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ON DECOMPOSITION IN BANACH SPACES

BILAL T. BILALOV AND CHINGIZ M. HASHIMOV

Abstract. Decomposition of arbitrary element with respect to the given system is considered. It is proved that if the arbitrary element can be expanded with respect to the given system and if this system becomes complete after exclusion of a finite number of elements, then the arbitrary element can be expanded with respect to the resulting system, too. This property stays true for the t-decomposition generated by the tensor product of Hilbert spaces X and Y. The obtained results are also of interest for the theory of frames.

1. Introduction

The problem of decomposition of the elements of Banach space or its subspace with respect to the given system plays a crucial role in many areas of mathematics such as approximation theory, spectral theory of operators, theory of bases, theory of frames, etc. Different branches of natural science showed a keen interest in the theory frames since recently. Many review articles and monographs (see, e.g., R.Young [19], O.Christensen [6, 7, 8, 9], Heil Ch. [17], Chui Ch. [11], etc.) have been dedicated to this theory. For applications we refer the reader to the monograph by Daubechies I. [12] and a review article by Dremin I.M., Ivanov O.V., Nechitailo V.A. [13]. One of the central points in the theory of frames is a decomposition of an arbitrary element with respect to a frame. In Hilbert spaces, unlike Banach spaces, such decomposition is always the case. In Banach spaces, there are frames which don't allow decomposition for an arbitrary element (more details on this matter can be found in [10]). Therefore, the problem of decomposition with respect to the given system in Banach spaces is of special interest.

It should be noted that the growing interest in frames gave rise to many generalizations of this concept, as well as to various methods for constructing frames. For more details we refer the reader to [1, 2, 3, 14, 15, 16, 18].

In this paper, we consider the problem of decomposability of an arbitrary element with respect to the given system. We prove that if any element in Banach space can be expanded in some system which becomes complete after exclusion of a finite number of its elements, then any element is also decomposable with

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respect to the newly created system. We extend this result to the case of t-decomposition generated by the tensor product of Hilbert spaces X and Y. The results obtained in this work can be used also in the theory of frames.

2. Needful Information

We will use the standard notation. $\mathbb{N}$ will be a set of all positive integers; $\mathbb{C}$ will be the set of complex numbers; $\mathbb{Z}_+ = \mathbb{N}\bigcup\{0\}$; K will denote a field of scalars; $L[M]$ will denote the linear span of the set M, and $\overline{M}$ will stand for the closure of M; Banach space will be referred to as B-space; Hilbert space will be referred to as H-space; $\|\cdot\|_X$ will denote a norm in the space X; $(\cdot\,;\,\cdot)_X$ will denote a scalar product in X; K-space will stand for a B-space of sequences of scalars. By $\vec{x}$ we will always mean a sequence $\{x_n\}_{n\in\mathbb{N}}$, i.e. $\{x_n\}_{n\in\mathbb{N}} \equiv \vec{x}$.

Let X be a B-space and $\vec{x} \subset X$ be some system. We will call this system nondegenerate if $x_n \neq 0$, $\forall n \in \mathbb{N} \Leftrightarrow \vec{x} \neq 0$. Assume

$$\mathscr{K}_{\vec{x}} \equiv \left\{ \vec{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}} \in K : \sum_{n=1}^{\infty} \lambda_n x_n \text{ is convergent in } X \right\}.$$

For $\vec{x} \neq 0$ define the norm in $\mathscr{K}_{\vec{x}}$ as follows

$$\left\|\vec{\lambda}\right\|_{\mathscr{K}_{\vec{x}}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_n x_n \right\|.$$

The following theorem is true.

Theorem 2.1. *Let X be a B-space and $\vec{x} \subset X$ be some system. If $\vec{x} \neq 0$, then $\mathscr{K}_{\vec{x}}$ is a K-space.*

More details about this result can be found e.g. in [17, 4, 5].

