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SOLVABILITY OF BOUNDARY VALUE PROBLEM FOR ELLIPTIC OPERATOR-DIFFERENTIAL EQUATIONS OF FOURTH ORDER WITH OPERATOR BOUNDARY CONDITIONS

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In memory of M. G. Gasymov on his 75th birthday

Abstract. Sufficient conditions for well-posed and unique solvability of a boundary value problem for elliptic operator-differential equations of fourth order with an unbounded operator in boundary conditions are obtained. Estimates for the norms of intermediate derivatives operators closely related to the solvability conditions are derived. Note that the solvability conditions are expressed only in terms of operator coefficients of the boundary value problem.

1. Introduction

Let H be a separable Hilbert space with a scalar product (x, y), $x, y \in H$, and A be a positive-definite self-adjoint operator in H ($A = A^* \geq cE$, $c > 0$, E is a unit operator). By H_γ ($\gamma \geq 0$) we will mean a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x,\ y \in D(A^\gamma)$. In case $\gamma = 0$ we assume that $H_0 = H$, $(x, y)_0 = (x, y)$, $x, y \in H$.

Denote by $L_2([a, b]; H)$, $-\infty \leq a < b \leq +\infty$, the Hilbert space of all vector-functions defined on $[a, b]$ with values in H and the norm

$$\|f\|_{L_2([a,b];H)} = \left(\int_a^b \|f(t)\|_H^2 \, dt \right)^{1/2}.$$

Following [8, Chapter 1], we introduce the Hilbert space

$$W_2^4([a, b]; H) = \left\{ u(t) : \, u^{(4)}(t) \in L_2([a, b]; H), \, A^4 u(t) \in L_2([a, b]; H) \right\}$$

equipped with the norm

$$\|u\|_{W_2^4([a,b];H)} = \left(\left\| u^{(4)} \right\|_{L_2([a,b];H)}^2 + \left\| A^4 u \right\|_{L_2([a,b];H)}^2 \right)^{1/2}.$$

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Throughout this paper, the derivatives $u^{(j)} \equiv \frac{d^j u}{dt^j}$ are understood in the sense of the theory of distributions in Hilbert space [8]. For $a = -\infty$, $b = +\infty$ we will assume that $L_2\left((-\infty, +\infty); H\right) \equiv L_2\left(\mathbb{R}; H\right)$, $W_2^4\left((-\infty, +\infty); H\right) \equiv W_2^4\left(\mathbb{R}; H\right)$, and for $a = 0$, $b = +\infty$ – $L_2\left([0, +\infty); H\right) \equiv L_2\left(\mathbb{R}_+; H\right)$, $W_2^4\left([0, +\infty); H\right) \equiv W_2^4\left(\mathbb{R}_+; H\right)$.

Next, by $L\left(X, Y\right)$ we will mean a set of linear bounded operators from the Hilbert space X to another Hilbert space Y. Fix some operator $K \in L(H_{5/2}, H_{3/2})$. Consider the following subspace of $W_2^4\left(\mathbb{R}_+; H\right)$:

$$W_{2,K}^4(\mathbb{R}_+; H) = \left\{ u\left(t\right) : u\left(t\right) \in W_2^4\left(\mathbb{R}_+; H\right),\ u''(0) = Ku'(0),\ u'''(0) = 0 \right\}.$$

The fact that $W_{2,K}^4(\mathbb{R}_+; H)$ is a subspace of $W_2^4\left(\mathbb{R}_+; H\right)$, follows from the trace theorem (see [8, Chapter 1]).

By $\sigma(\cdot)$ we denote the spectrum of the operator $(\cdot)$.

Consider in H the following boundary value problem:

$$u^{(4)}(t) + A^4 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t) = f(t),\ t \in \mathbb{R}_+, \tag{1.1}$$

$$u''\left(0\right) = Ku'\left(0\right),\ u'''(0) = 0, \tag{1.2}$$

where $A = A^* \geq cE$, $c > 0$, A_j, $j = 1, 2, 3, 4$ are linear and in general unbounded operators, $K \in L(H_{5/2}, H_{3/2})$, $f(t) \in L_2\left(\mathbb{R}_+; H\right)$, $u(t) \in W_2^4\left(\mathbb{R}_+; H\right)$.

