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OSCILLATION PROPERTIES OF THE EIGENVECTOR-FUNCTIONS OF THE ONE-DIMENSIONAL DIRAC'S CANONICAL SYSTEM

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In memory of M. G. Gasymov on his 75th birthday

Abstract. We consider the boundary value problem for one-dimensional Dirac's canonical system. The general characteristics of the location of the eigenvalues on the real axis and oscillation properties of eigenvector-functions of this problem are investigated.

1. Introduction

We consider the following boundary value problem for one-dimensional Dirac's canonical system

$$\vartheta' - \{\lambda + p(x)\}\, u = 0, \;\; u' + \{\lambda + r(x)\}\, \vartheta = 0, \; 0 < x < \pi, \tag{1.1}$$

$$\vartheta(0)\cos\alpha + u(0)\sin\alpha = 0, \tag{1.2}$$

$$\vartheta(\pi)\cos\beta + u\,(\pi)\sin\beta = 0, \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, the functions $p(x)$ and $r(x)$ are continuous on the interval $[0,\ \pi]$, $\alpha,\ \beta$ are real constans such that $0 \leq \alpha,\ \beta < \pi$.

If the boundary value problem (1.1)-(1.3) has non-trivial solution

$$U(x,\lambda) = (u\,(x,\lambda),\ \vartheta(x,\lambda))$$

for some $\lambda = \tilde{\lambda}$, then the number $\tilde{\lambda}$ is called eigenvalue, and the corresponding solution $U(x,\tilde{\lambda})$ is called eigenvector-function.

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons intended to make new mathematical results accesible to a wider audience. It treats in some depth relativistic of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scatering states and the external field problem in quantum electrodynamics, without neglecting the interpretational difficults and limitations of the theory. For the case in which $p(x) = V(x) + m$, $\;r(x) = V(x) - m$, where $V(x)$ is a potencial

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function, and m is the mass of a particle, the system (1.1) is known in relativistic quantum theory in a stationary one-dimensional Dirac system or first canonical form of Dirac system [10].

The basic comprehensive results (except the oscillation properties) about Dirac's system (1.1)-(1.3) were given in [10]. Inverse problems for Dirac system had been investigated by Gasymov and Levitan [4], Panakhov [12] and other. In recent being considered inverse nodal problems, that lies in constructing operators from the given zeros of their eigenvector-functions (see. [11] and also references in this). In [11], deal with an inverse nodal problem of reconstructing the Dirac system with the spectral parameter in the boundary conditions and is proved that a set of nodal points of one of the components of the eigenfunctions uniquely determines all the parameters of the boundary conditions and the coefficients of the Dirac equations.

We note that the oscillation properties of the eigenfunctions of the Sturm-Liouville problem completely studied by various methods (see, e.g. [1, 2, 5, 7, 10]). However, oscillation properties of eigenvector-functions of the Dirac system is subject to a detailed study. In [8] (see also [11]) studied oscillation properties of eigenvector-functions of the Dirac system with a spectral parameter in the boundary conditions. It should be noted that these studies did not specify the exact number of zeros of the components of the eigenvector-function corresponding nth eigenvalue (although for sufficiently large n).

In the present paper, we study the general characteristics of the location of the eigenvalues on the real axis and oscillation properties of eigenvector-functions of the spectral problem (1.1)-(1.3).

2. Comparison theorems

The following theorem is basic in circle of topics.

Theorem 2.1. *Suppose that we are given two systems*

$$\vartheta_1' - p_1(x)\, u_1 = 0, \;\; u_1' + r_1(x)\, \vartheta_1 = 0, \tag{2.1}$$

$$\vartheta_2' - p_2(x)\, u_2 = 0, \;\; u_2' + r_2(x)\, \vartheta_2 = 0, \tag{2.2}$$

where the functions $p_\nu(x)$ *and* $r_\nu(x)$, $\nu = 1,\ 2$, *are continuous on the interval* $[0,\ \pi]$. *Let* $(u_1(x), \vartheta_1(x))$ *and* $(u_2(x), \vartheta_2(x))$ *are arbitrary solutions of these systems respectively. If* $p_2(x) > p_1(x) > 0$ *and* $r_2(x) > r_1(x) > 0$, *or* $p_2(x) < p_1(x) < 0$ *and* $r_2(x) < r_1(x) < 0$ *over the entire interval* $[0,\ \pi]$, *then between every two consecutive zeros of function* $u_1(x)$ $(\vartheta_1(x))$ *there is at least one zero of function* $u_2(x)$ $(\vartheta_2(x))$.

