Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Number 2, 2014, Pages 22–33

MULTI-SUBLINEAR FRACTIONAL MAXIMAL OPERATOR AND MULTILINEAR FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES

VAGIF S. GULIYEV AND AFAG F. ISMAYILOVA

Abstract. In this paper the authors study the boundedness of multi-sublinear fractional maximal operator $M_{\alpha,m}$ and multilinear fractional integral operators $I_{\alpha,m}$ on product generalized Morrey spaces $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n)\times \ldots \times \mathcal{M}_{p_m,\varphi_m}$. We find the sufficient conditions on $(\varphi_1,\ldots,\varphi_m,\varphi)$ which ensures the boundedness of the operators $M_{\alpha,m}$ and $I_{\alpha,m}$ from $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n)\times \ldots \times \mathcal{M}_{p_m,\varphi_m}$ to $\mathcal{M}_{q,\psi}$. In all cases, the conditions for the boundedness of $M_{\alpha,m}$ are given in terms of supremal type inequalities on $(\varphi_1,\ldots,\varphi_m,\psi)$ and the conditions for the boundedness of $I_{\alpha,m}$ are given in terms of Zygmund-type integral inequalities on $(\varphi_1,\ldots,\varphi_m,\varphi)$, which do not assume any assumption on monotonicity of $\varphi_1,\ldots,\varphi_m,\psi$ in r.

1. Introduction

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderon-Zygmund operators was done by Coifman and Meyer in [4] and was later systematically studied by Grafakos and Torres in [8]-[10].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n\times\ldots\times\mathbb{R}^n$ be the m-fold product space ($m\in\mathbb{N}$). For $x\in\mathbb{R}^n$ and $r>0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\overrightarrow{f}$ the m-tuple $(f_1,f_2,\ldots,f_m)$, $\overrightarrow{y}=(y_1,\ldots,y_n)$ and $d\overrightarrow{y}=dy_1\cdots dy_n$.

The multilinear theory has been well developed in the past twenty years. In 1992, Grafakos [6] first study the following multilinear integrals, defined by

$$I_\alpha^m(\overrightarrow{f})(x)=\int_{\mathbb{R}^n}\frac{1}{|y|^{n-\alpha}}f_1\left(x-\theta_1 y\right)\ldots f_m\left(x-\theta_m y\right)dy,$$

where $\theta_i (i=1,\ldots,m)$ are fixed distinct and nonzero real numbers and $0<\beta<n$. Grafakos proved that the operator I_α^m is bounded from $L_{p_1}(\mathbb{R}^n)\times\ldots\times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $0<1/q=1/p_1+\ldots+1/p_m-\beta/n<1$, which can be regarded as an extension result for the classical fractional integral on Lebesgue spaces.

2010 *Mathematics Subject Classification.* 42B20, 42B25, 42B35.

Key words and phrases. Multi-sublinear fractional maximal operator; generalized Morrey space; multilinear fractional integral operator.

In [14, 15] was proved a certain O'Neil type inequality for dilated multi-linear convolution operators, including permutations of functions. This inequality was used to extend Grafakoss result [6] to more general multi-linear operators of potential type and the relevant maximal operators.

Let $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$. The multi-sublinear fractional maximal operator $M_{\alpha,m}$ is defined by

$$M_{\alpha,m}(\overrightarrow{f})(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{n}} \prod_{j=1}^{m} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_i(y_i) dy_i, \qquad 0 \le \alpha < nm.$$

In 1999, Kenig and Stein [17] studied the following multilinear fractional integral,

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\ldots f_m(y_m)}{|(x-y_1,\ldots,x-y_m)|^{nm-\alpha}} dy_1 dy_2 \ldots dy_m,$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p_1 + \ldots + 1/p_m - \beta/n > 0$ for each $p_i > 1 (i = 1, \ldots, m)$. If some $p_i = 1$, then $I_{\alpha,m}$ is bounded $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$. Obviously, the multilinear fractional integral $I_{\alpha,m}$ is a natural generalization of the classical fractional integral $I_\alpha \equiv I_{\alpha,1}$.