Definition. If the vector-function $u(t) \in W_2^4(\mathbb{R}_+; H)$ satisfies the equation (1.1) almost everywhere in $\mathbb{R}_+$, and the boundary conditions (1.2) are fulfilled in the sense

$$\lim_{t \to 0} \left\| \ u''(t) - Ku'(t) \right\|_{H_{3/2}} = 0,\ \lim_{t \to 0} \left\| u'''(t) \right\|_{H_{1/2}} = 0,$$

then $u(t)$ will be called a *regular solution* of the boundary value problem (1.1), (1.2).

The purpose of this paper is to find the conditions for existence and uniqueness of a regular solution of the boundary value problem (1.1), (1.2) under some restrictions on its operator coefficients.

Quite a good number of research works have been dedicated to the solvability of boundary value problems for second order elliptic operator-differential equations with operator boundary conditions (see, e.g., [3, 4, 7, 12-15] and the references therein); however, these studies are far from the full completion. The works dedicated to such boundary value problems for operator-differential equations of higher order are relatively few. For example, in [1, 2] the conditions for existence and uniqueness of a regular solution of boundary value problems for third order operator-differential equations on the semi-axis with an operator in one of the boundary conditions are obtained. As for the solvability of boundary value problems for operator-differential equations of higher order in case when the coefficients in the boundary conditions are complex numbers only, this matter has been extensively studied in [5, 6, 9-11, 16] and references therein.

It should be noted that, as the boundary conditions (1.2) include an unbounded operator, the obtained abstract results are applicable to the solvability of a new class of boundary value problems for elliptic partial differential equations of fourth order, which gives another reason to study the boundary value problems like (1.1), (1.2).

2. Main results

First, let $A_j = 0$, $j = 1, 2, 3, 4$ in (1.1) and consider a simpler equation

$$u^{(4)}(t) + A^4 u(t) = f(t), \ t \in \mathbb{R}_+. \tag{2.1}$$

Denote by P_0 the operator which acts from $W_{2,K}^4(\mathbb{R}_+; H)$ to $L_2(\mathbb{R}_+; H)$ according to the following rule:

$$P_0 u(t) = u^{(4)}(t) + A^4 u(t), u(t) \in W_{2,K}^4(\mathbb{R}_+; H).$$

The following lemma is true.

Lemma 1. *Let $B = A^{3/2} K A^{-5/2}$ and $-\frac{1}{\sqrt{2}} \notin \sigma(B)$. Then the equation $P_0 u(t) = 0$ has a unique zero solution in $W_{2,K}^4(\mathbb{R}_+; H)$.*

Proof. It is not difficult to see that the general solution of the equation $P_0 u(t) = 0$ belonging to the space $W_2^4(\mathbb{R}_+; H)$ has the following form:

$$u_0(t) = e^{\omega_1 t A}\varphi_0 + e^{\omega_2 t A}\varphi_1,$$

where

$$\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i, \ \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i,$$

and φ_0, $\varphi_1 \in H_{7/2}$. From the conditions (1.2) we have:

$$\begin{cases} \omega_1^2 A^2 \varphi_0 + \omega_2^2 A^2 \varphi_1 = K A(\omega_1 \varphi_0 + \omega_2 \varphi_1), \\ \omega_1^3 \varphi_0 + \omega_2^3 \varphi_1 = 0. \end{cases} \tag{2.2}$$

From the system (2.2) we obtain:

$$\varphi_1 = -\frac{\omega_1^3}{\omega_2^3}\varphi_0, \tag{2.3}$$

$$\left(E + \sqrt{2}B\right) A^{7/2}\varphi_0 = 0. \tag{2.4}$$

By condition, $-\frac{1}{\sqrt{2}} \notin \sigma(B)$. Then it follows from the equation (2.4) that $\varphi_0 = 0$. Hence from (2.3) we have $\varphi_1 = 0$. Consequently, $u_0(t) = 0$. The lemma is proved.