Proof. By virtue of Picone formula [7, p. 151] (see, also [5]) we have

$$\frac{d}{dx}\left\{\frac{u_1}{u_2}\,(u_1\vartheta_2 - u_2\vartheta_1)\right\}$$

$$= \frac{r_1}{r_2}\,(r_2 - r_1)\;\vartheta_1^2 + (p_2 - p_1)\;u_1^2 + r_2\left(\frac{u_1}{u_2}\vartheta_2 - \frac{r_1}{r_2}\vartheta_1\right)^2, \tag{2.3}$$

where in considering interval should be $u_2(x) \neq 0$.

Let us denote two successive zeros of $u_1(x)$ of by x_1 and x_2, and assume that $u_2(x)$ does not equal zero anywhere in the interval $(x_1,\ x_2)$. Note the quotient $u_1(x)/u_2(x)$ has a limit at the endpoints. For example the limit at x_1 is zero if $u_2(x_1) \neq 0$, and the limit is $u_1'(x_1)/u_2'(x_1) = r_1(x_1)\vartheta_1(x_1)/r_2(x_1)\vartheta_2(x_1)$, if $u_2(x_1) = 0$.

Then integrating the identity (2.3) from x_1 to x_2, we obtain the relation

$$\{(u_1/u_2)\ (u_1\vartheta_2 - u_2\vartheta_1)\}_{x_1}^{x_2} =$$

$$= \int_{x_1}^{x_2} \{(r_1/r_2)\,(r_2 - r_1)\ \vartheta_1^2 + (p_2 - p_1)\ u_1^2 + r_2\,((u_1/u_2)\vartheta_2 - (r_1/r_2)\vartheta_1)^2\}\,dx$$

which implies that the left-hand side integrates to zero while the right-hand side integrates to a positive number, or negative number. This contradiction proves that the function $u_2(x)$ has at least one zero in the interval $(x_1,\ x_2)$. Assertion of the theorem relating to the function $\vartheta_2(x)$ is proved similarly. It uses the following Picone type formula:

$$\frac{d}{dx}\left\{\frac{\vartheta_1}{\vartheta_2}\,(u_1\vartheta_2 - u_2\vartheta_1)\right\} =$$

$$= \frac{p_1}{p_2}\,(p_2 - p_1)\ u_1^2 + (r_2 - r_1)\ \vartheta_1^2 + p_2\left(\frac{\vartheta_1}{\vartheta_2}u_2 - \frac{p_1}{p_2}u_1\right)^2. \tag{2.4}$$

The proof of Theorem 2.1 is complete.

Theorem 2.2. *Let* $(u_1(x), \vartheta_1(x))$ *be the solution of the system* (2.1) *satisfying the initial conditions*

$$u_1(0) = \cos\alpha, \qquad \vartheta_1(0) = -\sin\alpha, \tag{2.5}$$

and be $(u_2(x), \vartheta_2(x))$ *the solution of system* (2.2) *satisfying the same initial conditions. Moreover, suppose that* $p_2(x) > p_1(x) > 0$ *and* $r_2(x) > r_1(x) > 0$, *or* $p_2(x) < p_1(x) < 0$ *and* $r_2(x) < r_1(x) < 0$ *over the entire interval* $[0,\ \pi]$.

If $u_1(x)$ $(\vartheta_1(x))$ *has* m *zeros in the interval* $0 < x \leq \pi$, *then* $u_2(x)$ $(\vartheta_2(x))$ *has not less than* m *zeros in the same interval, and the* k*th zero of* $u_2(x)$ $(\vartheta_2(x))$ *is less than the* k*th zero of* $u_1(x)$ $(\vartheta_1(x))$.