In this work, we prove the boundedness of the multi-sublinear fractional maximal operator $M_{\alpha,m}$ and multilinear fractional integral operators $T_{\alpha,m}$ from product generalized Morrey space $\mathcal{M}_{p_1,\varphi_1} \times \ldots \times \mathcal{M}_{p_m,\varphi_m}$ to $\mathcal{M}_{q,\varphi}$, if $1 < p_1, \ldots, p_m < \infty$ and $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$, and from the space $\mathcal{M}_{p_1,\varphi_1} \times \ldots \times \mathcal{M}_{p_m,\varphi_m}$ to the weak space $WM_{1,\varphi}$, if $1 \le p_1, \ldots, p_m < \infty$, $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$ and at least one p_i equals one.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [5], [18]. Introduced by C. Morrey [20] in 1938, they are defined by the norm

$$\|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$.

We also denote by $WM_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\varphi} \equiv \mathcal{M}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, |B(x,r)|^{-\frac{1}{p}}\, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, |B(x,r)|^{-\frac{1}{p}}\, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $\mathcal{M}_{p,\lambda}$ and $W\mathcal{M}_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$\mathcal{M}_{p,\lambda} = \mathcal{M}_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$W\mathcal{M}_{p,\lambda} = W\mathcal{M}_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In [21], the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\,\varphi(x,r) \tag{2.1}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty t^{\alpha p}\, \varphi(x,t)^p \frac{dt}{t} \le C\, r^{\alpha p}\, \varphi(x,r)^p, \tag{2.2}$$

for the fractional maximal operator or fractional integral operator, where $C(> 0)$ does not depend on r and $x \in \mathbb{R}^n$.

In [21] the following statements were proved.

Theorem 2.1. [21] *Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi(x,r)$ satisfies the conditions* (2.1)-(2.2). *Then for $p > 1$ the operators M_α and I_α are bounded from $M_{p,\varphi}(\mathbb{R}^n)$ to $M_{q,\varphi}(\mathbb{R}^n)$ and for $p = 1$ from $M_{1,\varphi}(\mathbb{R}^n)$ to $WM_{q,\varphi}(\mathbb{R}^n)$.*

The following statement, containing results obtained in [19], [21] was proved in [11] (see also [12, 13, 22]).

Theorem 2.2. *Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ) satisfies the condition*

$$\int_r^\infty t^{\alpha}\, \varphi_1(x,t) \frac{dt}{t} \le C\, \varphi(x,r), \tag{2.3}$$

where C does not depend on x and r. Then the operators M_α and I_α are bounded from M_{p,φ_1} to $M_{q,\varphi}$ for $p > 1$ and from M_{p,φ_1} to $WM_{q,\varphi}$ for $p = 1$.

The following statements, containing results Theorems 2.1 and 2.2 was proved in [1], see also [16].

Theorem 2.3. *Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ) satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \le C\, \varphi(x,r), \tag{2.4}$$

where C does not depend on x and r. Let the operator M_α is bounded from M_{p,φ_1} to $M_{q,\varphi}$ for $p>1$ and from M_{p,φ_1} to $WM_{q,\varphi}$ for $p=1$.

Theorem 2.4. *Let $0<\alpha<n$, $1\le p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and (φ_1,φ) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}dt \le C\,\varphi(x,r), \tag{2.5}$$

where C does not depend on x and r. Let the operator I_α is bounded from M_{p,φ_1} to $M_{q,\varphi}$ for $p>1$ and from M_{p,φ_1} to $WM_{q,\varphi}$ for $p=1$.

