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# ON THE ERROR ESTIMATION OF THE EIGENVALUES OF SOME BOUNDARY VALUE PROBLEMS

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*In memory of M. G. Gasymov on his 75th birthday*

**Abstract**. A posteriori error estimates for the eigenvalues of boundary value problems for ordinary differential and ordinary integro-differential equations computed by the method of mechanical quadratures are obtained in this paper.

## 1. Introduction

The matter of error estimation when solving boundary value problems for linear ordinary differential and ordinary integro-differential equations by approximate methods has been widely studied, while the same cannot be said about the error estimation for the eigenvalues of these problems. The case of integral equations is different. The problem of error estimation is successfully solved both when solving non-homogeneous equations (by the method of replacing the kernel by a similar one, for example, by a degenerate one, and by the method of mechanical quadratures (see, e.g., [2, 3])) and when computing the eigenvalues. For example, in [4] the problem of error estimation for the eigenvalues of the Fredholm integral equation of the second kind with Hermitian kernel computed by the method of mechanical quadratures was considered. We should also note [1] which used the method of replacing the kernel by a similar one.

In this work, we obtain a posteriori error estimates for the eigenvalues of linear ordinary differential and ordinary integro-differential equations computed by the method of mechanical quadratures.

## 2. Main results

**1.** Consider the ordinary differential equation

$$Lx\,(t) \equiv x^{(m)}\,(t) + \sum_{s=0}^{m-1} p_s\,(t)\,x^{(s)}\,(t) = \lambda q\,(t)\,x\,(t) \tag{2.1}$$

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with the conditions

$$l_i x = \sum_{j=0}^{m-1} \left[ \alpha_{ij} x^{(j)}(0) + \beta_{ij} x^{(j)}(1) \right] = 0, \ i = 1, 2, ..., m, \tag{2.2}$$

where $\alpha_{ij}$, $\beta_{ij} = const$, $i = 1, 2, ..., m$, $q(t) > 0$, and the coefficients $q(t)$, $p_s(t)$, $s = 0, 1, ..., m-1$, are continuous on the interval $[0, 1]$.

Denote by $g(t, s)$ the Green's function of the differential expression $Lx(t)$ under the boundary conditions (2.2).

Let the problem

$$Lx = 0, \ \ l_i x = 0, \ \ i = 1, 2, ..., m,$$

be self-adjoint. Then it is not difficult to show that the boundary value problem (2.1), (2.2) is equivalent to the integral equation

$$y(t) = \lambda \int_0^1 G(t, s) y(s) ds, \tag{2.3}$$

where

$$G(t, s) = g(t, s) \sqrt{q(t) q(s)}, \ \ y(t) = x(t) \sqrt{q(t)}.$$

Consider the quadrature formula

$$\int_0^1 \varphi(t) dt = \sum_{k=1}^{n} A_k \varphi(t_k) + R_n(t), \tag{2.4}$$

where $0 \leq t_1 < t_2 < ... < t_k \leq 1, \ \ A_k > 0$.

Applying the quadrature formula (2.4) to the integral on the right-hand side of (2.3), we find:

$$y(t) = \lambda \sum_{k=1}^{n} A_k G(t, t_k) y(t_k) + \lambda R_n(t). \tag{2.5}$$

Replacing $t$ by $t_i$ $(i = 1, 2, ..., n)$ in (2.5) leads us to the following system of equations:

$$y(t_i) = \lambda \sum_{k=1}^{n} A_k G(t_i, t_k) y(t_k) + \lambda R_n(t_i), \ \ (i = 1, 2, ...., n). \tag{2.6}$$

Removing the term with $R_n(t_i)$ in (2.6) and replacing $y(t_k)$ by $\tilde{y}(t_k)$, we obtain the linear algebraic system

$$\tilde{y}_i = \lambda \sum_{k=1}^{n} A_k G_{ik} \tilde{y}_k,$$

where $G_{ik} = G(t_i, t_k)$, $\tilde{y}(t_k) = \tilde{y}_k$ $(i = 1, 2, ...., n)$.