Proof. Let x_1 denote the zero of $u_1(x)$ $(\vartheta_1(x))$ closest to (but different from) the point 0. On the basis of the preceding theorem it suffices to prove that $u_2(x)$ $(\vartheta_2(x))$ has at least one zero in the interval $(0,\ x_1)$. Assume the contrary, i.e. let $u_2(x) \neq 0$ $(\vartheta_2(x) \neq 0)$ for $x \in (0,\ x_1)$. Integrating the identity (2.3) ((2.4)) from 0 to x_1, we obtain

$$\{(u_1/u_2)\ (u_1\vartheta_2 - u_2\vartheta_1)\}_0^{x_1} =$$

$$= \int_0^{x_1} \{(r_1/r_2)\,(r_2 - r_1)\ \vartheta_1^2 + (p_2 - p_1)\ u_1^2 + r_2\,((u_1/u_2)\vartheta_2 - (r_1/r_2)\vartheta_1)^2\}\,dx,$$

$$(\{(\vartheta_1/\vartheta_2)\ (u_1\vartheta_2 - u_2\vartheta_1)\}_0^{x_1} =$$

$$= \int_0^{x_1} \{(p_1/p_2)\,(p_2 - p_1)\ u_1^2 + (r_2 - r_1)\ \vartheta_1^2 + p_2\,((\vartheta_1/\vartheta_2)u_2 - (p_1/p_2)u_1)^2\}\,dx)$$

which implies by (2.5) the left-hand side integrates to zero while the right-hand side integrates to a positive number, or negative number. This contradiction proves the Theorem 2.2.

One can readily show that there exists a unique solution $(u(x,\lambda), \vartheta(x,\lambda))$ of system (1.1) satisfying the initial condition

$$u(0,\lambda) = \cos\alpha, \ \ \vartheta(0,\lambda) = -\sin\alpha, \tag{2.6}$$

moreover, for each fixed $x \in [0,\pi]$, the functions $u(x,\lambda)$ and $\vartheta(x,\lambda)$ are entire functions of the argument λ. The proof of this assertion reproduces that of Theorem 1.1 in [10, p. 3] with obvious modifications.

Since the functions $u(x,\lambda)$ and $\vartheta(x,\lambda)$ satisfy the boundary condition (1.2), to find the eigenvalues of the boundary value problem (1.1)-(1.3) we have to insert the functions $u(x,\lambda)$ and $\vartheta(x,\lambda)$ in the boundary condition (1.3) and find the roots of this equation. So, the eigenvalues of problem (1.1)-(1.3) are the roots of the following equation

$$\vartheta(\pi,\lambda)\cos\beta + u\,(\pi,\lambda)\sin\beta = 0. \tag{2.7}$$

Let

$$M = \inf\{\lambda \in \mathbb{R} \,:\, \lambda + p(x) > 0, \ \lambda + r(x) > 0, x \in [0,\pi]\}$$

and

$$m = \sup\{\lambda \in \mathbb{R} \,:\, \lambda + p(x) < 0, \ \lambda + r(x) < 0, \ x \in [0,\pi]\}.$$

From Theorems 2.1 and 2.2 imply

Corollary 2.1. *If $\lambda'' > \lambda' > M$, or $\lambda'' < \lambda' < m$, then the function $u(x,\lambda'')$ $(\vartheta(x,\lambda''))$ in the interval $0 < x \leq \pi$ has at least the same number of zeros, how many and function $u(x,\lambda')$ $(\vartheta(x,\lambda'))$, and the kth zero of $u(x,\lambda'')$ $(\vartheta(x,\lambda''))$ is less than the kth zero of $u(x,\lambda')(\vartheta(x,\lambda'))$.*

Consider the equation

$$u(x,\lambda) = 0 \quad (\vartheta(x,\lambda) = 0), \quad 0 \leq x \leq \pi. \tag{2.8}$$

The zeros of this equation are obviously functions of λ.

In the following two statements, we assume that $\lambda \notin (m, M)$.

Lemma 2.1. *If x_0 $(0 < x_0 < \pi)$ is a zero of function $u(x,\lambda_0)\,(\vartheta(x,\lambda_0))$, then for any sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ the function $u(x,\lambda)$ $(\vartheta(x,\lambda))$ has exastly one zero in the interval $|x - x_0| < \varepsilon$.*

The proof of this fact is similar to the proof of Lemma 3.1 [10, p. 16].