The following theorem is true.

Theorem 1. *Let $B = A^{3/2} K A^{-5/2}$ and $-\frac{1}{\sqrt{2}} \notin \sigma(B)$. Then for every $f(t) \in L_2(\mathbb{R}_+; H)$ the boundary value problem* (2.1), (1.2) *has a unique regular solution.*

Proof. Due to Lemma 1, the problem

$$u^{(4)}(t) + A^4 u(t) = 0, \ t \in \mathbb{R}_+,$$

$$u''(0) = K u'(0), \ u'''(0) = 0$$

has only zero solution belonging to the space $W_{2,K}^4(\mathbb{R}_+; H)$.

Let's show that the equation $P_0 u(t) = f(t)$ has the solution $u(t) \in W_{2,K}^4(\mathbb{R}_+; H)$ for every $f(t) \in L_2(\mathbb{R}_+; H)$.

First, let's extend the vector-function $f(t)$ by zero for $t<0$ and denote the extended function by $F(t)$. Let $\hat{F}(\xi)$ be the Fourier transform for the vector-function $F(t)$, i.e.

$$\hat{F}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(t)e^{-i\xi t}dt,$$

where the integral on the right-hand side is understood in the sense of mean convergence in H.

Applying direct and indirect Fourier transforms, we obtain that the vector-function

$$\upsilon(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\xi^4E+A^4\right)^{-1}\left(\int_0^{+\infty}f(s)\,e^{-i\xi s}ds\right)e^{it\xi}d\xi,\ t\in\mathbb{R},$$

satisfies the equation

$$\upsilon^{(4)}(t)+A^4\upsilon(t)=F(t)$$

almost everywhere in R. Let us prove that $\upsilon(t)\in W_2^4(\mathbb{R};H)$. Let $\hat{\upsilon}(\xi)$ be the Fourier transform of the vector-function $\upsilon(t)$. According to the Plancherel theorem,

$$\begin{gathered}\|\upsilon(t)\|^2_{W_2^4(\mathbb{R};H)}=\left\|\upsilon^{(4)}(t)\right\|^2_{L_2(\mathbb{R};H)}+\left\|A^4\upsilon(t)\right\|^2_{L_2(\mathbb{R};H)}=\\ \left\|\xi^4\hat{\upsilon}(\xi)\right\|^2_{L_2(\mathbb{R};H)}+\left\|A^4\hat{\upsilon}(\xi)\right\|^2_{L_2(\mathbb{R};H)}=\\ \left\|\xi^4\left(\xi^4E+A^4\right)^{-1}\hat{F}(\xi)\right\|^2_{L_2(\mathbb{R};H)}+\left\|A^4\left(\xi^4E+A^4\right)^{-1}\hat{F}(\xi)\right\|^2_{L_2(\mathbb{R};H)}\le\\ \sup_{\xi\in\mathbb{R}}\left\|\xi^4\left(\xi^4E+A^4\right)^{-1}\right\|^2_{H\to H}\left\|\hat{F}(\xi)\right\|^2_{L_2(\mathbb{R};H)}+\\ \sup_{\xi\in\mathbb{R}}\left\|A^4\left(\xi^4E+A^4\right)^{-1}\right\|^2_{H\to H}\left\|\hat{F}(\xi)\right\|^2_{L_2(\mathbb{R};H)}=\\ \left(\sup_{\xi\in\mathbb{R}}\left\|\xi^4\left(\xi^4E+A^4\right)^{-1}\right\|^2_{H\to H}+\sup_{\xi\in\mathbb{R}}\left\|A^4\left(\xi^4E+A^4\right)^{-1}\right\|^2_{H\to H}\right)\|F(t)\|^2_{L_2(\mathbb{R};H)}.\end{gathered}\tag{2.5}$$

From the spectral theory of self-adjoint operators it is known that for $\xi\in\mathbb{R}$

$$\left\|\xi^4\left(\xi^4E+A^4\right)^{-1}\right\|\le\sup_{\sigma\in\sigma(A)}\left|\xi^4\left(\xi^4+\sigma^4\right)^{-1}\right|<1,$$

$$\left\|A^4\left(\xi^4E+A^4\right)^{-1}\right\|\le\sup_{\sigma\in\sigma(A)}\left|\sigma^4\left(\xi^4+\sigma^4\right)^{-1}\right|<1.$$

Then from (2.5) we have $\upsilon(t)\in W_2^4(\mathbb{R};H)$.