We also need the concept of factor space. Let X be a B-space and $X_0 \subset X$ be its closed subspace. Introduce an equivalence relation " $\sim$ " in X: $x \sim y \Leftrightarrow x - y \in X_0$. The relation " $\sim$ " divides X into equivalence classes, and we denote the set of these classes by $X/_{X_0}$. Class $\tilde{X}$ containing the element x is denoted by $\tilde{X}_x$. With regard to the operations

$$\lambda \tilde{X}_x = \tilde{X}_{\lambda x}; \tilde{X}_x + \tilde{X}_y = \tilde{X}_{x+y},$$

$X/_{X_0}$ turns into a linear space. Moreover, if equipped with the norm

$$\left\|\tilde{X}\right\|_{X/_{X_0}} = \inf_{x\in X} \|x\|,$$

$X/_{X_0}$ becomes a B-space.

We also give some concepts and facts concerning the Hilbert tensor product. Let $X; Y$ be H-spaces and $Z = X \otimes Y$ be their tensor product. For simplicity, the tensor product $x \otimes y$ of elements $x \in X$ and $y \in Y$ will be denoted by $xy = x \otimes y$. Let $M \subset Y$ be some set. Assume

$$L_t[M] \equiv \left\{ z \in Z : \exists \{x_k; y_k\}_1^m \subset X \times M \Rightarrow z = \sum_{k=1}^{m} x_k y_k \right\}.$$

$L_t[M]$ is called a t- *span* of set M. Let $\vec{y} \subset Y$ be some system. Define

$$\Lambda^{(t)} \equiv \left\{ \vec{x} \subset X : \sum_{k=1}^{\infty} x_k y_k < +\infty \right\},$$

where $\sum(\cdot) < +\infty$ means the convergence of the series in Z.

System $\vec{y} \subset Y$ *is said to be* t- *complete in* Z *if for* $\forall z \in Z$, $\exists \left\{ x_k^{(n)} \right\}_{k=1}^{m_n} \subset X$, $\forall n \in \mathbb{N}$:

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} x_k^{(n)} y_k = Z.$$

Let us define linear operations in $\Lambda^{(t)}$, as usual, componentwise. Let $\vec{x} \in \Lambda^{(t)}$. Put

$$\|\vec{x}\|_{\Lambda^{(t)}} = \sup_m \left\| \sum_{k=1}^{m} x_k y_k \right\|_Z . \tag{2.1}$$

It is not difficult to see that $\|\cdot\|_{\Lambda^{(t)}}$ is a norm in $\Lambda^{(t)}$ if the system $\vec{y}$ is non-degenerate. It suffices to verify that $\|\vec{x}\|_{\Lambda^{(t)}} = 0 \Rightarrow \vec{x} = 0$. Let $\|\vec{x}\|_{\Lambda^{(t)}} = 0 \Rightarrow \sum_{k=1}^{m} x_k y_k = 0$, $\forall m \in \mathbb{N} \Rightarrow x_k = 0, \forall k \in \mathbb{N}$. Let us show that $\Lambda^{(t)}$ is complete with respect to $\|\cdot\|_{\Lambda^{(t)}}$. Let $\|\vec{x}_m - \vec{x}_n\|_{\Lambda^{(t)}} \to 0$, $n, m \to \infty$, where $\vec{x}^n \equiv \left\{ x_k^{(n)} \right\}_{k\in\mathbb{N}}$, $\forall n \in \mathbb{N}$. Thus

$$\left\| \sum_{k=1}^{m} \left(x_k^{(n)} - x_k^{(n+p)} \right) y_k \right\|_Z \to 0, \quad n \to \infty, \forall m; p \in \mathbb{N}.$$

Take $\forall k_0 \in \mathbb{N}$. We have

$$\left\| x_{k_0}^{(n)} - x_{k_0}^{(n+p)} \right\|_X = \frac{\left\| x_{k_0}^{(n)} - x_{k_0}^{(n+p)} \right\|_X \|y_{k_0}\|_Y}{\|y_{k_0}\|_Y} = \frac{\left\| \left(x_{k_0}^{(n)} - x_{k_0}^{(n+p)} \right) y_{k_0} \right\|_Z}{\|y_{k_0}\|_Y} =$$

$$= \frac{\left\| \sum_{k=1}^{k_0} \left(x_k^{(n)} - x_k^{(n+p)} \right) y_k - \sum_{k=1}^{k_0-1} \left(x_k^{(n)} - x_k^{(n+p)} \right) y_k \right\|_Z}{\|y_{k_0}\|_Y} \leq$$

$$\leq \frac{2 \|z_n - z_{n+p}\|_{\Lambda^{(t)}}}{\|y_{k_0}\|_Y} \to 0, \ n \to \infty, \ \forall p \in \mathbb{N}.$$