From the Lemma 2.1 there follows an important corollary.

Corollary 2.2. *As λ varies, the function $u(x,\lambda)\,(\vartheta(x,\lambda))$ can lose zeros or gain zeros only by these zeros leasving or entering the interval $[0,\pi]$ through its endpoints 0 and π.*

3. Oscillation properties of the eigenvector-functions of the problem (1.1)-(1.3)

Now consider the problem (1.1)-(1.3) for $p(x) \equiv r(x) \equiv 0$, i.e. consider the problem

$$\vartheta' - \lambda\, u = 0, \ \ u' + \lambda\vartheta = 0, \ \ 0 < x < \pi, \tag{3.1}$$

$$\vartheta(0)\cos\alpha + u(0)\sin\alpha = 0,$$

$$\vartheta(\pi)\cos\beta + u\,(\pi)\sin\beta = 0.$$

As is difficult to see, in this case

$$u(x,\lambda)=\cos(\lambda x-\alpha),\ \vartheta(x,\lambda)=\sin(\lambda x-\alpha). \tag{3.2}$$

As was already mentioned above, the eigenvalues of the boundary value problem (1.1)-(1.3) coincide with the roots of the equation (2.7). Then, by (3.2), we have

$$\sin(\lambda\pi-\alpha)\cos\beta+\cos(\lambda\pi-\alpha)\sin\beta=0$$

which implies that

$$\sin(\lambda\pi-\alpha+\beta)=0\,.$$

Consequently, the eigenvalues of problem (3.1), (1.2), (1.3) are

$$\lambda_n=n+(\alpha-\beta)/\pi,\ \ n=0,\,\pm1,\,\pm2,\ \ldots\,,$$

and the corresponding eigenvector-functions are

$$(u_n(x),\,\vartheta_n(x))=(\cos(\lambda_n x-\alpha),\sin(\lambda_n x-\alpha))=$$

$$=(\cos(n+((\alpha-\beta)/\pi)x-\alpha),\sin(n+((\alpha-\beta)/\pi)x-\alpha)),\ \ n=0,\,\pm1,\,\pm2,\ \ldots\,.$$

Remark 3.1. We have: $\lambda_0>0$ for $\alpha>\beta$, $\lambda_0=0$ for $\alpha=\beta$, $\lambda_0<0$ for $\alpha<\beta$.

It is known (e.g. [10, p. 57]) that eigenvalues of the boundary value problem (1.1)-(1.3) are real and simple and the values range from $-\infty$ to $+\infty$ and can be numerated in increasing order:

$$\ldots<\lambda_{-n}<\ldots<\lambda_{-1}<\lambda_0<\lambda_1<\ldots.<\lambda_n<\ldots.$$

Denote by μ_n and $\nu_n,\ n=0,\ \pm1,\pm2,\ldots$ the eigenvalues of the problem (1.1)-(1.3) for $\beta=0$ and $\beta=\pi/2$, respectively. Note that the function

$$G(\lambda)=\frac{u(\pi,\lambda)}{\vartheta(\pi,\lambda)}$$

is defined for

$$\lambda\in\mathrm{B}\equiv(\mathbb{C}\backslash\mathbb{R})\bigcup\left(\bigcup_{n=-\infty}^{n=+\infty}(\mu_{n-1},\mu_n\right)$$

and is meromorphic function finite order, and μ_n and $\nu_n,\ n=0,\ \pm1,\pm2,\ldots$, are poles and zeros of this function respectively.

Lemma 3.1. *The following formula holds:*

$$\frac{\partial}{\partial\lambda}\left(\frac{u(\pi,\lambda)}{\vartheta(\pi,\lambda)}\right)=-\frac{\int_0^\pi\{u^2(x,\lambda)+\vartheta^2(x,\lambda)\}\,dx}{\vartheta^2(\pi,\lambda)},\quad \lambda\in\mathrm{B}. \tag{3.3}$$