Denote by $u_1(t)$ the restriction of the function $\upsilon(t)$ to $\mathbb{R}_+$. Then $u_1(t)$ belongs to $W_2^4(\mathbb{R}_+;H)$ and satisfies the equation (2.1) almost everywhere in $\mathbb{R}_+$. And the trace theorem [8, Chapter 1] implies $u_1^{(j)}(0)\in H_{7/2-j}$, $j=0,1,2,3$.

The solution to the boundary value problem (2.1), (1.2) will be searched for in the following form:

$$u(t)=u_1(t)+e^{\omega_1tA}\psi_0+e^{\omega_2tA}\psi_1,$$

where $\psi_0\,,\psi_1\,\in H_{7/2}$ are subject to be found from the conditions (1.2):

$$\left\{\begin{array}{c} u_1''(0)+\omega_1^2A^2\psi_0+\omega_2^2A^2\psi_1=K\left(u_1'(0)+\omega_1A\psi_0+\omega_2A\psi_1\right), \\ u_1'''(0)+\omega_1^3A^3\psi_0+\omega_2^3A^3\psi_1=0. \end{array}\right. \tag{2.6}$$

Taking into account

$$\psi_1=-\frac{\omega_1^3}{\omega_2^3}\psi_0-\frac{1}{\omega_2^3}A^{-3}u_1'''(0)$$

in the first equation of system (2.6) and considering the condition $-\frac{1}{\sqrt{2}}\notin\sigma(B)$, we uniquely determine

$$\psi_0=A^{-7/2}\left(E+\sqrt{2}B\right)^{-1}A^{7/2}\eta\in H_{7/2},$$

where

$$\eta=-\frac{1}{\sqrt{2}}A^{-2}\left[\omega_2Ku_1'(0)-\frac{1}{\omega_2}KA^{-2}u_1'''(0)-\omega_2u_1''(0)+A^{-1}u_1'''(0)\right]\in H_{7/2}.$$

Thus, $u(t)$ belongs to $W_2^4(\mathbb{R}_+;H)$, satisfies the equation (2.1) almost everywhere in $\mathbb{R}_+$ and the conditions (1.2).

On the other hand, the operator $P_0:W_{2,K}^4(\mathbb{R}_+;H)\to L_2(\mathbb{R}_+;H)$ is bounded:

$$\|P_0u\|_{L_2(\mathbb{R}_+;H)}^2=\left\|u^{(4)}+A^4u\right\|_{L_2(\mathbb{R}_+;H)}^2\le 2\,\|u\|_{W_2^4(\mathbb{R}_+;H)}^2\,.$$

As a result, by virtue of Banach inverse operator theorem, there exists the operator $P_0^{-1}:L_2(\mathbb{R}_+;H)\to W_{2,K}^4(\mathbb{R}_+;H)$ and this operator is bounded. It follows that

$$\|u\|_{W_2^4(\mathbb{R}_+;H)}\le const\ \|f\|_{L_2(\mathbb{R}_+;H)}\,.$$

The theorem is proved.

