Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Number 2, 2014, Pages 115–124

THE GENERAL SOLUTION OF THE HOMOGENEOUS RIEMANN PROBLEM IN THE WEIGHTED SMIRNOV CLASSES

SABINA R. SADIGOVA

Abstract. In this work Riemann problem of theory of analytic functions in weighted Smirnov classes is considered. Under certain conditions on the coefficients of this problem its Noetherian is proved. In the case of the solvability the general solution of the homogeneous Riemann problem is constructed. Sufficient condition on the weight function for the solvability of the corresponding problem is obtained.

1. Introduction

Faber polynomials in complex domains are well known in the approximation theory. They are good tools while investigating many problems in the approximation theory and in theory of conformal mappings. More information about these problems can be found in [8, 11, 12, 13, 14, 15, 16, 23, 24] (and references therein). Faber polynomials are natural generalizations of classical exponential systems $\{z^n\}_{n\in\mathbb{Z}}$ ($\mathbb{Z}$ are integers) for the case of arbitrary domain with boundary Γ considered inside (or outside) the unit circle on the complex plane. On the unit circle, the system $\{z^n\}_{n\in\mathbb{Z}}$ generates a system of exponents $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ that plays an important role in solving many problems for partial differential equations by Fourier method. Trivial example of this is the Dirichlet problem for the analytic function u:

$$\left.\begin{array}{l}\frac{\partial u}{\partial \bar{z}}=0\ ,\ \ z\in\omega,\\ u\left(z\right)=f\left(z\right)\ ,\ z\in\partial\omega,\end{array}\right\} \qquad (1.1)$$

where $\omega\equiv\{z:\ |z|<1\}$. If we assume that f belongs to the weighted space $L_{p,\rho}\left(\partial\omega\right)$, then the solution u is sought in the Hardy class $H^+_{p,\rho}$, i.e.

$$u\left(z\right)=\sum_{n=0}^{\infty}u_n z^n\ ,\ \ z\in\omega.$$

Since the system $\left\{z^n/_{\partial\omega}\right\}_{n\in\mathbb{Z}}$ forms a basis for $L_{p,\rho}\left(\partial\omega\right)$, then

$$f\left(z\right)=\sum_{n=-\infty}^{+\infty}f_n z^n\ ,\ \ z\in\partial\omega.$$

2010 *Mathematics Subject Classification.* 30E25, 30D55, 32A55.
Key words and phrases. weighted Smirnov classes, Riemann problem.

Thus, if $f_n = 0$, $\forall n < 0$, then the Dirichlet problem (1.1) for the analytic function $u \in H^+_{p,\rho}$ is solvable and $u(z) = \sum_{n=0}^{\infty} f_n z^n$. To solve the similar problem in an arbitrary simply-connected domain D with boundary Γ

$$\left.\begin{array}{l} \frac{\partial u}{\partial \bar{z}} = 0\,, \quad z \in D, \\ u/_{\Gamma} = f\,, \end{array}\right\}$$

by the same method ($f \in L_{p,\rho}(\Gamma)$ is some given function) we need to study basis properties of the system of Faber polynomials in Lebesgue or Sobolev spaces of functions on the curve Γ. In this case, the weighted Smirnov spaces of analytic functions quite naturally play the role of the weighted Hardy space.

For studying the basicity of the system of generalized Faber polynomials in weighted Lebesgue spaces, we intend to apply the method of the theory of the Riemann boundary value problems for analytic functions. Therefore, first, it should to study the solvability of these problems in weighted Smirnov spaces. In this paper we consider the homogeneous Riemann problem in the weighted Smirnov classes. First, we define these weighted classes. Under certain conditions on the weight function we prove that these spaces are Banach spaces. A sufficient condition for the solvability of the Riemann problem is obtained and the general solution of the homogeneous and non-homogeneous problem are constructed. It should be noted that this method previously used by the authors of [1, 2, 3, 4, 21, 22] for the study of basicity of the perturbed systems of exponents, sines and cosines. In [5] the basicity of systems of generalized polynomials is proved by the same method in Lebesgue or Sobolev spaces and in $L_p(\Gamma)$.

It should be noted that similar problems previously were considered in [18, 19, 20].