Proof. Since vector-functions $(u(x,\mu),\ \vartheta(x,\mu))$ and $(u(x,\lambda),\ \vartheta(x,\lambda))$, $\mu,\ \lambda\in\mathrm{B}$, are solutions of the problem (1.1), we have

$$\begin{aligned}
&\vartheta'(x,\mu)-\{\mu+p(x)\}\,u(x,\mu)=0,\\
&u'(x,\mu)+\{\mu+r(x)\}\,\vartheta(x,\mu)=0,\\
&\vartheta'(x,\lambda)-\{\lambda+p(x)\}\,u(x,\lambda)=0,\\
&u'(x,\lambda)+\{\lambda+r(x)\}\,\vartheta(x,\lambda)=0.
\end{aligned}$$

Multiplying these equations by $u(x,\lambda)$,$-\vartheta(x,\lambda)$,$-u(x,\mu)$ and $\vartheta(x,\mu)$, respectively, and adding, we obtain

$$\frac{d}{dx}\{\vartheta(x,\mu)u(x,\lambda)-u(x,\mu)\vartheta(x,\lambda)\}=(\mu-\lambda)\{u(x,\mu)\,u(x,\lambda)+\vartheta(x,\mu)\,\vartheta(x,\lambda)\}\,.$$

Integrating this relation from 0 to π, we find that

$$\{\vartheta(x,\mu)u\,(x,\lambda) - u(x,\mu)\vartheta(x,\lambda)\}_0^\pi =$$

$$= (\mu-\lambda)\int_0^\pi \{u\,(x,\mu)\,u\,(x,\lambda) + \vartheta(x,\mu)\,\vartheta(x,\lambda)\}\,dx,$$

whence, by (2.6), it follows that

$$\vartheta(\pi,\mu)u\,(\pi,\lambda) - u\,(\pi,\mu)\,\vartheta(\pi,\lambda) =$$

$$= (\mu-\lambda)\int_0^\pi \{u\,(x,\mu)\,u\,(x,\lambda) + \vartheta(x,\mu)\,\vartheta(x,\lambda)\}\,dx\,.$$

Thus,

$$-\left(\frac{u\,(\pi,\mu)}{\vartheta(\pi,\mu)} - \frac{u\,(\pi,\lambda)}{\vartheta(\pi,\lambda)}\right) = (\mu-\lambda)\frac{\int_0^\pi \{u\,(x,\mu)\,u\,(x,\lambda) + \vartheta(x,\mu)\,\vartheta(x,\lambda)\}\,dx}{\vartheta(\pi,\mu)\,\vartheta(\pi,\lambda)}.$$

Dividing this equality by $(\mu-\lambda)$ and passing to the limit as $\mu \to \lambda$ we obtain (3.3). The Lemma 3.1 is proved.

Corollary 3.1. *The function $G(\lambda)$ is continuous and strictly decreasing on each interval (μ_{n-1},μ_n), $n = 0, \pm1, \pm2, \ \dots$.*

By $m\,(\lambda)$ and $s\,(\lambda)$, $\lambda \in \mathbb{R}$, we denote the number of zeros in the interval $(0,\pi)$ of functions $u(x,\lambda)$ and $\vartheta(x,\lambda)$, respectively.

Lemma 3.2. Let $p(x) \equiv r(x) \equiv 0$. If $\lambda \in (\mu_{n-1},\mu_n]$ for $n > 0$, then

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (n-1, n-1) & \text{for } \lambda \in (\mu_{\mathrm{n}-1}, \nu_{\mathrm{n}}], \\ (n, n-1) & \text{for } \lambda \in (\nu_{\mathrm{n}}, \mu_{\mathrm{n}}], \end{cases} \quad \text{in the case } \alpha = 0,$$

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (n-1, n) & \text{for } \lambda \in (\mu_{\mathrm{n}-1}, \nu_{\mathrm{n}}], \\ (n, n) & \text{for } \lambda \in (\nu_{\mathrm{n}}, \mu_{\mathrm{n}}], \end{cases} \quad \text{in the case } \alpha \in (0, \pi/2],$$

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (n, n) & \text{for } \lambda \in (\mu_{\mathrm{n}-1}, \nu_{\mathrm{n}}], \\ (n+1, n) & \text{for } \lambda \in (\nu_{\mathrm{n}}, \mu_{\mathrm{n}}], \end{cases} \quad \text{in the case } \alpha \in (\pi/2, \pi)\,;$$

if $\lambda \in [\mu_{n-1},\mu_n)$ for $n < 0$, then

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (|n|+1, |n|) & \text{for } \lambda \in [\mu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \\ (|n|, |n|) & \text{for } \lambda \in [\nu_{\mathrm{n}}, \mu_{\mathrm{n}}), \end{cases} \quad \text{in the case } \alpha \in [0, \pi/2),$$

