *it is necessary that for each* $\hat{u}(t)\in U[x^0(\cdot)](t)$, $t\in T\setminus\{t_1-1\}$ *the equality* (4.6) *and inequality* (4.10) *be fulfilled for all* $v\in U_0\left[x^0\,(\cdot)\right](t_1-1)$, $\tilde{v}\in E^r$, $\theta\in\{t_0,\ldots,t_1-2\}$.

Conclusion 4.1. Unlike the continuous one [8], in the discrete case the optimality conditions formulated by the first and second variations of the quality functional, allow strengthenings in the forms (4.3), (4.4). The optimality conditions (4.6), (4.10) are the strengthening and generalization of the Euler equation [1,3,13] respectively and optimality conditions from [7].

As far as we know, the notion of zero variation of the quality functional and optimality condition (4.2), have no analogues. If in addition to assumptions of Section 2 the set $U(t_1-1)$ is open, then from the condition (4.2) we can get the optimality condition of type (4.6), (4.10). It is easy to show that optimality conditions (4.3) and (4.6) are equivalent. We also note that: 1) the considered scheme for obtaining necessary optimality conditions is easily applicable for more complex discrete control systems; 2) using a new formula of the second variation $\delta^2S(\cdot)$ (see (3.24)) and the method from [1] we can get a stronger multipoint optimality condition of type [1].

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