Thus, the approximate computation of eigenvalues of the boundary value problem (2.1), (2.2) is reduced by the mechanical quadrature method to the computation of eigenvalues of the matrix

$$U = \begin{pmatrix} A_1 G_{11} & A_2 G_{12} & ... & A_n G_{1n} \\ A_1 G_{21} & A_2 G_{22} & ... & A_n G_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ A_1 G_{n1} & A_2 G_{n2} & ... & A_n G_{nn} \end{pmatrix}. \tag{2.7}$$

Denote the eigenvalues of the boundary value problem (2.1), (2.2) and those of the matrix (2.7) by $\lambda_{jL}$ and $\lambda_{jU}$, respectively. We will assume that the eigenvalues $\lambda_{jL}$ and $\lambda_{jU}$ are numbered in non-descending order of their moduli (with multiplicity).

Construct the iterated kernel

$$G_2(t,s)=\int_0^1 G(t,z)\,G(z,s)\,dz. \tag{2.8}$$

Applying the quadrature formula (2.4) to the right-hand side of (2.8), we get

$$G_2(t,s)=N(t,s)+R_n(t,s),$$

where

$$N(t,s)=\sum_{k=1}^{n} A_k G(t,t_k)\,G(t_k,s) \tag{2.9}$$

is a degenerate Hermitian kernel, $R_n(t,s)$ is a small kernel and $A_k>0$. Then the Hermitian kernel (2.9) is positive and all its eigenvalues are also positive.

Let $\lambda_{jN}$ be the eigenvalues of the kernel $N(t,s)$.

Applying the well-known Weyl theorem (see, e.g., [5]) to the operators $N(t,s)$ and $G_2(t,s)=N(t,s)+R_n(t,s)$, we find:

$$\left|\lambda_{jL}^2-\lambda_{jN}\right|\le\frac{\lambda_{jN}^2\cdot\|R_n\|}{1-|\lambda_{jN}|\cdot\|R_n\|}, \tag{2.10}$$

where $\|R_n\|$ is the norm of an integral operator with the kernel $R(t,s)$. Obviously, the eigenvalues $\lambda_{jN}$ of the kernel $N(t,s)$ coincide with those of the matrix $A$. The element in the $i$-th row and $k$-th column of the matrix $A$ is defined by the formula

$$a_{ik}=\sqrt{A_i}\cdot\sqrt{A_k}\cdot\int_0^1 G(t_i,s)\,G(s,t_k)\,ds. \tag{2.11}$$

Now, applying the quadrature formula (2.4) to the right-hand side of (2.11), we obtain

$$a_{ik}=\sqrt{A_i}\cdot\sqrt{A_k}\cdot\left(\sum_{r=1}^{n}A_rG_{ir}G_{rk}+R_{ik}\right). \tag{2.12}$$

Take the square of the matrix $U$:

$$U^2=(b_{ik})=\left(\sum_{r=1}^{n}A_rG_{ir}A_kG_{rk}\right),$$

where

$$b_{ik}=\left(U^2\right)_{ik}=\sum_{r=1}^{n}A_rG_{ir}A_kG_{rk}$$

is the element in the $i$-th row and $k$-th column of the square of the matrix $U$.

Consider the diagonal matrix

$$D=(d_{ik})=\begin{pmatrix}\sqrt{A_1} & 0 & \dots & 0\\ 0 & \sqrt{A_2} & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \sqrt{A_n}\end{pmatrix},$$

where

$$d_{ik} = \begin{cases} \sqrt{A_i} & for \quad i = k, \\ 0 & for \quad i \neq k. \end{cases}$$

It is not difficult to show that

$$DU^2D^{-1} = (C_{ik}), \tag{2.13}$$

where

$$C_{ik} = \left(DU^2D^{-1}\right)_{ik} = \sqrt{A_i} \cdot \sqrt{A_k} \cdot \sum_{r=1}^{n} A_r G_{ir} G_{rk}.$$

Then it follows from (2.12) and (2.13) that

$$A = DU^2D^{-1} + B,$$

where

$$B = \begin{pmatrix} A_1R_{11} & \sqrt{A_1A_2}R_{12} & \dots & \sqrt{A_1A_n}R_{1n} \\ \sqrt{A_2A_1}R_{21} & A_2R_{22} & \dots & \sqrt{A_2A_n}R_{2n} \\ \dots & \dots & \dots & \dots \\ \sqrt{A_nA_1}R_{n1} & \sqrt{A_nA_2}R_{n2} & \dots & A_nR_{nn} \end{pmatrix}.$$