2. Auxiliary facts and concepts

In what follows we'll need some concepts and facts. Denote by $O_r(z)$ a circle with radius r and center in z_0 in the complex plane, i.e. $O_r(z_0) \equiv \{z \in \mathbb{C} : |z - z_0| < r\}$ ($\mathbb{C}$ is the complex plane). $|M|$ will denote the Lebesgue measure of (linear) sets $M \subset \Gamma$, where $\Gamma \subset \mathbb{C}$ is some rectifiable curve.

Definition 2.1. The Jordan rectifiable curve Γ is said to be Carleson or regular if

$$\sup_{z \in \Gamma} \left| \Gamma \bigcap O_r(z) \right| \leq cr\,, \quad \forall r > 0\,,$$

where c is a constant independent of r.

We refer the reader to [7, 9, 10, 17] for further information about these and related results.

Let Γ be a rectifiable Jordan curve and ω be a weight function on Γ, i.e. $\omega(z) > 0$ a.e. $z \in \Gamma$.

Definition 2.2. We say that a weight ω belongs to the Muckenhoupt class $A_p(\Gamma)$ ($p > 1$) on the curve Γ, if

$$\sup_{z \in \Gamma} \sup_{r > 0} \left(\frac{1}{r} \int_{\Gamma \bigcap O_r(z)} \omega(\xi)\ |d\xi| \right) \left(\frac{1}{r} \int_{\Gamma \bigcap O_r(z)} |\omega(\xi)|^{-\frac{1}{p-1}}\ |d\xi| \right)^{p-1} < +\infty.$$

As usual, denote by $L_p(\Gamma;\omega)$ a weighted Lebesgue space of functions with the norm $\|\cdot\|_{p,\omega}$:

$$\|f\|_{L_p(\Gamma;\omega)} = \left(\int_\Gamma |f(\xi)|^p \omega(\xi)|d\xi|\right)^{\frac{1}{p}}.$$

Consider the Cauchy singular operator S_Γ:

$$S_\Gamma(f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi-\tau}d\xi\,,\ \tau\in\Gamma.$$

The key role in obtaining our main results is played by the following theorem of G. David [7].

Theorem D. *The operator S_Γ is bounded in $L_p(\Gamma)$, $1<p<+\infty$, if and only if Γ is a regular curve. Moreover, if Γ is a regular curve, then the singular operator S_Γ is bounded in $L_p(\Gamma;\omega)$, $1<p<+\infty$, if and only if $\omega\in A_p(\Gamma)$.*

More detailed about these and related results can be found in [10].

3. Main assumptions

Let $A(\xi)\equiv|A(\xi)|e^{i\alpha(\xi)}$, $B(\xi)\equiv|B(\xi)|e^{i\beta(\xi)}$ be complex-valued functions given on the curve Γ. We'll assume that they satisfy the following basic conditions:

i) $|A|^{\pm1}, |B|^{\pm1}\in L_\infty(\Gamma)$;

ii) $\alpha(\xi)$, $\beta(\xi)$ *are piecewise-continuous on* Γ *and let* $\left\{\xi_k\,,\, k=\overline{1,r}\right\}\subset\Gamma$ *be discontinuity points of the function* $\theta(\xi)\equiv\beta(\xi)-\alpha(\xi)$.

For the curve Γ we require the following conditions be fulfilled:

iii) Γ *is any Lyapunov or Radon curve (i.e. it is a curve of bounded rotation without cusps). We'll assume that the direction along* Γ *is positive, i.e. while moving in this direction, the domain* D *remains in the left side. Let* $a\in\Gamma$ *be a start point (also an end point) of the curve* Γ. *We'll assume that* $\xi\in\Gamma$ *follows the point* $\tau\in\Gamma$, *i.e.* $\tau\prec\xi$, *if* ξ *follows* τ *while moving in positive direction along* $\Gamma\backslash a$, *where* $a\in\Gamma$ *is the junction of two points* $a^+=a^-$, a^+ *is the start point and* a^- *is the end point of the curve* Γ.