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (|n|, |n|) & \text{for } \lambda \in [\mu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \\ (|n|-1, |n|) & \text{for } \lambda \in [\nu_{\mathrm{n}}, \mu_{\mathrm{n}}), \end{cases} \quad \text{in the case } \alpha \in [\pi/2, \pi)\,;$$

if $\lambda \in [\mu_{-1},\mu_0)$, then

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (1, 0) & \text{for } \lambda \in [\mu_{-1}, \nu_0), \\ (0, 0) & \text{for } \lambda \in [\nu_0, \mu_0), \end{cases} \quad \text{in the case } \alpha \in [0, \pi/2),$$

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (0, 0) & \text{for } \lambda \in [\mu_{-1}, \nu_0), \\ (0, 0) & \text{for } \lambda \in (\nu_0, \mu_0), \end{cases} \quad \text{in the case } \alpha = \pi/2,$$

$$(m\,(\lambda), s(\lambda)) = \begin{cases} (0, 0) & \text{for } \lambda \in [\mu_{-1}, \nu_0), \\ (1, 0) & \text{for } \lambda \in [\nu_0, \mu_0), \end{cases} \quad \text{in the case } \alpha \in [\pi/2, \pi)\,.$$

Proof. Let $p(x) \equiv r(x) \equiv 0$. In this case

$$\mu_n = n + (\alpha/\pi) \in [n, n+1), \ (u_n(x), \vartheta_n(x)) = (\cos(\mu_n x - \alpha), \, \sin(\mu_n x - \alpha)),$$

$$\nu_n = n - \frac{1}{2} + \frac{\alpha}{\pi} \in [n - \frac{1}{2}, n + \frac{1}{2}), \ (u_n(x), \vartheta_n(x)) = (\cos(\nu_n x - \alpha), \, \sin(\nu_n x - \alpha)).$$

The following relation is valid:

$$... < \nu_{-1} < \mu_{-1} < \nu_0 < \mu_0 < \nu_1 < \mu_1 < ... < \nu_n < \mu_n < ... . \tag{3.4}$$

Note that $\mu_0 = 0$ and $(u_0(x), \vartheta_0(x)) = (1, 0)$ for $\alpha = 0$. It is obvious that if $\alpha \in (0, \pi)$, then $\mu_0 x - \alpha \in (-\alpha, 0)$. Therefore,

$$(m\,(\mu_0), s(\mu_0)) = \begin{cases} (0,0) & \text{for } \alpha \in (0, \pi/2], \\ (1,0) & \text{for } \alpha \in (\pi/2, \pi). \end{cases} \tag{3.5}$$

Obviously, $\mu_n x - \alpha \in (-\alpha, n\pi)$, if $n > 0$, $\mu_n x - \alpha \in (n\pi, -\alpha)$, if $n < 0$. Consequently, the following relations hold:

$$(m\,(\mu_n), s(\mu_n)) = \begin{cases} (n, n-1) & \text{for } \alpha = 0, \\ (n, n) & \text{for } \alpha \in (0, \pi/2], \\ (n+1, n) & \text{for } \alpha \in (\pi/2, \pi), \end{cases} \quad \text{in the case } n > 0, \tag{3.6}$$

$$(m(\mu_n), s(\mu_n)) = \begin{cases} (|n|, |n|-1) & \text{for } \alpha \in [0, \pi/2), \\ (|n|-1, |n|-1) & \text{for } \alpha \in [\pi/2, \pi), \end{cases} \quad \text{in the case } n < 0. \tag{3.7}$$

Note that $\nu_0 = 0$ and $(u_0(x), \vartheta_0(x)) = (0, -1)$ for $\alpha = \pi/2$. Moreover, if $\alpha \in [0, \pi/2)$, then $(\nu_0 x - \alpha) \in (-\pi/2, -\alpha)$; if $\alpha \in (\pi/2, \pi)$, then $(\nu_0 x - \alpha) \in (-\alpha, -\pi/2)$. Therefore,

$$(m\,(\nu_0), s(\nu_0)) = \begin{cases} (0,0) & \text{for } \alpha \in [0, \pi/2), \\ (0,0) & \text{for } \alpha \in (\pi/2, \pi). \end{cases} \tag{3.8}$$