Consequently, $\left\{ x_{k_0}^{(n)} \right\}_{n\in\mathbb{N}} \subset X$ is fundamental. Let $x_{k_0}^{(n)} \to x_{k_0}$, $n \to \infty$. Let us show that the series $\sum_{k=1}^{\infty} x_k y_k$ is convergent in Z. Let $\varepsilon > 0$ be an arbitrary number. In fact, then $\exists n_0 \in \mathbb{N} : \|z_n - z_{n+p}\|_Z < \varepsilon, \forall n \geq n_0, \forall p \in \mathbb{N}. \Rightarrow$ $\left\| \sum_{k=1}^{m} \left(x_k^{(n)} - x_k^{(n+p)} \right) y_k \right\|_Z < \varepsilon, \forall n \geq n_0, \forall m, p \in \mathbb{N}$. Passing to the limit as $p \to \infty$, we get

$$\left\| \sum_{k=1}^{m} \left(x_k^{(n)} - x_k \right) y_k \right\|_Z \leq \varepsilon, \forall n \geq n_0, \forall m \in \mathbb{N}. \tag{2.2}$$

It is clear that

$$\left\| \sum_{k=m}^{m+p} \left(x_k^{(n)} - x_k \right) y_k \right\|_Z \leq 2\varepsilon, \forall n \geq n_0, \forall m, p \in \mathbb{N}.$$

As the series $\sum_{k=1}^{\infty} x_k^{(n)} y_k$ is convergent, $\exists m_n \in \mathbb{N}$:

$$\left\| \sum_{k=m}^{m+p} x_k^{(n)} y_k \right\|_Z < \varepsilon, \forall m \geq m_n \,, \forall p \in \mathbb{N}.$$

We have

$$\left\| \sum_{k=m}^{m+p} x_k y_k \right\|_Z \leq \left\| \sum_{k=m}^{m+p} \left(x_k^{(n)} - x_k \right) y_k \right\|_Z +$$

$$+ \left\| \sum_{k=m}^{m+p} x_k^{(n)} y_k \right\|_Z < 3\varepsilon, \forall m \geq m_n \,, \forall p \in \mathbb{N}.$$

The latter implies the convergence of the series $\sum_{k=1}^{\infty} x_k y_k$. Let $\vec{x} \equiv \{x_k\}_{k\in\mathbb{N}}$. Let us show that $\|\vec{x}_n - \vec{x}\|_{\Lambda^{(t)}} \to 0 \,, n \to \infty$. This follows directly from (2.2) because

$$\sup_m \left\| \sum_{k=1}^{m} \left(x_k^{(n)} - x_k \right) y_k \right\|_Z \leq \varepsilon, \forall n \geq n_0 \,,$$

and, as a result, $\|\vec{x}_n - \vec{x}\|_{\Lambda^{(t)}} \leq \varepsilon$, $\forall n \geq n_0$. Thus, $\Lambda^{(t)}$ is a B-space. We call it a *t-space* of coefficients of the system $\vec{y}$. Thus, the following theorem is true.

Theorem 2.2. *Let $\vec{y} \subset Y$ be a non-degenerate system. Then its t-space of coefficients $\Lambda^{(t)}$ is a B-space with respect to the norm* (2.1).