By $\mathscr{LR}$ *we denote a class of curves satisfying condition iii).*

So, without loss of generality, we'll assume that $a^+\prec\xi_1\prec...\prec\xi_r\prec b=a^-$. By $g(\xi_0\pm0)$ we denote one-sided limits $\lim\limits_{\substack{\xi\to\xi_0\pm0\\ \xi\in\Gamma}} g(\xi)$ of the function $g(\xi)$ at the point $\xi_0\in\Gamma$ generated by the positive direction of Γ, respectively. The jumps $\theta(\xi)$ at the points ξ_k, $k=\overline{1,r}$, are denoted by h_k:$h_k=\theta(\xi_k+0)-\theta(\xi_k-0)$, $k=\overline{1,r}$. Let the following condition hold with respect to the jumps:

iv) $\left\{h_k-\frac{2\pi}{p}\,:\,k=\overline{0,r}\right\}\bigcap\mathbb{Z}=\emptyset$, *where* $h_0=\theta(a+0)-\theta(a-0)$, $p\in(1,+\infty)$ *is some number.*

Let $D^+\subset\mathbb{C}$ be a bounded domain with the boundary $\Gamma=\partial D^+$, with respect to which the condition i) holds. By $E_p(D^+)$, $1<p<\infty$, we denote Smirnov's Banach space of analytic functions in D^+ with the norm $\|\cdot\|_{E_p(D^+)}$:

$$\|f\|_{E_p(D^+)}=:\left\|f^+\right\|_{L_p(\Gamma)},\ \forall f\in E_p\left(D^+\right), \tag{3.1}$$

where $f^{+}=f/_{\Gamma}$ are non-tangential boundary values of the function f on Γ. Similarly we define the Smirnov class $E_p(D^-)$ in an unbounded domain D^- with the boundary $\Gamma=\partial D^-$ and with the norm

$$\|f\|_{E_p(D^-)}=:\left\|f^-\right\|_{L_p(\Gamma)},\ \forall f\in E_p\left(D^-\right),$$

where $f^-=f/_{\Gamma}$ are non-tangential boundary values of f on Γ.

With respect to the norm (3.1) the weighted Smirnov class is defined. Let $\rho\in L_1(\Gamma)$ be some weight function. Let us define the weighted Smirnov class $E_{p,\rho}(D^+)$:

$$E_{p,\rho}\left(D^+\right)\equiv\left\{f\in E_1\left(D^+\right):\ \left\|f^+\right\|_{L_{p,\rho}(\Gamma)}<+\infty\right\},$$

and let

$$\|f\|_{E_{p,\rho}(D^+)}=\left\|f^+\right\|_{L_{p,\rho}(\Gamma)}. \tag{3.2}$$

The weighted Smirnov classes are similarly defined in an unbounded domain. Let $D^-\subset\mathbb{C}$ be an unbounded domain containing the point at infinity (∞). Denote by ${}_mE_1(D^-)$ the class of functions from $E_1(D^-)$, which are analytic in D^- with their orders $k\leq m$ at infinity, i.e. the function $f\in E_1(D^-)$ in the neighbourhood of an infinitely remote point $z=\infty$ has the Laurent expansion $f(z)=\sum_{k=-\infty}^{m}a_kz^k$, where m is an integer.

If there exits a weight function $\rho\in L_1(\Gamma)$, then the weighted class ${}_mE_{p,\rho}(D^-)$ is defined as

$${}_mE_{p,\rho}\left(D^-\right)\equiv\left\{f\in_m E_1\left(D^-\right):\ \left\|f^-\right\|_{L_{p,\rho}(\Gamma)}\right\},$$

and

$$\|f\|_{{}_mE_{p,\rho}(D^-)}=\left\|f^-\right\|_{L_{p,\rho}(\Gamma)},$$

where f^- are non-tangential boundary values of the function f on Γ.

The following lemma is true.

Lemma 3.1. *Let $\rho^{-\frac{q}{p}}\in L_1(\Gamma)$. Then the class $E_{p,\rho}(D^+)$ is a Banach space with respect to the norm* (3.2).

Similarly we prove the following

Lemma 3.2. *Let $\rho^{-\frac{q}{p}}\in L_1(\Gamma)$. Then the weighted class ${}_mE_{p,\rho}(D^-)$ is a Banach space with respect to the norm $\|\cdot\|_{{}_mE_{p,\rho}(D^-)}$.*

4. Riemann's homogeneous problem in weighted Smirnov classes

Consider the following homogeneous Riemann problem in the weighted classes $E_{p,\rho}(D^+)\times_m E_{p,\rho}(D^-)$:

$$A(\xi)F^+(\xi)+B(\xi)F^-(\xi)=0,\ \ a.e.\ \ \xi\in\Gamma. \tag{4.1}$$

By the solution of problem (4.1) we mean the pair of analytic functions $(F^+;F^-)\in E_{p,\rho}(D^+)\times_m E_{p,\rho}(D^-)$, whose non-tangential boundary values $F^{\pm}(\xi)$ satisfy relation (4.1) a.e. on Γ. In the case without weight this is a well-studied problem, and its theory has been elucidated in I.I.Danilyuk's monograph [6]. In obtaining our main results we will use the following lemma from [6].