Theorem 1, combined with Lemma 1, implies that the operator P_0, under condition $-\frac{1}{\sqrt{2}}\notin\sigma(B)$ with $B=A^{3/2}KA^{-5/2}$, performs an isomorphism between the spaces $W_{2,K}^4(\mathbb{R}_+;H)$ and $L_2\left(\mathbb{R}_+;H\right)$. Consequently, the norm $\|P_0u\|_{L_2(\mathbb{R}_+;H)}$ is equivalent to the original norm $\|u\|_{W_2^4(\mathbb{R}_+;H)}$ in $W_{2,K}^4(\mathbb{R}_+;H)$. Then, as the intermediate derivative operators

$$A^j\frac{d^{4-j}}{dt^{4-j}}:W_{2,K}^4(\mathbb{R}_+;H)\to L_2(\mathbb{R}_+;H),\;\; j=1,2,3,4,$$

are continuous [8], their norms can be estimated by $\|P_0u\|_{L_2(\mathbb{R}_+;H)}$. Estimates for these norms are required when establishing solvability conditions for the boundary value problem (1.1), (1.2). But, before we proceed to the estimates for these norms, we prove the following lemma.

Lemma 2. *Let $B=A^{3/2}KA^{-5/2}$ and $ReB\ge 0$. Then for every $u\left(t\right)\in W_{2,K}^4(\mathbb{R}_+;H)$ there holds the inequality*

$$\|P_0u\|_{L_2(\mathbb{R}_+;H)}^2\ge\|u\|_{W_2^4(\mathbb{R}_+;H)}^2+2\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)}^2\,. \tag{2.7}$$

Proof. Note that integration by parts for $u\left(t\right)\in W_{2,K}^4(\mathbb{R}_+;H)$ yields

$$Re\left(u^{(4)},A^4u\right)_{L_2(\mathbb{R}_+;H)}=Re\left(BA^{5/2}u'(0),A^{5/2}u'(0)\right)+\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)}^2. \tag{2.8}$$

Taking into account (2.8), we have:

$$\|P_0u\|^2_{L_2(\mathbb{R}_+;H)} = \left\|u^{(4)}\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} + 2Re\left(u^{(4)}, A^4u\right)_{L_2(\mathbb{R}_+;H)} =$$

$$\|u\|^2_{W_2^4(\mathbb{R}_+;H)} + 2Re\left(BA^{5/2}u'(0), A^{5/2}u'(0)\right) + 2\left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)}. \tag{2.9}$$

As $ReB \geq 0$, the equality (2.9) implies the validity of the lemma. The lemma is proved.

Theorem 2. *Let $B = A^{3/2}KA^{-5/2}$ and $ReB \geq 0$. Then for every $u(t) \in W^4_{2,K}(\mathbb{R}_+;H)$ there hold the following estimates:*

$$\left\|A^ju^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)} \leq c_j \|P_0u\|_{L_2(\mathbb{R}_+;H)}, \; j = 1, 2, 3, 4, \tag{2.10}$$

where

$$c_1 = \frac{1}{\sqrt{2}}, \; c_2 = \frac{1}{2}, \; c_3 = c_4 = 1.$$

Proof. Scalar multiplication of both sides of (2.1) by $A^4u(t)$ in $L_2(\mathbb{R}_+;H)$ and integration by parts with consideration of conditions $u(t) \in W^4_{2,K}(\mathbb{R}_+;H)$ and $ReB \geq 0$ yield:

$$Re\left(P_0u, A^4u\right)_{L_2(\mathbb{R}_+;H)} = Re\left(u^{(4)} + A^4u, A^4u\right)_{L_2(\mathbb{R}_+;H)} =$$

$$\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} + Re\left(BA^{5/2}u'(0), A^{5/2}u'(0)\right) + \left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} \geq$$

$$\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)}. \tag{2.11}$$

Applying Cauchy-Schwarz inequality to the left-hand side of (2.11), and then using Young inequality, we have:

$$\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)} \left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\frac{\varepsilon}{2}\|P_0u\|^2_{L_2(\mathbb{R}_+;H)} + \frac{1}{2\varepsilon}\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)}, \; \varepsilon > 0. \tag{2.12}$$

Assuming $\varepsilon = \frac{1}{2}$ in (2.12), we get

$$\left\|A^2u''\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{4}\|P_0u\|^2_{L_2(\mathbb{R}_+;H)}$$

or

$$\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\|P_0u\|_{L_2(\mathbb{R}_+;H)}. \tag{2.13}$$