To obtain our main results, we also need the concept of a *t-scalar* product for the pair $(x; z) \in X \times Z$. Take $\forall y \in Y$ and consider the linear functional $\vartheta_{(x;z)}(y) = (xy; z)_Z$. We have

$$\left|\vartheta_{(x;z)}(y)\right| \leq \|xy\|_Z \|z\|_Z = \|x\|_X \|z\|_Z \|y\|_Y \,. \tag{2.3}$$

Consequently, $\vartheta_{(x;z)}$ is a linear continuous functional on Y. As a result, $\exists! \tilde{y} \in Y$:

$$\vartheta_{(x;z)}(y) = (y; \tilde{y})_Y \,, \forall y \in Y.$$

We call $\tilde{y}$ a *t-scalar* product of elements x and z, and denote it as $\tilde{y} = \langle x; z\rangle_Y$. It is not difficult to see that *t-scalar* product possesses the following properties with regard to $\forall x, x_k \in X$, $\forall z, z_k \in Z$, $k = 1, 2$:

1) $\langle x_1 + x_2; z\rangle_Y = \langle x_1; z\rangle_Y + \langle x_2; z\rangle_Y$;
2) $\langle x; z_1 + z_2\rangle_Y = \langle x; z_1\rangle_Y + \langle x; z_2\rangle_Y$;
3) $\langle \lambda x; z\rangle_Y = \lambda \langle x; z\rangle_Y = \langle x; \bar{\lambda} z\rangle_Y$, $\forall \lambda \in \mathbb{C}$;
4) $\langle x_1; x_2 \otimes y_1\rangle_Y = \overline{(x_1; x_2)}_X y_1$, $\forall y_1 \in Y$.

The properties 1)-3) follow directly from the definition of *t-scalar* product. So we only have to prove the validity of property 4) . We have

$$\vartheta^{(y)}_{(x_1;x_2\otimes y_1)} = (x_1 \otimes y; x_2 \otimes y_1)_Z = (x_1; x_2)_X (y; y_1)_Y =$$

$$= \left(y; \overline{(x_1; x_2)}_X y_1\right) = (y; \langle x_1; \langle x_2 \otimes y_1\rangle\rangle_Y) \,, \ \forall y \in Y \Rightarrow \text{the validity of 4)}.$$

Moreover, from the relation (2.3) it follows directly that $\langle x; z\rangle_Y$ depends continuously on x and z.

3. Main results

3.1. On decomposition in B-spaces. Let us state the following theorem on decomposition in B-spaces, which is directly related to the frames and is of interest in itself.

Theorem 3.1. *Let X be a B-space equipped with the norm $\|\cdot\|$, and suppose that the arbitrary element in X can be expanded with respect to the system $\{x_n\}_{n\in\mathbb{Z}_+} \subset X$. If $\{x_n\}_{n\in\mathbb{N}}$ is complete in X, then the arbitrary element in X can be expanded with respect to the system $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. It suffices to consider the case $x_n \neq 0$, $\forall n \in \mathbb{Z}_+$. If x_0 can be expanded in terms of $\{x_n\}_{n\in\mathbb{N}}$, then the assertion of theorem is obviously true. Suppose x_0 can not be expanded with respect to $\{x_n\}_{n\in\mathbb{N}}$. Then it is not difficult to see that in decomposition $x = \lambda_0 x_0 + \sum_{n=1}^{\infty} \lambda_n x_n$, of an arbitrary element $x \in X$ the coefficient λ_0 is determined uniquely, and hence, $\lambda_0 = \vartheta(x)$, where $\vartheta : X \to \mathbb{C}$ is a linear functional with $\vartheta(x_0) = 1$, $\vartheta(x_n) = 0$, $\forall n \in \mathbb{N}$. Thus

$$x = \vartheta(x)\, x_0 + \sum_{n=1}^{\infty} \lambda_n x_n. \tag{3.1}$$

We denote the space of coefficients of the system $\{x_n\}_{n\in\mathbb{N}}$ by $\mathscr{K}$, and the corresponding norm is defined as

$$\left\| \{\lambda_n\}_{n\in\mathbb{N}} \right\|_{\mathscr{K}} = \sup_m \left\| \sum_{n=1}^{m} \lambda_n x_n \right\|.$$