Remark 2.1. It is obvious that if condition (2.3) holds, then condition (3.5) holds too. In general, condition (3.5) does not imply condition (2.3). For example, the functions

$$\varphi_1(r)=\frac{1}{\chi_{(1,\infty)}(r)r^{\frac{n}{p}-\beta}},\ \varphi_2(r)=r^{-\frac{n}{q}}\left(1+r^\beta\right),\ 0<\beta<\frac{n}{p}$$

satisfy condition (3.5) but do not satisfy condition (2.3) (see [16]).

3. The multi-sublinear fractional maximal operator in the product spaces $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n)\times\ldots\times\mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$

Let v be a weight. We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t>0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)}=\sup_{t>0} v(t)|g(t)|$$

and $L_\infty(0,\infty)\equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A}=\left\{\varphi\in\mathfrak{M}^+(0,\infty;\uparrow):\lim_{t\to 0+}\varphi(t)=0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g\in\mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t):=\|u\,g\|_{L_\infty(t,\infty)},\ t\in(0,\infty).$$

The following theorem was proved in [2].

Theorem 3.1. *Let v_1, v_2 be non-negative measurable functions satisfying $0<\|v_1\|_{L_\infty(t,\infty)}<\infty$ for any $t>0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\|v_2\overline{S}_u\left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\right)\right\|_{L_\infty(0,\infty)}<\infty. \tag{3.1}$$

In this section, we will prove the boundedness of multi-sublinear maximal operators on product generalized Morrey space, first we prove the following theorem.

Theorem 3.2. *Let $1 \le p_1, \dots, p_m < \infty$ and $0 < \alpha < mn$ with $1/q = 1/p_1 + \dots + 1/p_m - \alpha/n$ and $\alpha = \sum_{i=1}^m \alpha_i$ where each α_i satisfies $0 < \alpha_i < \frac{n}{p_i}$. Then, for $1 < p_1, \dots, p_m < \infty$ the inequality*

$$\|M_{\alpha,m}(\overrightarrow{f})\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^m \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))} \tag{3.2}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \dots \times L^{loc}_{p_m}(\mathbb{R}^n)$.

Moreover, if at least one p_i equals one, the inequality

$$\|M_{\alpha,m}(\overrightarrow{f})\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^m \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))} \tag{3.3}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \dots \times L^{loc}_{p_m}(\mathbb{R}^n)$.

Proof. $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f_j = f_j^0 + f_j^\infty, \quad f_j^0 = f_j \chi_{2B}, \quad f_j^\infty = f_j \chi_{\complement_{(2B)}}, \quad j = 1, \dots, m. \tag{3.4}$$

Thus for $y \in B(x_0,r)$ we get

$$M_{\alpha,m}(\overrightarrow{f})(y) = \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} \prod_{i=1}^m \left(\frac{1}{|B(y,t)|} \int_{B(y,t)} |f_i^0(z_i) + f_i^\infty(z_i)| dz_i \right)$$

$$\le \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} \prod_{i=1}^m \left(\frac{1}{|B(y,t)|} \int_{B(y,t)} |f_i^0(z_i)| dz_i + \frac{1}{|B(y,t)|} \int_{B(y,t)} |f_i^\infty(z_i)| dz_i \right)$$

$$\le \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} \left(\prod_{i=1}^m A_{B(y,t)} f_i^0 \right) + \sup_{t>0} |B(x,r)|^{\frac{\alpha}{n}} \left(\sum{}' A_{B(y,t)} f_1^{\beta_1} \cdots A_{B(y,t)} f_m^{\beta_m} \right)$$

$$= I_1(y) + I_2(y),$$

where $\beta_1, \dots, \beta_m \in \{0, \infty\}$ and each term in the sum $\sum'$ contains at least one $\beta_i = 1$, and where we denote

$$A_{B(y,t)} f_i^{\beta_i} = \frac{1}{|B(y,t)|} \int_{B(y,t)} |f_i^{\beta_i}(z_i)| dz_i.$$

By the boundedness of $M_{\alpha,m} : L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ we have

$$\begin{aligned} \|I_1\|_{L_q(B(x_0,r))} &\le \|M_{\alpha,m}(\overrightarrow{f}^0)\|_{L_q(B(x_0,r))} \\ &\le C \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} = C \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,2r))} \\ &\le C r^{\frac{n}{q}} \prod_{i=1}^m \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))}. \end{aligned}$$

To treat the term $I_2(y)$, we first consider the case $\beta_1 = \beta_2 = \dots = \beta_m = \infty$.