Now, applying the Weyl theorem [5] to the matrices $DU^2D^{-1}$ and $A = DU^2D^{-1} + B$, we get

$$\left|\lambda_{jN} - \lambda_{jU}^2\right| \leq \frac{\lambda_{jU}^4 \cdot \|B\|}{1 - \lambda_{jU}^2 \cdot \|B\|}. \tag{2.14}$$

Consequently,

$$|\lambda_{jN}| \leq \frac{\lambda_{jU}^2}{1 - \lambda_{jU}^2 \cdot \|B\|}. \tag{2.15}$$

From the inequalities (2.10) and (2.15) we have

$$\left|\lambda_{jU}^2 - \lambda_{jN}\right| \leq \frac{\lambda_{jU}^4 \cdot \|R_n\|}{\left(1 - \lambda_{jU}^2 \cdot \|B\|\right)\left[1 - \lambda_{jU}^2\left(\|B\| + \|R_n\|\right)\right]}. \tag{2.16}$$

Now from the inequalities (2.14) and (2.16) we find

$$\left|\lambda_{jU}^2 - \lambda_{jL}^2\right| \leq \left|\lambda_{jU}^2 - \lambda_{jN}\right| + \left|\lambda_{jN} - \lambda_{jL}^2\right| < \delta_j, \tag{2.17}$$

where

$$\delta_j = \frac{\lambda_{jU}^4\left(\|B\| + \|R_n\|\right)}{1 - \lambda_{jU}^2\left(\|B\| + \|R_n\|\right)} \tag{2.18}$$

with

$$\lambda_{jU}^2\left(\|B\| + \|R_n\|\right) < 1.$$

In view of

$$|\lambda_{jU} + \lambda_{jL}| \geq 2|\lambda_{jU}| - |\lambda_{jU} - \lambda_{jL}|,$$

from (2.17) we have

$$|\lambda_{jU} - \lambda_{jL}|\left(2|\lambda_{jU}| - |\lambda_{jU} - \lambda_{jL}|\right) \leq \delta_j.$$

Hence

$$|\lambda_{jU} - \lambda_{jL}| \leq \frac{\delta_j}{|\lambda_{jU}| + \sqrt{\lambda_{jU}^2 - \delta_j}}, \tag{2.19}$$

where $\delta_j$ is defined by (2.18).

Thus, we have proved the following

**Theorem.** *Under the above conditions, the error estimate* (2.19) *for the eigenvalues of the boundary value problem* (2.1), (2.2) *in terms of the eigenvalues of the matrix* (2.7) *is true, obtained by applying the mechanical quadrature method to the boundary value problem* (2.1), (2.2).

**2.** Consider the ordinary integro-differential equation

$$Kx(t) \equiv x^{(m)}(t) + \sum_{s=1}^{m-2} p_s(t)\, x^{(s)}(t) + \int_0^1 b(t,\sigma)\, x(\sigma)\, d\sigma = \lambda a(t)\, x(t) \quad (2.20)$$

with the conditions

$$l_j x = \sum_{i=0}^{m-1} \left[\alpha_{ij} x^{(i)}(0) + \beta_{ij} x^{(i)}(1)\right] + \int_0^1 \gamma_j(t)\, x(t)\, dt = 0,\ j = 1,2,...,m, \quad (2.21)$$

where $\alpha_{ij}$, $\beta_{ij}$ $=$ $const$, $j = 1,2,...,m$, $a(t) > 0$, the coefficients $a(t)$, $p_s(t)$, $s = 1,2,...,m-2$, are continuous on the interval $[0,\ 1]$, and $\gamma_j(t)$, $j = 1,2,...,m$, are the summable functions on $[0,\ 1]$.

Denote by $\Gamma\ (t,s)$ the Green's function of the differential expression $Kx(t)$ under the conditions (2.21) and let the problem $Kx(t) = 0$, $l_j(x) = 0$ be Hermitian self-adjoint. Then the problem (2.20), (2.21) is equivalent to the integral equation

$$z(t) = \lambda \int_0^1 G(t,s)\, z(s)\, ds,$$

where

$$G(t,s) = \Gamma(t,s)\sqrt{a(t)\, a(s)},\quad z(t) = x(t)\sqrt{a(t)}.$$

Continuing in the same way as we did when considering problem (2.1), (2.2), we arrive at the inequality (2.19) for the problem (2.20), (2.21), where $\lambda_{jU}$ and $\lambda_{jL}$ are the eigenvalues of the matrix $U = (A_k G_{ik})$ and the kernel $G(t\,,s)$, respectively.

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