On the other hand, from (2.12) we have

$$\left\|A^4u\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)} \left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)}.$$

Consequently,

$$\left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)}. \tag{2.14}$$

The validity of inequality (2.14) can be obtained from (2.7), too. Besides, it follows from the inequality (2.7) that

$$\left\|u^{(4)}\right\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)}. \tag{2.15}$$

Now let's estimate $\left\|A^3u'\right\|_{L_2(\mathbb{R}_+;H)}$. It is shown in [11] that for $u\,(t) \in W_2^4(\mathbb{R}_+;H)$ there holds

$$\left\|A^3u'\right\|^2_{L_2(\mathbb{R}_+;H)} \leq 2\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)}\left\|A^4u\right\|_{L_2(\mathbb{R}_+;H)}. \tag{2.16}$$

Considering (2.13), (2.14) in (2.16), we have

$$\left\|A^3u'\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|^2_{L_2(\mathbb{R}_+;H)}$$

or

$$\left\|A^3u'\right\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)}.$$

Finally, we pass on to estimation of $\|Au'''\|_{L_2(\mathbb{R}_+;H)}$. Integrating by parts with the consideration of $u\,(t) \in W^4_{2,K}(\mathbb{R}_+;H)$, applying Cauchy-Schwarz inequality and then using inequalities (2.13), (2.15), we get

$$\left\|Au'''\right\|^2_{L_2(\mathbb{R}_+;H)} = \int_0^{+\infty}(Au''', Au''')_H dt = \left.(Au'', Au''')_H\right|_0^{+\infty} -$$

$$\int_0^{+\infty}(A^2u'', u^{(4)})_H dt = -\int_0^{+\infty}(A^2u'', u^{(4)})_H dt \leq$$

$$\left\|A^2u''\right\|_{L_2(\mathbb{R}_+;H)}\left\|u^{(4)}\right\|_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\|P_0u\|^2_{L_2(\mathbb{R}_+;H)}.$$

Consequently,

$$\left\|Au'''\right\|_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{\sqrt{2}}\|P_0u\|_{L_2(\mathbb{R}_+;H)}.$$

The theorem is proved.

Now we consider the case when $A_j \neq 0$, $j = 1, 2, 3, 4$.

Denote by P the operator which acts from $W^4_{2,K}(\mathbb{R}_+;H)$ to $L_2\,(\mathbb{R}_+;H)$ according to the following rule:

$$Pu(t) = u^{(4)}(t) + A^4u(t) + \sum_{j=1}^{4}A_ju^{(4-j)}(t), \;\; u\,(t) \in W^4_{2,K}\,(\mathbb{R}_+;H)\,.$$

The following lemma is true.

Lemma 3. *Let $A_jA^{-j} \in L(H,H)$, $j = 1, 2, 3, 4$. Then the operator P is a bounded operator from $W^4_{2,K}\,(\mathbb{R}_+;H)$ to $L_2\,(\mathbb{R}_+;H)$.*

Proof. For every $u\,(t) \in W^4_{2,K}(\mathbb{R}_+;H)$ there holds

$$\|Pu\|_{L_2(\mathbb{R}_+;H)} \leq \|P_0u\|_{L_2(\mathbb{R}_+;H)} + \left\|\sum_{j=1}^{4}A_ju^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\sqrt{2}\,\|u\|_{W_2^4(\mathbb{R}_+;H)} + \sum_{j=1}^{4}\left\|A_ju^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\sqrt{2}\,\|u\|_{W_2^4(\mathbb{R}_+;H)} + \sum_{j=1}^{4}\left\|A_jA^{-j}\right\|_{H\to H}\left\|A^ju^{(4-j)}\right\|_{L_2(\mathbb{R}_+;H)}. \tag{2.17}$$

Then, by virtue of the theorem for intermediate derivatives [8, Chapter 1], from the inequality (2.17) we obtain

$$\|Pu\|_{L_2(\mathbb{R}_+;H)} \leq const\,\|u\|_{W_2^4(\mathbb{R}_+;H)}.$$

The lemma is proved.