Let y be an arbitrary point from B. If $B(y,t) \cap {}^\complement(2B) \ne \emptyset$, then $t > r$. Indeed, if $z_i \in B(y,t) \cap {}^\complement(2B)$, then $t > |y - z_i| \ge |x - z_i| - |x - y| > 2r - r = r$ for $i = 1, \dots, m$.

On the other hand, $B(y,t) \cap \complement(2B) \subset B(x_0,2t)$. Indeed, $z_i \in B(y,t) \cap \complement(2B)$, then we get $|x_0 - z_i| \le |y - z_i| + |x_0 - y_i| < t + r < 2t$ for $i = 1, \dots, m$.

$$
\begin{aligned}
&\sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} A_{B(y,t)} f_1^\infty \dots A_{B(y,t)} f_m^\infty \\
&= \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} \prod_{i=1}^m \frac{1}{|B(y,t)|} \int_{B(y,t) \cap \complement B(x_0,2r)} |f_i(z_i)| dz_i \\
&\le 2^{nm-\alpha} \sup_{t>r} |B(x_0,2t)|^{\frac{\alpha}{n}} \prod_{i=1}^m \frac{1}{|B(x_0,2t)|} \int_{B(x_0,2t)} |f_i(z_i)| dz_i \\
&\le 2^{nm-\alpha} \sup_{t>2r} |B(x_0,t)|^{\frac{\alpha}{n}} \prod_{i=1}^m \frac{1}{|B(x_0,t)|} \int_{B(x_0,t)} |f_i(z_i)| dz_i \\
&\lesssim \sup_{t>2r} \prod_{i=1}^m t^{\alpha_i - \frac{n}{p_i}} \, \|f_i\|_{L_{p_i}(B(x_0,t))}.
\end{aligned}
$$

Therefore, for all $y \in B$ we have

$$
\sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} A_{B(y,t)} f_1^\infty \dots A_{B(y,t)} f_m^\infty \lesssim \sup_{t>2r} \prod_{i=1}^m t^{\alpha_i - \frac{n}{p_i}} \, \|f_i\|_{L_{p_i}(B(x_0,t))}.
$$

Then

$$
\| \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} A_{B(y,t)} f_1^\infty \dots A_{B(y,t)} f_m^\infty \|_{L_q(B)} \lesssim r^{\frac{n}{q}} \sup_{t>2r} \prod_{i=1}^m t^{\alpha_i - \frac{n}{p_i}} \, \|f_i\|_{L_{p_i}(B(x_0,t))}.
$$

For the case that $\beta_{j1} = \dots = \beta_{jl} = 0$ for some $\{j1, \dots, jl\} \subset \{1, \dots, m\}$ where $1 \le l < m$, we only consider the case $\beta_1 = \infty$ since the other ones follow in analogous way. Note that

$$
\begin{aligned}
&\sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} A_{B(y,t)} f_1^\infty \dots A_{B(y,t)} f_m^\infty \\
&\lesssim r^{\frac{n}{q_1}} \sup_{t>2r} t^{\alpha_1 - \frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \, M_{\alpha_2} f_2^0(x_0) \dots M_{\alpha_m} f_m^0(x_0).
\end{aligned}
$$