Lemma 4.1. *Let D^+ be an arbitrary domain bounded by a rectifiable curve $\Gamma = \partial D^+$. Homogeneous problem*

$$F^+(\xi) - F^-(\xi) = 0, \ a.e. \ \xi \in \Gamma,$$

has only trivial solutions in the form of polynomials in the classes $E_1(D^+) \times {}_mE_1(D^-)$, whose degree does not exceed m, and when $m < 0$, has only the trivial solution.

We'll solve the homogeneous problem (4.1) on the scheme proposed in the I.I.Danilyuk's monograph [6]. Let S be a length of the curve Γ and $z = z(s)$, $0 \leq s \leq S$, be a parametric representation of Γ with respect to the length of the arch s. Problem (4.1) can be written as

$$F^+[z(s)] - D(s) F^-[z(s)] = 0, \ a.e. \ s \in [0, S], \tag{4.2}$$

where $D(s) = -\frac{B[z(s)]}{A[z(s)]}$. Put $\Omega(s) \equiv argD(s)$, $0 \leq s \leq S$, and let $h_k = z(s_k + 0) - z(s_k - 0)$, $k = \overline{1, r}\,; h_0 = \Omega(+0) - \Omega(S - 0)$. Consider the piecewise holomorphic function

$$Z_{(1)}(z) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \ln|D(s)| \frac{dz(s)}{z(s) - z}\right\},$$

$$\tilde{Z}(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \Omega(s) \frac{dz(s)}{z(s) - z}\right\}.$$

Denote by $Z : Z(z) \equiv Z_{(1)}(z)\tilde{Z}(z)$ the product of these functions. As follows from the Sokhotskii-Plemelj formula, this function satisfies the homogeneous equation (4.2) a.e. on Γ , i.e.

$$Z^+[z(s)] - D(s) Z^-[z(s)] = 0, \ a.e. \ s \in [0, S].$$

Regarding the first multiplier $Z_{(1)}(z)$ we have the following

Lemma 4.2. [6] *Let with respect to the coefficients $A(\xi)$, $B(\xi)$ and the curve Γ satisfy the conditions i)-iv). Then the functions $Z_{(1)}(z)$; $\left[Z_{(1)}(z)\right]^{-1}$ are bounded in each of the domains $D^\pm$.*

To conduct further research, we'll represent the function $\Omega(s)$ in the form

$$\Omega(s) = \Omega_0(s) + \Omega_1(s) \ , 0 \leq s \leq S,$$

where $\Omega_0(s)$ is a continuous part, and $\Omega_1(s)$ is a jump function, which is determined by the expression

$$\Omega_1(0) = 0,$$

$$\Omega_1(s) = [\Omega(+0) - \Omega(0)] + \sum_{0 < s_k < s} h_k + [\Omega(s) - \Omega(s - 0)] \ , \ 0 < s < S.$$

Denote

$$h_0^{(0)} = \Omega_0(S) - \Omega_0(0) \ , \ h_0^{(1)} = \Omega_1(+0) - \Omega_1(s - 0).$$

Let

$$Z_{(2)}(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \Omega_0(s) \frac{dz(s)}{z(s) - z}\right\},$$

and

$$Z_{(3)}(z) = \exp\left\{\frac{1}{2\pi}\int_\Gamma \Omega_1(s)\,\frac{dz(s)}{z(s)-z}\right\}.$$

In [6] it is shown that the following inclusion holds.

$$\tilde{Z}^{\pm}_{(2)}(s) \equiv |z(s)-z(0)|^{\pm\frac{h_0^{(0)}}{2\pi}}\left|Z^{\pm}_{(2)}[z(s)]\right|^{\pm 1} \in L_q(\Gamma)\ ,\ \forall q\in(0,+\infty)\,. \qquad (4.3)$$

Modulus of the boundary values of the function $Z_{(3)}(z)$ has the representation [6]:

$$\left|Z^{+}_{(3)}[z(\sigma)]\right| \equiv |z(0)-z(\sigma)|^{-\frac{h_0^{(1)}}{2\pi}} \prod_{0<s_k<S} |z(s_k)-z(\sigma)|^{-\frac{h_k}{2\pi}}\,. \qquad (4.4)$$

The following lemma is true.