It is easy to verify that $(\nu_n x - \alpha) \in (-\alpha, (n-1/2)\pi)$, if $n > 0$; $(\nu_n x - \alpha) \in (-\alpha, (n-1/2)\pi)$, if $n < 0$. Consequently, the following relations hold:

$$(m\,(\nu_n), s(\nu_n)) = \begin{cases} (n-1, n-1) & \text{for } \alpha = 0, \\ (n-1, n) & \text{for } \alpha \in (0, \pi/2], \\ (n, n) & \text{for } \alpha \in (\pi/2, \pi), \end{cases} \quad \text{in the case } n > 0, \tag{3.9}$$

$$(m\,(\nu_n), s(\nu_n)) = \begin{cases} (|n|, |n|) & \text{for } \alpha \in [0, \pi/2), \\ (|n|-1, |n|) & \text{for } \alpha \in [\pi/2, \pi), \end{cases} \quad \text{in the case } n < 0. \tag{3.10}$$

The lemma is obtained by applying Corollary 2.1, using the relations (3.4)-(3.10). The proof of Lemma 3.2 is complete.

Theorem 3.1. *Eigenvector-functions* $(u_n(x), \vartheta_n(x))$, $n \in \mathbb{Z}$, *of the problem* (3.1), (1.2), (1.3), *corresponding to the eigenvalues* λ_n, *have the following oscillation properties:*

a_1) *if* $\alpha = 0$, $\beta \in [0, \pi/2)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (n,\, n-1)$ *for* $n > 0$*;*

b_1) *if* $\alpha \in (0, \pi/2]$, $\beta \in [0, \pi/2)$, *or* $\alpha \in (\pi/2, \pi)$, $\beta \in [\pi/2, \pi)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (n,\, n)$ *for* $n \geq 0$;

c_1) *if* $\alpha = 0$, $\beta \in [\pi/2, \pi)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (n-1,\, n-1)$ *for* $n > 0$;

d_1) *if* $\alpha \in (0, \pi/2]$, $\beta \in [\pi/2, \pi)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (n-1,\, n)$ *for* $n > 0$;

e_1) *if* $\alpha \in (\pi/2, \pi)$, $\beta \in [0, \pi/2)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (n+1,\, n)$ *for* $n \geq 0$;

a_2) *if* $\alpha \in [0, \pi/2)$, $\beta = 0$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (|n|,\, |n|-1)$ *for* $n < 0$;

b_2) *if* $\alpha \in [0, \pi/2)$, $\beta \in (0, \pi/2]$, *or* $\alpha \in [\pi/2, \pi)$, $\beta \in (\pi/2, \pi)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (|n|,\, |n|)$ *for* $n \leq 0$;

c_2) *if* $\alpha \in [0, \pi/2)$, $\beta \in (\pi/2, \pi)$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (|n|+1,\, |n|)$ *for* $n \leq 0$;

d_2) *if* $\alpha \in [\pi/2, \pi)$, $\beta = 0$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (|n|-1,\, |n|-1)$ *for* $n < 0$;

e_2) *if* $\alpha \in [\pi/2, \pi)$, $\beta \in (0, \pi/2]$, *then* $(m\,(\lambda_n), s\,(\lambda_n)) = (|n|-1,\, |n|)$ *for* $n < 0$.

Proof. By virtue of (3.4) and Lemma 3.1 the following location of the eigenvalues of problem (3.1), (1.2), (1.3) on the real axis is true:

$$... < \mu_{-2} < \nu_{-1} < \lambda_{-1} < \mu_{-1} < \nu_0 < \lambda_0 < \mu_0 < \nu_1 < \lambda_1 < \mu_1 < \; ..., \tag{3.11}$$

if $\beta \in (0, \pi/2)$,

$$... < \mu_{-2} < \; \lambda_{-1} < \nu_{-1} < \mu_{-1} < \lambda_0 < \nu_0 < \mu_0 < \lambda_1 < \nu_1 < \mu_1 < ..., \tag{3.12}$$

if $\beta \in (\pi/2, \pi)$.