Then combine the estimates above we can easily get that

$$
\begin{aligned}
&\| \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}} A_{B(y,t)} f_1^\infty A_{B(y,t)} f_2^0 \dots A_{B(y,t)} f_m^0 \|_{L_q(B)} \\
&\lesssim r^{\frac{n}{q_1}} \sup_{t>2r} t^{\alpha_1 - \frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \prod_{i=2}^m \|M_{\alpha_i} f_i^0\|_{L_{p_i}(B)} \\
&\le r^{\frac{n}{q_1}} \sup_{t>2r} t^{\alpha_1 - \frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \prod_{i=2}^m \left(|B|^{\frac{1}{q_i}} \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i^0\|_{L_{p_i}(B(x_0,t))} \right) \\
&\approx r^{\frac{n}{q_1}} \sup_{t>2r} t^{\alpha_1 - \frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \prod_{i=2}^m r^{\frac{n}{q_i}} \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \, \|f_i\|_{L_{p_i}(B(x_0,2r))} \\
&\lesssim r^{\frac{n}{q}} \prod_{i=1}^m \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))}.
\end{aligned}
$$

Hence we have obtained

$$\|M_{\alpha,m}(\overrightarrow{f})\|_{L_p(B)} \le \|I_1\|_{L_p(B)} + \|I_2\|_{L_p(B)}$$
$$\lesssim r^{\frac{n}{q}} \prod_{i=1}^{m} \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))}.$$

Thus we obtain (3.2).

For the case that at least one p_i equals one, repeat the estimates above and note that $\overrightarrow{f} \to M_{\alpha,m}(\overrightarrow{f})$ is boundedness from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$, the proof of (3.3) can be treated similarly and we omit the details here.

□

Next we give the boundedness of multilinear fractional maximal operator $\overrightarrow{f} \to M_{\alpha,m}(\overrightarrow{f})$ on product generalized Morrey space.

Theorem 3.3. *Let* $1 \le p_1, \ldots, p_m < \infty$ *and* $0 < \alpha < mn$ *with* $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$ *and* $\alpha = \sum_{i=1}^m \alpha_i$ *where each* α_i *satisfies* $0 < \alpha_i < \frac{n}{p_i}$. *Suppose that* (φ_1, φ_2) *satisfies the condition*

$$\prod_{i=1}^{m} \sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_i(x,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{q_i}}} \le C\psi(x,r), \tag{3.5}$$

where C *does not depend on* x *and* r. *Then, if all* $p_i > 1$, *it follows*

$$\|M_{\alpha,m}(\overrightarrow{f})\|_{\mathcal{M}_{q,\psi}} \le C\|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \cdots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}},$$

and if at least one $p_i = 1$, *it follows*

$$\|M_{\alpha,m}(\overrightarrow{f})\|_{WM_{q,\psi}} \le C\|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \ldots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}},$$

with the constant C *independent of* $\overrightarrow{f}$.

Proof. Let $1 \le p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $\overrightarrow{f} \in \mathcal{M}_{p_1,\varphi_1} \times \ldots \times \mathcal{M}_{p_m,\varphi_1}$. By Theorems 3.1 and 3.2 we obtain

$$\begin{aligned}
\|M_{\alpha,m}(\overrightarrow{f})\|_{\mathcal{M}_{q,\psi}} &= \sup_{x\in\mathbb{R}^n,\, r>0} \psi^{-1}(x,r) r^{-\frac{n}{q}} \|M_{\alpha,m}(\overrightarrow{f})\|_{L_p(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \prod_{i=1}^{m} \psi^{-\frac{1}{m}}(x,r) \sup_{t>2r} t^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \prod_{i=1}^{m} \varphi_i^{-1}(x,r) r^{\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x,r))} \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \prod_{i=1}^{m} \|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \cdots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}}
\end{aligned}$$

by (3.5), which completes the proof for $1 < p_1, \ldots, p_m < \infty$ and $0 < \alpha < mn$ with $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$.