Lemma 4.3. [6] *Let the curve* Γ *satisfy the condition iii) and* $\Omega_1(s)-$ *be an arbitrary jump function with jumps* $h_0^{(1)} = \Omega_1(+0)-\Omega_1(S-0)$ *at point* $z(0)$*. Then the modulus of the boundary values of the function* $Z_{(3)}(z)$ *is representable by the formula* (4.4) *a.e.* $\sigma\in[0,S]$*.*

It is clear that

$$Z^{\pm}[z(s)] = Z^{\pm}_{(1)}[z(s)]\,Z^{\pm}_{(2)}[z(s)]\,Z^{\pm}_{(3)}[z(s)]\,.$$

Future $Z(z)$ *will be called a canonical solution of the homogeneous problem* (4.2)*.*
Assume

$$\Phi(z) \equiv \frac{F(z)}{Z(z)}. \qquad (4.5)$$

We have

$$\Phi^{+}(\tau) = \Phi^{-}(\tau)\,,\ a.e.\ \tau\in\Gamma.$$

Let us show that the function Φ satisfy all the conditions of Lemma 4.1. So, $Z(z)$ has no zeros and no poles at $z\notin\Gamma$. Therefore, the functions $\Phi(z)$ and $F(z)$ have the same order at infinity. By definition of solution, we have $F\in E_1(D^{+})$. From the results of I.I.Danilyuk's monograph [6] (see e.g. Lemma 16.5, page 148) it follows that if the conditions i)-iii) are fulfilled, then the function $Z(z)$ belongs to classes $E_\delta(D^{\pm})$ for sufficiently small $\delta>0$. Then from the relation (4.5) we obtain that the function $\Phi(z)$ belongs to class $E_\mu(D^{\pm})$ for sufficiently small $\mu>0$. Thus, as it follows from Smirnov's theorem [6], if $\Phi^{+}\in L_1(\Gamma)$, then it is clear that $\Phi\in E_1(D^{+})$. As, $\Phi^{+}(\tau)=\Phi^{-}(\tau)$, a.e. $\tau\in\Gamma$, then it is sufficient to show that $\Phi^{-}(\tau)$ belongs to space $L_1(\Gamma)$. We have

$$\left|\Phi^{-}(\tau)\right| = \left|F^{-}(\tau)\right|\left|Z^{-}(\tau)\right|^{-1},\ a.e.\ \tau\in\Gamma.$$

By definition of solution, we have $|F^{-}|\in L_{p,\rho}(\Gamma)$. Therefore, if $|Z^{-}|^{-1}\in L_{q;\tilde{\rho}}(\Gamma)$, then $\Phi^{-}\in L_1(\Gamma)$, where $\tilde{\rho}=\rho^{-\frac{q}{p}}$. This follows directly from the Hölder's inequality

$$\int_\Gamma \left|\Phi^{-}(\tau)\right||d\tau| \le \left(\int_\Gamma \left|F^{-}(\tau)\right|^{p}\rho(\tau)\,|d\tau|\right)^{\frac{1}{p}}\left(\int_\Gamma \left|Z^{-}(\tau)\right|^{-q}\rho^{-\frac{q}{p}}(\tau)\,|d\tau|\right)^{\frac{1}{q}}.$$

Then by Smirnov's theorem [6] we obtain that the function $\Phi(z)$ belongs to $E_1(D^{\pm})$. Since, $\Phi^+(\tau) = \Phi^-(\tau)$, a.e. $\tau \in \Gamma$, then from the uniqueness theorem (i.e. from Lemma 4.1) follows that $\Phi(z)$ is a polynomial of degree $k \leq m$, i.e. $\Phi(z) \equiv P_m(z)$, where $P_m(z)$ is a polynomial of degree $k \leq m$. As a result, we obtain the following representation

$$F(z) \equiv Z(z) P_m(z) . \tag{4.6}$$

In the subsequent need to find out the belonging of the functions $F(z)$ to the desired class. Suppose that the inequality

$$\frac{h_k}{2\pi} < 1 , \ k = \overline{0, r}, \tag{4.7}$$

is fulfilled. It is clear that $\exists p_0 \in (1, +\infty)$:

$$\frac{h_k}{2\pi} p_0 < 1 , \ k = \overline{0, r}. \tag{4.8}$$

Let

$$\sigma(s) \equiv |z(0) - z(s)|^{-\frac{h_0}{2\pi}} \prod_{0<s_k<S} |z(s_k) - z(s)|^{-\frac{h_k}{2\pi}} ,$$

where $h_0 = h_0^{(1)} - h_0^{(0)}$. Then the modulus of the boundary values of $|Z^+(z(s))|$ is representable by the formula

$$\left|Z^+(z(s))\right| \equiv \left|Z^+_{(1)}(z(s))\right| \left|\tilde{Z}^+_{(2)}(z(s))\right| \sigma(s) .$$

Paying attention to Lemma 4.2, we have

$$\left|Z^+(z(s))\right| \sim \sigma(s) \left|\tilde{Z}^+_{(2)}(z(s))\right| , \ s \in [0, S] . \tag{4.9}$$

Applying Hölder's inequality, we get

$$\int_\Gamma \left|Z^+(z(s))\right| |dz(s)| \leq \left(\int_\Gamma |\sigma(s)|^{p_0} |dz(s)|\right)^{\frac{1}{p_0}} \left(\int_\Gamma \left|\tilde{Z}^+_{(2)}(s)\right|^{p_0'} |dz(s)|\right)^{\frac{1}{p_0'}} , \tag{4.10}$$

where $\frac{1}{p_0} + \frac{1}{p_0'} = 1$. It is known that it holds (see e.g. [6])

$$\left|\frac{dz}{ds}\right| = 1, \ a.e. \ s \in [0, S] ,$$

moreover, $\exists \delta_0; k_0 > 0$:

$$k_0 |s - \sigma| \leq |z(s) - z(\sigma)| \leq |s - \sigma| , \ \forall s, \sigma : |s - \sigma| \leq \delta_0.$$

Taking into account these relations, and paying attention to the inclusion (4.3) and the inequality (4.8), from (4.10) we have $Z^+ \in L_1(\Gamma)$, and as a result, $F^+ \in L_1(\Gamma)$. Then from Smirnov's theorem it follows that the function $F(z)$ belongs to Smirnov class $E_1(D^+)$. It is absolutely clear that the boundary values $F^{\pm}$ of function F on Γ satisfy the relation (4.2). From the condition i) follows $|D|^{\pm 1} \in L_\infty$, therefore it is clear that

$$\left|Z^+(z(s))\right| \sim \left|Z^-(z(s))\right| , \ s \in (0, S) .$$

Beginning from this relation it is easy to enclose that $F^- \in L_1(\Gamma)$, and as a result, $F \in_m E_1(D^-)$. We'll find conditions under which the boundary values $F^{\pm}$

belong to the space $L_{p,\rho}(\Gamma)$. It is clear that if $Z^+ \in L_{p,\rho}(\Gamma)$, then $F^\pm \in L_{p,\rho}(\Gamma)$. Assume that $\exists p_1 \in (1,+\infty)$:

$$\int_0^S \sigma^{pp_1}(s)\,\rho^{p_1}(z(s))\,ds < +\infty. \tag{4.11}$$

Then taking into account the expression (4.9) we have

$$\int_0^S \left|Z^+(z(s))\right|^p \rho(z(s))\,ds \le$$

$$M\left(\int_0^S \sigma^{pp_1}(s)\,\rho^{p_1}(z(s))\,ds\right)^{\frac{1}{p_1}}\left(\int_0^S \left|\tilde{Z}^+_{(2)}(s)\right|^{q_1} ds\right)^{\frac{1}{q_1}},$$

where M is some constant and $\frac{1}{p_1}+\frac{1}{q_1}=1$. Paying attention to the relation (4.3) we obtain from this $Z^+ \in L_{p,\rho}(\Gamma)$. It remains to verify the fulfillment of the condition $|Z^-|^{-1} \in L_{p;\tilde{\rho}}(\Gamma)$. Similarly we establish that if $\exists p_2 \in (1,+\infty)$:

$$\int_0^S \sigma^{-qp_2}(s)\,\rho^{-\frac{q}{p}p_2}(z(s))\,ds < +\infty, \tag{4.12}$$

then again from the relation (4.3) follows immediately $|Z^-|^{-1} \in L_{p;\tilde{\rho}}$. Summing up the obtained results we arrive at the following conclusion.