$(\mathscr{K};\, \|\cdot\|_{\mathscr{K}})$ is a B-space. Let

$$\Lambda_0 \equiv \left\{ \{\lambda_n\}_{n\in\mathbb{N}} \in \mathscr{K} : \sum_{n=1}^{\infty} \lambda_n x_n = 0 \right\}.$$

It is absolutely clear that Λ_0 is a linear subspace of $\mathscr{K}$. Let us show that Λ_0 is closed. Let $\left\{\vec{\lambda}^{(m)}\right\}_{m\in\mathbb{N}} \subset \Lambda_0$ be some fundamental sequence with $\vec{\lambda}^{(m)} \equiv \left\{\lambda_n^{(m)}\right\}_{n\in\mathbb{N}} \in \Lambda_0$, $\forall m \in \mathbb{N}$. As $\mathscr{K}$ is complete, it is clear that $\exists \vec{\lambda} \in \mathscr{K}$: $\vec{\lambda}^{(m)} \to \vec{\lambda}$ in $\mathscr{K}$, where $\vec{\lambda} \equiv \{\lambda_n\}_{n\in\mathbb{N}}$. Consequently, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ is convergent in X. We have

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \left\| \sum_{n=1}^{\infty} \left(\lambda_n - \lambda_n^{(m)}\right) x_n \right\| \leq$$

$$\leq \sup_l \left\| \sum_{n=1}^{l} \left(\lambda_n - \lambda_n^{(m)}\right) x_n \right\| = \left\| \vec{\lambda} - \vec{\lambda}^{(m)} \right\|_{\mathscr{K}} \to 0,\ m \to \infty.$$

As a result, $\sum_{n=1}^{\infty} \lambda_n x_n = 0 \Rightarrow \vec{\lambda} \in \Lambda_0 \Rightarrow \Lambda_0$ is closed. Let us consider the factor space $\mathscr{K}/_{\Lambda_0}$ with the norm

$$\|\Lambda\|_{(1)} = \inf_{\vec{\lambda}\in\Lambda} \left\|\vec{\lambda}\right\|_{\mathscr{K}},\ \forall \Lambda \in \mathscr{K}/_{\Lambda_0}.$$

Let

$$\tilde{X} \equiv \left\{ \tilde{x} \in X : \exists \vec{\lambda} \in \mathscr{K} \ , \ \tilde{x} = \sum_{n=1}^{\infty} \lambda_n x_n \right\}.$$

It is clear that $\tilde{X}$ is a linear space. Let $\tilde{x} \in X$ and $\sum_{n=1}^{\infty} \lambda_n x_n = \tilde{x}$. By $\Lambda_{\tilde{x}}$we denote the factor-class that contains $\vec{\lambda} \equiv \{\lambda_n\}_{n \in \mathbb{N}}$. It is not difficult to see that the mapping $\tilde{x} \to \Lambda_{\tilde{x}}$ is bijective. Introduce the following norm in $\tilde{X}$:

$$\| \tilde{x} \|_{(2)} = \|\Lambda_{\tilde{x}}\|_{(1)} \ , \ \forall \tilde{x} \in \tilde{X}.$$

With such a norm, $\tilde{X}$ is a Banach space. From (3.1) it follows directly that $X = L[x_0] \dot{+} \tilde{X}$. Introduce another norm in X:

$$\|x\|_{(3)} = \| \alpha(x) x_0 \| + \|\tilde{x}\|_{(2)} \ , \ \forall x \in X,$$

where $x = \alpha(x) x_0 + \tilde{x}$, $\tilde{x} \in \tilde{X}$. We have

$$\|\tilde{x}\| = \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \le \sup_m \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \left\| \vec{\lambda} \right\|_{\mathscr{K}} \ , \ \forall \vec{\lambda} \equiv \{\lambda_n\}_{n \in \mathbb{N}} \in \Lambda_{\tilde{x}}.$$

It follows

$$\|\tilde{x}\| \le \inf_{\vec{\lambda} \in \Lambda_{\tilde{x}}} \left\| \vec{\lambda} \right\|_{\mathscr{K}} = \|\Lambda_{\tilde{x}}\|_{(1)} = \|\tilde{x}\|_{(2)} .$$