For $p_i = 1$ and $f_i \in \mathcal{M}_{1,\varphi_1} (i = 1, \ldots, m)$, by the definition of $\mathcal{M}_{1,\varphi}$ and a similar argument as before we can get

$$\|M_{\alpha,m}(\overrightarrow{f})\|_{W\mathcal{M}_{q,\psi}} \le C\|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \ldots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}}.$$

The theorem has been proved. □

Remark 3.1. Note that in the case $m = 1$ Theorems 3.2 and 3.3 were proved in [1] (see also [16]). Theorem 3.3 do not impose the pointwise doubling condition (2.1) and (2.2). In the case $\varphi_1(x,r) = \varphi_2(x,r) = \varphi(x,r)$ Theorem 3.3 containing the results Theorem 2.1.

4. The multilinear fractional integral operators in the product spaces $\mathcal{M}_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times \mathcal{M}_{p_m,\varphi_m}(\mathbb{R}^n)$

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr,\ 0 < t < \infty.$$

Theorem 4.1. ([3]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0}\frac{w(t)}{t}\int_0^t \frac{dr}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

In this section, we will prove the boundedness of multilinear singular integral operators on product generalized Morrey space, first we prove the following theorem.

Theorem 4.2. *Let $1 \leq p_1,\ldots,p_m < \infty$ and $0 < \alpha < mn$ with $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$ and $\alpha = \sum_{i=1}^m \alpha_i$ where each α_i satisfies $0 < \alpha_i < \frac{n}{p_i}$. Then, for $1 < p_1,\ldots,p_m < \infty$ the inequality*

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}}\prod_{i=1}^m \int_{2r}^\infty t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x_0,t))}dt \tag{4.1}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Moreover, if at least one p_i equals one, the inequality

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}}\prod_{i=1}^m \int_{2r}^\infty t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x_0,t))}dt \tag{4.2}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. We just consider the case $p_i > 1$ for $i = 1,\ldots,m$ and write $f_i = f_i^0 + f_i^\infty$. Then we split $I_{\alpha,m}(\overrightarrow{f})$ as follows

$$I_{\alpha,m}(\overrightarrow{f})(x) = I_{\alpha,m}(f_1^0,\ldots,f_m^0)(x) + \sum_{\beta_1,\ldots,\beta_m}{}' I_{\alpha,m}(f_1^{\beta_1},\ldots,f_m^{\beta_m})(x),$$

where $\beta_1,\dots,\beta_m \in \{0,\infty\}$ and each term of $\sum'$ contains at least $\beta_i \neq 0$. Then,

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{L_p(B(x,r))} \le \|I_{\alpha,m}(f_1^0,\dots,f_m^0)\|_{L_p(B(x,r))} + \|\sum_{\beta_1,\dots,\beta_m}{}' I_{\alpha,m}(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_p(B(x,r))}$$
$$\le I + II.$$

For I, by the boundedness of $I_{\alpha,m}$ from product $L_{p_1}(\mathbb{R}^n)\times\dots\times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $0<\alpha<mn$ with $1/q = 1/p_1+\dots+1/p_m-\alpha/n$ for each $p_i>1(i=1,\dots,m)$, we have,

$$\|I_{\alpha,m}(\overrightarrow{f^0})\|_{L_q(B(x,r))} \le \|I_{\alpha,m}(\overrightarrow{f^0})\|_{L_q(\mathbb{R}^n)}$$
$$\lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,2r))}.$$

Taking into account that

$$\|f_i\|_{L_{p_i}(B(x,2r))} \lesssim r^{\frac{n}{q_i}} \int_{2r}^\infty t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x,t))}dt,\ i=1,\dots,m \tag{4.3}$$

we get

$$\|I_{\alpha,m}(\overrightarrow{f^0})\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^m \int_{2r}^\infty t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x,t))}dt. \tag{4.4}$$

For II, first we consider the case $\beta_1=\dots=\beta_m=\infty$.