Theorem 4.1. *Let the conditions i)-iii) be fulfilled with respect to the complex-valued functions $A(\xi)$, $B(\xi)$ and the curve Γ. Assume that with respect to jumps $\{h_k\}$ and the weight function $\rho(\xi)$ the conditions* (4.7), (4.11), *and* (4.12) *are fulfilled. Then the general solution of the homogenous problem* (4.1) *has a representation*

$$F(z) \equiv Z(z)\,P_m(z),$$

in classes $E_{p,\rho}(D^+)\times_m E_{p,\rho}(D^-)$, where $Z(z)$ is a canonical solution, and $P_m(z)$ is an arbitrary polynomial of order $k \le m$.

From this theorem follows immediately following

Corollary 4.1. *Suppose that all the conditions of Theorem 4.1 are fulfilled. Then under the condition $F(\infty)=0$ the problem* (4.1) *has only a trivial, i.e. zero solution in classes $E_{p,\rho}(D^+)\times_m E_{p,\rho}(D^-)$.*

Let us consider some special cases concerning the weight function ρ .

Example 4.1. *Let ρ be of the following form*

$$\rho(z(s)) = \prod_{k=1}^m |s-t_k|^{\alpha_k}, \tag{4.13}$$

where $\{t_k\}_1^m \subset (0,S)$ are different points, $\{\alpha_k\}_1^m \subset \mathbb{R}$ is some number. The union of the sets $\{s_k\}_0^r$ and $\{t_k\}_1^m$ denote by $\{\tau_k\}_1^l : \{\tau_k\}_1^l \equiv \{s_k\}_0^r \cup \{t_k\}_1^m$. Let $\chi_A(\cdot)$ be the characteristic function of the set A. Denote by $T_k : T_k \equiv \{\tau_k\}$, $k=\overline{1,l}$ the singleton $\{\tau_k\}$, $k=\overline{1,l}$. Assume

$$\beta_k = -\frac{p}{2\pi}\sum_{i=1}^r h_i \chi_{T_k}(s_i) + \sum_{i=1}^m \alpha_i \chi_{T_k}(t_i),\ k=\overline{1,l}. \tag{4.14}$$

Let us assume that the following inequality hold

$$-1<\beta_k<\frac{p}{q}, k=\overline{1,l}. \tag{4.15}$$

It is easy to show that when the inequalities (4.15) are fulfilled, the relations (4.11), (4.12) hold and as a result, the assertion of Theorem 4.1 is true, i.e. we have

Corollary 4.2. *Let the functions* $A(\xi)$, $B(\xi)$ *and the curve* Γ *satisfy the conditions i)-iii), and the weight function is of the form* (4.13)*. Assume that the inequalities* (4.15) *are fulfilled , where the quantities* β_k *are defined by the expressions* (4.14)*. Then the general solution of the problem* (4.1) *in classes* $E_{p,\rho}(D^+)\times_m E_{p,\rho}(D^-)$ *has the representation* (4.6)*.*

Example 4.2. *As a weight function* ρ *we again take* (4.13)*, but this time we assume that* $\{s_k\}_1^r\cap\{t_k\}_1^m=\emptyset$*. In this case the following corollary is true.*

Corollary 4.3. *Let all the conditions of Corollary 4.2 be fulfilled and* $\{s_k\}_1^r\cap\{t_k\}_1^m=\emptyset$*.*

If the inequalities

$$-\frac{1}{q}<\frac{h_k}{2\pi}<\frac{1}{p}, k=\overline{1,r};$$
$$-1<\alpha_i<\frac{q}{p}, i=\overline{1,m},$$

hold, then the general solution of the problem (4.1) *has a representation* (4.6) *in classes* $E_{p,\rho}(D^+)\times_m E_{p,\rho}(D^-)$*.*

Acknowledgements

The author would like to express her deepest gratitude to Professor Bilal T. Bilalov for his encouragement and valuable guidance throughout this research.

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan — Grant No EIF/GAM-2-2013-2(8)-25/02/1.

References

[1] B. T. Bilalov, Basicity of some systems of exponents, cosines and sines, *Diff. Uravn.*, **26** (1990), no. 1, 10–16.