Consequently, $\|x\| \le \|x\|_{(3)}$, $\forall x \in X$. It is clear that X stays complete with regard to the norm $\|\cdot\|_{(3)}$. As a result, we get that the norms $\|\cdot\|$ and $\|\cdot\|_{(3)}$ are equivalent, i.e.

$$\exists c > 0 : \|\cdot\|_{(3)} \le c \|\cdot\| \Rightarrow |\vartheta(x)| \le \frac{c}{\|x_0\|} \|x\| \ , \ \forall x \in X \Rightarrow \vartheta \in X^* \Rightarrow \{x_n\}_{n \in \mathbb{N}} ,$$

is not complete in X. This contradiction proves the theorem. □

This theorem has the following corollary.

Corollary 3.1. *Let X a be B-space and suppose that the arbitrary element in this space can be expanded with respect to the system $\{x_n\}_{n \in \mathbb{N}} \subset X$. If the system $\{x_n\}_{n \in \mathbb{N} \setminus F}$ is complete in X, where $F \subset \mathbb{N} : card\, F < +\infty$, then the arbitrary element in X can be expanded also with respect to the system $\{x_n\}_{n \in \mathbb{N} \setminus F}$.*

Note that if $card\, F = +\infty$, then Corollary 1, in general, is not true. In fact, let X be a B-space with the basis $\{x_n\}_{n \in \mathbb{N}} \subset X$, and let $\{y_n\}_{n \in \mathbb{N}} \subset X$ be some complete and minimal system in X, which doesn't form a basis for X. Consider the system $\{z_n\}_{n \in \mathbb{N}}$, where

$$z_n = \begin{cases} x_k \ , \ if \ \ n = 2k - 1, \\ y_k , \ if \ \ n = 2k. \end{cases}$$

It is absolutely clear that the arbitrary element $x \in X$ can be expanded with respect to the system $\{z_n\}_{n \in \mathbb{N}}$. Assume $F = \{2k - 1\}_{k \in \mathbb{N}}$. It is clear that the system $\{z_n\}_{n \in \mathbb{N} \setminus F}$ is complete in X, but not every $x \in X$ can be expanded with regard to it. While if $F_0 \subset F : card\, F_0 < +\infty$, then $\forall x \in X$ can be expanded with regard to $\{z_n\}_{n \in \mathbb{N} \setminus F_0}$. In other words, the arbitrary element preserves its decomposability if the finite number of elements in an arbitrary basis is replaced with a complete and minimal system.

3.2. On t-decomposition. Let $X; Y$ be H-spaces and $Z = X \otimes Y$ be their tensor product. Let $\vec{y} \subset Y$ be some system. We will say that the element $z \in Z$ can be *t-decomposed* with respect to the system $\vec{y}$, if $\exists \vec{x} \subset X : z = \sum_{k=1}^{\infty} x_k y_k$ in Z.

Let us state the main result of this subsection.

Theorem 3.2. *Suppose that the arbitrary element in Z can be t-decomposed with respect to the system $\{y_n\}_{n \in \mathbb{Z}_+} \subset Y$ and let $\vec{y}$ be t-complete in Z. Then the arbitrary element in Z can be t-decomposed also with respect to the system $\vec{y}$.*

Proof. It suffices to consider the case $y_n \neq 0\,,\ \forall n \in \mathbb{Z}_+$. If y_0 can be expanded with regard to $\vec{y}$, then the assertion of theorem is obviously true. So let's suppose thaty_0 doesn't expand with$\vec{y}$. Then in the decomposition of arbitrary $z \in Z$:

$$z = x_0 y_0 + \sum_{n=1}^{\infty} x_n y_n,$$

the coefficient x_0 is determined uniquely. In fact, suppose we have another expansion

$$z = x'_0 y_0 + \sum_{n=1}^{\infty} x'_n y_n,$$

where $x_0 \neq x'_0$. We have

$$\left(x_0 - x'_0\right) y_0 = \sum_{n=1}^{\infty} \left(x'_n - x_n\right) y_n.$$