When $|x-y_i|\le r$, $|z-y_i|\ge 2r$, we have $\frac{1}{2}|z-y_i| \le |x-y_i| \le \frac{3}{2}|z-y_i|$, and so we get

$$|I_{\alpha,m}(\overrightarrow{f^\infty})(z)| \lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} d\overrightarrow{y}$$
$$\lesssim \prod_{i=1}^m \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}dy_i$$

and

$$\|I_{\alpha,m}(\overrightarrow{f^\infty})\|_{L_q(B(x,r))} \le \prod_{i=1}^m \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}dy_i\ \|\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n)}$$
$$\lesssim r^{\frac{n}{q}} \prod_{i=1}^m \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}dy_i.$$

By Fubini's theorem we have

$$\int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}dy_i \approx \int_{\complement_{B(x,2r)}} |f_i(y_i)| \int_{|x_0-y_i|}^\infty \frac{dt}{t^{n-\alpha_i+1}}dy_i$$
$$\approx \int_{2r}^\infty \int_{2r\le|x_0-y_i|<t} |f_i(y_i)|dy_i \frac{dt}{t^{n-\alpha_i+1}}$$
$$\lesssim \int_{2r}^\infty \int_{B(x_0,t)} |f_i(y_i)|dy_i \frac{dt}{t^{n-\alpha_i+1}}.$$

Applying Hölder's inequality, we get

$$\int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}dy_i \lesssim \int_{2r}^\infty t^{\alpha_i-\frac{n}{p_i}-1}\|f_i\|_{L_{p_i}(B(x,t))}dt. \tag{4.5}$$

Moreover, for all $p_i \in [1,\infty)$, $i = 1,\dots,m$ the inequality

$$\|I_{\alpha,m}(\overrightarrow{f^{\infty}})\|_{L_q(B(x,r))} \lesssim r^{\frac{n}{q}} \prod_{i=1}^{m} \int_{2r}^{\infty} t^{\alpha_i - \frac{n}{p_i} - 1} \|f_i\|_{L_{p_i}(B(x,t))} dt \tag{4.6}$$

is valid.

Next we consider the case that some $\beta_i = 0$ and other $\beta_j = \infty$. To this end we may assume that $\beta_1 = \beta_2 = \infty$ and $\beta_3 = \cdots = \beta_m = 0$. Recall the fact that $|x - y_i| \approx |z - y_i|$ for $z \in B(x,r)$ and $y_i \in {}^{\complement}B(x,2r)$, we have that

$$\begin{aligned}
& I_{\alpha,m}(f_1^{\infty}, f_2^{\infty}, f_3^{0}, \dots, f_m^{0})(z) \\
& \lesssim \int_{{}^{\complement}B(x,2r)\times {}^{\complement}B(x,2r)} \frac{|f_1(y_1)||f_2(y_2)|}{\{|x-y_1| + |x-y_2|\}^{mn-\alpha}} dy_1 dy_2 \prod_{i=3}^{m} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
& \lesssim \int_{{}^{\complement}B(x,2r)} \frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha_1}} dy_1 \int_{{}^{\complement}B(x,2r)} \frac{|f_2(y_2)|}{|x-y_2|^{n-\alpha_2}} dy_2 \prod_{i=3}^{m} r^{\alpha_i - n} \int_{B(x,2r)} |f_i(y_i)| dy_i.
\end{aligned}$$

By the inequalities (4.3), (4.5) and use the Hölder's inequality for integrals,

$$\begin{aligned}
& \|I_{\alpha,m}(f_1^{\infty}, f_2^{\infty}, f_3^{0}, \dots, f_m^{0})\|_{L_q(B(x,r))} \\
& \lesssim r^{\frac{n}{q}} \int\limits_{{}^{\complement}B(x,2r)} \frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha_1}} dy_1 \int\limits_{{}^{\complement}B(x,2r)} \frac{|f_2(y_2)|}{|x-y_2|^{n-\alpha_2}} dy_2 \prod_{i=3}^{m} r^{\alpha_i - \frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x,2r))} \\
& \lesssim r^{\frac{n}{q}} \int\limits_{{}^{\complement}B(x,2r)} \frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha_1}} dy_1 \int\limits_{{}^{\complement}B(x,2r)} \frac{|f_2(y_2)|}{|x-y_2|^{n-\alpha_2}} dy_2 \prod_{i=3}^{m} \int_{r}^{\infty} t^{\alpha_i - \frac{n}{p_i} - 1} \|f_i\|_{L_{p_i}(B(x,t))} dt \\
& \leq r^{\frac{n}{q}} \prod_{i=1}^{m} \int_{r}^{\infty} t^{\alpha_i - \frac{n}{p_i} - 1} \|f_i\|_{L_{p_i}(B(x,t))} dt.
\end{aligned}$$