Based on the above discussion, we are now able to state our main result.

The following main theorem is true.

Theorem 3. *Let* $ReB \geq 0$, $A_j A^{-j} \in L(H,H)$, $j = 1,2,3,4$, *and the inequality*

$$\alpha = \sum_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H \to H} < 1$$

hold, where the numbers c_j, $j = 1,2,3,4$, *are defined by Theorem 2, i.e.*

$$c_1 = \frac{1}{\sqrt{2}},\ c_2 = \frac{1}{2},\ c_3 = c_4 = 1.$$

Then the boundary value problem (1.1), (1.2) *has a unique regular solution for every* $f(t) \in L_2\left(\mathbb{R}_+; H\right)$.

Proof. Let's rewrite the boundary value problem (1.1), (1.2) in the form of operator equation

$$P_0 u(t) + (P - P_0) u(t) = f(t),$$

where $f(t) \in L_2\left(\mathbb{R}_+; H\right)$, $u(t) \in W_{2,K}^4\left(\mathbb{R}_+; H\right)$.

The conditions $B = A^{3/2} K A^{-5/2}$, $ReB \geq 0$ guarantee the existence of a bounded inverse operator P_0^{-1} from $L_2\left(\mathbb{R}_+; H\right)$ to $W_{2,K}^4(\mathbb{R}_+; H)$. Substituting $u(t) = P_0^{-1} v(t)$, where $v(t) \in L_2\left(\mathbb{R}_+; H\right)$, we obtain the following equation in the space $L_2\left(\mathbb{R}_+; H\right)$:

$$v(t) + (P - P_0) P_0^{-1} v(t) = f(t).$$

Then for every $v(t) \in L_2\left(\mathbb{R}_+; H\right)$, in view of estimates (2.10), we have:

$$\left\| (P - P_0) P_0^{-1} v \right\|_{L_2(\mathbb{R}_+; H)} = \left\| (P - P_0) u \right\|_{L_2(\mathbb{R}_+; H)} \leq$$

$$\sum_{j=1}^{4} \left\| A_j A^{-j} \right\|_{H \to H} \left\| A^j u^{(4-j)} \right\|_{L_2(\mathbb{R}_+; H)} \leq$$

$$\sum_{j=1}^{4} c_j \left\| A_j A^{-j} \right\|_{H \to H} \left\| P_0 u \right\|_{L_2(\mathbb{R}_+; H)} = \alpha \left\| v \right\|_{L_2(\mathbb{R}_+; H)}.$$

As, by condition, $\alpha < 1$, the operator $E + (P - P_0) P_0^{-1}$ has an inverse in $L_2\left(\mathbb{R}_+; H\right)$. Consequently,

$$u(t) = P_0^{-1} \left(E + (P - P_0) P_0^{-1} \right)^{-1} f(t),$$

with

$$\left\| u \right\|_{W_2^4(\mathbb{R}_+; H)} \leq$$

$$\left\| P_0^{-1} \right\|_{L_2(\mathbb{R}_+; H) \to W_2^4(\mathbb{R}_+; H)} \left\| \left(E + (P - P_0) P_0^{-1} \right)^{-1} \right\|_{L_2(\mathbb{R}_+; H) \to L_2(\mathbb{R}_+; H)} \left\| f \right\|_{L_2(\mathbb{R}_+; H)}$$

$$\leq const \left\| f \right\|_{L_2(\mathbb{R}_+; H)}.$$

The theorem is proved.

Remark 1. In Theorem 3, the condition $ReB \geq 0$, where $B = A^{3/2} K A^{-5/2}$, allows the omission of condition $-\frac{1}{\sqrt{2}} \notin \sigma(B)$.

Remark 2. Separate consideration is required for the case when the operator ReB is not non-negative.