Hence we obtain

$$\left\langle x_0 - x'_0; \left(x_0 - x'_0\right) y_0 \right\rangle_Y = \left\| x_0 - x'_0 \right\|_X^2 y_0 =$$

$$\left\langle x_0 - x'_0; \sum_{n=1}^{\infty} \left(x'_n - x_n\right) y_n \right\rangle_Y = \sum_{n=1}^{\infty} \left\langle x_0 - x'_0; \left(x'_n - x_n\right) y_n \right\rangle_Y =$$

$$= \sum_{n=1}^{\infty} \left(x_0 - x'_0; x'_n - x_n\right)_X y_n.$$

As $\|x_0 - x'_0\|_X \neq 0$, we obtain that y_0 can be expanded with regard to $\vec{y}$, which is a contradiction. Thus, $x_0 = T(z)$, where $T : Z \to X$ is a linear operator. It is not difficult to see that the following relation holds

$$T(x y_0) = x\ ,\ \ T(x y_n) = 0\,,\ \ \forall x \in X\,,\ \forall n \in \mathbb{N}. \tag{3.2}$$

Consequently

$$z = (Tz)\, y_0 + \sum_{n=1}^{\infty} x_n y_n\ ,\ \forall z \in Z. \tag{3.3}$$

We denote the *t-space* of coefficients of the system $\vec{y}$ by $\Lambda^{(t)}$ and equip it with the norm

$$\|\vec{x}\|_{\Lambda^{(t)}} = \sup_m \left\| \sum_{n=1}^{\infty} x_n y_n \right\|_Z\ ,\ \forall \vec{x} \in \Lambda^{(t)}.$$

As proved earlier, $\left(\Lambda^{(t)}; \|\cdot\|_{\Lambda^{(t)}}\right)$ is a B-space. Assume

$$\Lambda_0^{(t)} \equiv \left\{ \vec{x} \in \Lambda^{(t)} \; : \; \sum_{n=1}^{\infty} x_n y_n = 0 \right\}.$$

$\Lambda_0^{(t)}$ is a linear subspace of $\Lambda^{(t)}$. Let us show that it is closed. Let $\left\{\vec{x}^{(m)}\right\}_{m\in\mathbb{N}} \subset \Lambda_0^{(t)}$ be some fundamental sequence with $\vec{x}^{(m)} \equiv \left\{x_n^{(m)}\right\}_{n\in\mathbb{N}} \in \Lambda_0^{(t)}$, $\forall m \in \mathbb{N}$. Since $\Lambda^{(t)}$ is complete, $\exists \vec{x} \in \Lambda^{(t)} : \vec{x}^{(m)} \to \vec{x}$, $m \to \infty$, in $\Lambda^{(t)}$. Consequently, the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent in Z. We have

$$\left\| \sum_{n=1}^{\infty} x_n y_n \right\|_Z = \left\| \sum_{n=1}^{\infty} \left(x_n - x_n^{(m)} \right) y_n \right\|_Z \le$$

$$\le \sup_l \left\| \sum_{n=1}^{l} \left(x_n - x_n^{(m)} \right) y_n \right\|_Z = \left\| \vec{x} - \vec{x}^{(m)} \right\|_{\Lambda^{(t)}} \to 0\,,\; m \to \infty.$$

Consequently, $\sum_{n=1}^{\infty} x_n y_n = 0 \Rightarrow \vec{x} \in \Lambda_0^{(t)} \Rightarrow \Lambda_0^{(t)}$ is closed. Consider the factor space $\Lambda^{(t)} \big/ \Lambda_0^{(t)}$ with the norm

$$\|\Lambda\|_{(1)} = \inf_{\vec{x}\in\Lambda} \|\vec{x}\|_{\Lambda^{(t)}}\,, \;\; \forall \Lambda \in \Lambda^{(t)} \big/ \Lambda_0^{(t)}.$$

Assume

$$\tilde{Z} \equiv \left\{ \tilde{z} \in Z : \exists \vec{x} \in \Lambda^{(t)} \Rightarrow \tilde{z} = \sum_{n=1}^{\infty} x_n y_n \right\}.$$