For the proof of the inequality (4.2), by a similar argument as in the proof of (4.1) and pay attention to the fact that $\overrightarrow{f} \to I_{\alpha,m}(\overrightarrow{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \cdots \times L_{p_m}(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$, we can similarly prove (4.2) and we omit the details here. □

Now we give the boundedness of multilinear fractional integral operators on product generalized Morrey space.

Theorem 4.3. *Let $1 \leq p_1, \dots, p_m < \infty$ and $0 < \alpha < mn$ with $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/n$ and $\alpha = \sum_{i=1}^{m} \alpha_i$ where each α_i satisfies $0 < \alpha_i < \frac{n}{p_i}$. Suppose that $(\varphi_1, \dots, \varphi_m, \psi)$ satisfies the condition*

$$\prod_{i=1}^{m} \int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_i(x,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{q_i}+1}} dt \lesssim \psi(x,r). \tag{4.7}$$

Then the operator $I_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times M_{p_m,\varphi_m}(\mathbb{R}^n)$ to $M_{q,\psi}(\mathbb{R}^n)$ for $p_i > 1$, $i = 1, \dots, m$, and from product space $M_{p_1,\varphi_1}(\mathbb{R}^n) \times \ldots \times M_{p_m,\varphi_m}(\mathbb{R}^n)$ to $WM_{q,\psi}(\mathbb{R}^n)$ for $p_i \geq 1$, $i = 1, \dots, m$.

Proof. Let $1 < p_1, \dots, p_m < \infty$ and $\overrightarrow{f} \in M_{p_1,\varphi_1}(\mathbb{R}^n) \times \dots \times M_{p_m,\varphi_m}(\mathbb{R}^n)$. By Theorems 4.1 and 4.2 we have

$$\begin{aligned}
\|I_{\alpha,m}(\overrightarrow{f})\|_{M_{q,\psi}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \prod_{i=1}^{m} \varphi(x,r)^{-\frac{1}{m}} \int_r^\infty t^{\alpha_i-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \prod_{i=1}^{m} \varphi_i^{-1}(x,r)\; r^{\frac{n}{p_i}}\; \|f_i\|_{L_{p_i}(B(x,r))} \\
&= \prod_{i=1}^{m} \|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \cdots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}}.
\end{aligned}$$

When $p_i = 1$, $i = 1, \dots, m$, the proof is similar and we omit the details here. □

Remark 4.1. As shown in [16], the condition (3.5) is weaker than (4.7): the latter implies the former, in particular, the functions

$$\varphi_i(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p_i}-\beta_i}},\ i = 1, \dots, m,\ \psi(r) = r^{-\frac{n}{q}}\big(1 + r^\beta\big),\ 0 < \beta < \frac{n}{p}$$

satisfy condition (3.5) but do not satisfy condition (4.7).

Acknowledgements

The research of Vagif S. Guliyev was partially supported by the grant of Ahi Evran University Scientific Research Projects (PYO.FEN.4001.13.012).

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Vagif S. Guliyev
Ahi Evran University, Department of Mathematics, Kirsehir, Turkey
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.
E-mail address: `vagif@guliyev.com`

Afag F. Ismayilova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.
E-mail address: `afaismayilova28@gmail.com`

Received: September 22, 2014; Accepted: October 30, 2014