[2] B. T. Bilalov, The basis properties of some systems of exponential functions, cosines, and sines, *Siberian Math. J.*, **45** (2004), no. 2, 214–221 (translated from *Sibirsk. Mat. Zh.*, **45** (2004), no. 2, 264–273).

[3] B. T. Bilalov, The basis properties of power systems in L_p, *Siberian Math. J.*, **47** (2006), no. 1, 18–27 (translated from *Sibirsk. Mat. Zh.*, **47** (2006), no. 1, 25–36).

[4] B. T. Bilalov, A system of exponential functions with shift and the Kostyuchenko problem, *Siberian Math. J.*, **50** (2009), no. 2, 223–230 (translated from *Sibirsk. Mat. Zh.*, **50** (2009), no. 2, 279–288).

[5] B. T. Bilalov and T. I. Najafov, On basicity of systems of generalized Faber polynomials, *Jaen J. Approx.*, **5** (2013), no. 1, 19–34.

[6] I. I. Danilyuk, *Nonregular boundary value problems in the plane*, Nauka, Moscow, 1975 (in Russian).

[7] G. David, Operateurs integraux singulars sur certains courbes du plan complexe, *Ann. Sci. Ecole Norm. Sup.,* **17** (1984), 157–189.

[8] E. M. Dynkin, Uniform approximation of functions in Jordan domains, Siberian Math. J., **18** (1977), 775–786.

[9] E. M. Dynkin, Methods of theory of singular integrals, *Itogi nauki i tekhniki., Ser. mat. anal.,* **15** (1987), 197–292.

[10] E. M. Dynkin, Methods of theory of singular integrals II. *Littlewood-Paley theory and its application, Itogi nauki i tekhniki,* Moscow, **42** (1989), 199–227.

[11] D. Gayer, *Lectures on the theory of approximation in a complex domain.* Moscow, Mir, 1986 (in Russian).

[12] D. M. Israfilov, Approximation by generalized Faber series in weighed Bergman spaces on finite domains with a quasiconformal boundary, *EAST Journal on Approximations,* **4** (1998), no. 1, 1–13.

[13] D. M. Israfilov, Approximation by p- Faber polynomials in the weighed Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials, *Const. Approx.,* **17** (2001), no. 3, 335–351.

[14] D. M. Israfilov, Faber series on weighted Bergman spaces, Complex Variables, *Theory and Applications,* **45** (2001), no. 2, 167–181.

[15] D. M. Israfilov, Approximation by p- Faber-Rational functions in weighted Lebesgue spaces, *Czechoslovak Math. Journal,* **54** (2004), no. 129, 751–765.

[16] D. M. Israfilov and N. P. Tozman, Approximation in Morrey-Smirnov classes, *Azerbaijan Journal of Mathematics,* **1** (2011), no. 1, 99–113.

[17] V. M. Kokilashvli, On approximation of analytic functions of the class E_p, *DAN SSSR,* **177** (1967), no. 2, 261–264.

[18] V. M. Kokilashvili, Boundary Value Problems of Analytic and Harmonic Functions in a Domain with Piecewise Smooth Boundary in the Frame of Variable Exponent Lebesgue Spaces, *Modern Aspects of the Theory of Partial Differential Equations Operator Theory: Advances and Application,* **216** (2011), 17–39.

[19] G. Manjavidze and N. Manjavidze, Boundary-value problems for analytic and generalized analytic functions, *Journal of Mathematical Sciences,* **160** (2009), no. 6, 745–821.

[20] Z. Meshveliani, The Riemann-Hilbert problem in weighted Smirnov classes of analytic functions, *Proc. A. Razmadze Math. Inst.,* **137** (2005), 65–86.

[21] E. I. Moiseev, On basicity of the systems of sines and cosines, *DAN SSSR,* **275** (1984), no. 4, 794–798.

[22] E. I. Moiseev, On basicity of a system of sines, *Diff. Uravn.,* **23** (1987), no. 1, 177–179.

[23] V. I. Smirnov and N. A. Lebedev, *Constructive Theory of Functions of a Complex Variable,* Nauka, Moscow-Leningrad, 1964 (in Russian).

[24] P. K. Suetin, *Series in Faber Polynomials,* Nauka, Moscow, 1984 (in Russian).

Sabina R. Sadigova

Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.

E-mail address: s_sadigova@mail.ru

Received: November 10, 2014; Accepted: December 12, 2014