It is clear that $\tilde{Z}$ is a linear space. Let $\tilde{z} \in \tilde{Z}$ and $\tilde{z} = \sum_{n=1}^{\infty} x_n y_n$. Denote by $\Lambda_{\tilde{z}}^{(t)}$ the factor-class containing $\tilde{z}$. It is not difficult to see that the mapping $\tilde{z} \to \Lambda_{\tilde{z}}^{(t)}$ is bijective. Introduce the norm in $\tilde{Z}$ as follows

$$\|\tilde{z}\|_{\tilde{Z}} = \left\| \Lambda_{\tilde{z}}^{(t)} \right\|_{(1)}, \quad \forall \tilde{z} \in Z.$$

With this norm, $\tilde{Z}$ is a B-space. From the representation (3.3) it follows directly

$$Z = L_t\,[y_0] \dot{+} \tilde{Z}.$$

Introduce another norm in Z:

$$\|\, z\,\|_{(2)} = \|(Tz)\, y_0\|_Z + \|\tilde{z}\|_{\tilde{Z}}\;, \forall z \in Z,$$

where $z = (Tz)\, y_0 + \tilde{z}$, $\tilde{z} \in \tilde{Z}$. We have

$$\|\tilde{z}\|_Z = \left\| \sum_{n=1}^{\infty} x_n y_n \right\|_Z \le \sup_m \left\| \sum_{n=1}^{m} x_n y_n \right\|_Z = \|\vec{x}\|_{\Lambda^{(t)}}\,, \;\; \forall \vec{x} \in \Lambda_{\tilde{z}}^{(t)}.$$

Hence we obtain

$$\|\tilde{z}\|_Z \le \inf_{\vec{x}\in\Lambda_{\tilde{z}}^{(t)}} \|\vec{x}\|_{\Lambda^{(t)}} = \|\Lambda_{\tilde{z}}\|_{(1)} = \|\tilde{z}\|_{\tilde{Z}}\,.$$

Consequently, $\|z\|_Z \le \|z\|_{(2)}$, $\forall z \in Z$. It is clear that Z stays complete with respect to the norm $\|\cdot\|_{(2)}$. As a result, we obtain that the norms $\|\cdot\|_Z$ and $\|\cdot\|_{(2)}$ are equivalent, i.e. $\exists M > 0$:

$$\|\cdot\|_{(2)} \le M\,\|\cdot\|_Z \Rightarrow \|(Tz)\,y_0\|_Z \le M\,\|z\|_Z \ ,\ \forall z \in Z.$$

As $\|(Tz)\,y_0\|_Z = \|Tz\|_X\,\|y_0\|_Y$, we obtain that

$$\|Tz\|_X \le \frac{M}{\|y_0\|_Y}\,\|z\|_Z\,, \forall z \in Z.$$

As a result, $T \in L\,(Z;X)$. It follows that $\vec{y}$ cannot be *t-complete* in Z. Since otherwise the second of relations (3.2) would yield $Tz = 0$, $\forall z \in Z$, which would contradict to the relation $T\,(xy_0) = x$, $\forall x \in X$. □

In particular, we obtain

Corollary 3.2. *Suppose that the arbitrary element in Z can be t-decomposed with respect to the system $\{y_n\}_{n\in\mathbb{Z}_+} \subset Y$ and let $\{y_n\}_{n\in\mathbb{Z}_+\setminus F}$ be t-complete in Z. If the system $\{y_n\}_{n\in\mathbb{Z}_+\setminus F}$ is t-complete in Z and $card\,F < +\infty$, then the arbitrary element in Z can be t-decomposed also with respect to the system $\{y_n\}_{n\in\mathbb{Z}_+\setminus F}$.*

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Bilal T. Bilalov
Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, AZ1141, Baku, Azerbaijan.
E-mail address: b_bilalov@mail.ru

Chingiz M. Hashimov
Ganja State University, Ganja, Azerbaijan.
E-mail address: chingiz.heshimov.88@mail.ru

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