References

[1] A. R. Aliev and S. F. Babayeva, On the boundary value problem with the operator in boundary conditions for the operator-differential equation of the third order, *Journal of Mathematical Physics, Analysis, Geometry*, **6** (2010), no. 4, 347–361.

[2] A. R. Aliev and N. L. Muradova, Third-order operator-differential equations with discontinuous coefficients and operators in the boundary conditions, *Electron. J. Differ. Equ.*, **2013** (2013), no. 219, 1–13.

[3] B. A. Aliev, A boundary value problem for a second order elliptic differential-operator equation with a spectral parameter and operator boundary conditions, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.*, **32(40)** (2010), 21–46.

[4] B. A. Aliev and Ya. Yakubov, Second order elliptic differential-operator equations with unbounded operator boundary conditions in UMD Banach spaces, *Integr. Equ. Oper. Theory*, **69** (2011), no. 2, 269–300.

[5] M. G. Gasymov, On the theory of polynomial operator pencils, *Soviet Math. Dokl.* **12** (1972), 1143–1147 (translated from *Dokl. Akad. Nauk SSSR*, **199** (1971), no. 4, 747–750).

[6] M. G. Gasymov, The solvability of boundary value problems for a class of operator-differential equations, *Dokl. Akad. Nauk SSSR*, **235** (1977), no. 3, 505–508 (in Russian).

[7] M. G. Gasymov and S. S. Mirzoev, Solvability of boundary value problems for second-order operator-differential equations of elliptic type, *Differ. Equ.*, **28** (1992), no. 4, 528–536 (translated from *Differentsial'nye Uravneniya*, **28** (1992), no. 4, 651–661).

[8] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Dunod, Paris, 1968; Mir, Moscow, 1971; Springer-Verlag, Berlin, 1972.

[9] S. S. Mirzoev, Conditions for the well-defined solvability of boundary-value problems for operator differential equations, *Soviet Math. Dokl.*, **28** (1983), 629–632 (translated from *Dokl. Akad. Nauk SSSR*, **273** (1983), no. 2, 292–295).

[10] S. S. Mirzoev, Multiple completeness of root vectors of polynomial operator pencils corresponding to boundary-value problems on the semiaxis, *Funct. Anal. Appl.*, **17** (1983), no. 2, 151–153 (translated from *Funkts. Anal. Prilozh.*, **17** (1983), no. 2, 84–85).

[11] S. S. Mirzoev, Correct solvability of boundary value problems for even-order operator-differential equations, *Spectral theory of differential operators*, 85–91, Azerbaidzhan. Gos. Univ., Baku, 1984 (in Russian).

[12] S. S. Mirzoev, A. R. Aliev, and L. A. Rustamova, Solvability conditions for boundary-value problems for elliptic operator-differential equations with discontinuous coefficient, *Math. Notes*, **92** (2012), no. 5-6, 722–726 (translated from *Mat. Zametki*, **92** (2012), no. 5, 789–793).

[13] S. S. Mirzoev, A. R. Aliev, and L. A. Rustamova, On the boundary value problem with the operator in boundary conditions for the operator-differential equation of second order with discontinuous coefficients, *Journal of Mathematical Physics, Analysis, Geometry*, **9** (2013), no. 2, 207–226.

[14] S. S. Mirzoev and M. Yu. Salimov, On the solvability of the boundary-value problem for second-order equations in Hilbert space with an operator coefficient in the boundary condition, *Math. Notes*, **91** (2012), no. 5-6, 808–815 (translated from *Mat. Zametki*, **91** (2012), no. 6, 861–869).

[15] S. S. Mirzoev and Kh. V. Yagubova, On the solvability of boundary value problems with operators in the boundary conditions for a class of second-order operator-differential equations, *Dokl. Nats. Akad. Nauk Azerb.*, **57** (2001), no. 1-3, 12–17 (in Russian).

[16] A. A. Shkalikov, Elliptic equations in a Hilbert space and related spectral problems, *J. Soviet Math.*, **51** (1990), no. 4, 2399–2467 (translated from *Trudy Sem. Petrovsk.*, **14** (1989), 140–224).

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