From Lemma 3.2 and relations (3.11), (3.12), we obtain:

(1) if $\alpha = 0$, then

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n-1) & \text{for } n > 0, \\ (|n|, |n|-1) & \text{for } n < 0, \end{cases} \text{ in the case } \beta = 0,$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n-1) & \text{for } n > 0, \\ (|n|, |n|) & \text{for } n \le 0, \end{cases} \text{ in the case } \beta \in (0, \pi/2),$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n-1, n-1) & \text{for } n > 0, \\ (|n|, |n|) & \text{for } n \le 0, \end{cases} \text{ in the case } \beta = \pi/2,$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n-1, n-1) & \text{for } n > 0, \\ (|n|+1, |n|) & \text{for } n \le 0, \end{cases} \text{ in the case } \beta \in (\pi/2, \pi);$$

(2) if $\alpha \in (0, \pi/2)$, then

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n) & \text{for } n \ge 0, \\ (|n|, |n|-1) & \text{for } n < 0, \end{cases} \text{ in the case } \beta = 0,$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n) & \text{for } n \ge 0, \\ (|n|, |n|) & \text{for } n < 0, \end{cases} \text{ in the case } \beta \in (0, \pi/2),$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n-1, n) & \text{for } n > 0, \\ (|n|, |n|) & \text{for } n \le 0, \end{cases} \text{ in the case } \beta = \pi/2,$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n-1, n) & \text{for } n > 0, \\ (|n|+1, |n|) & \text{for } n \le 0, \end{cases} \text{ in the case } \beta \in (\pi/2, \pi);$$

(3) if $\alpha = \pi/2$, then

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n) & \text{for } n \ge 0, \\ (|n|-1, |n|-1) & \text{for } n < 0, \end{cases} \text{ in the case } \beta = 0,$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n) & \text{for } n \ge 0, \\ (|n|-1, |n|) & \text{for } n < 0, \end{cases} \text{ in the case } \beta \in (0, \pi/2),$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n-1, n) & \text{for } n > 0, \\ (|n|-1, |n|) & \text{for } n < 0, \end{cases} \text{ in the case } \beta = \pi/2,$$

$$(m\,(\lambda_n), s(\lambda_n)) = \begin{cases} (n-1, n) & \text{for } n > 0, \\ (|n|, |n|) & \text{for } n \le 0, \end{cases} \text{ in the case } \beta \in (\pi/2, \pi);$$

(4) if $\alpha \in (\pi/2, \pi)$, then

$$(m(\lambda_n), s(\lambda_n)) = \begin{cases} (n+1, n) & \text{for } n \geq 0, \\ (|n|-1, |n|-1) & \text{for } n < 0, \end{cases} \text{ in the case } \beta = 0,$$

$$(m(\lambda_n), s(\lambda_n)) = \begin{cases} (n+1, n) & \text{for } n \geq 0, \\ (|n|-1, |n|) & \text{for } n < 0, \end{cases} \text{ in the case } \beta \in (0, \pi/2),$$

$$(m(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n) & \text{for } n \geq 0, \\ (|n|-1, |n|) & \text{for } n < 0, \end{cases} \text{ in the case } \beta = \pi/2,$$

$$(m(\lambda_n), s(\lambda_n)) = \begin{cases} (n, n) & \text{for } n > 0, \\ (|n|, |n|) & \text{for } n \leq 0, \end{cases} \text{ in the case } \beta \in (\pi/2, \pi).$$

Assertions $a_1) - e_2)$ of this theorem follow directly from the relations (1)-(4). The Theorem 3.1 is proved.

Now consider the following boundary value problem

$$\vartheta' - \{\lambda + \mu\, p(x)\}\, u = 0, \;\; u' + \{\lambda + \mu\, r(x)\} = 0, \;\; 0 < x < \pi,$$

$$\vartheta(0) \cos\alpha + u(0) \sin\alpha = 0, \tag{3.13}$$

$$\vartheta(\pi) \cos\beta + u(\pi) \sin\beta = 0,$$

where $0 \leq \mu \leq 1$.

Remark 3.2. By Theorem 7.1 of [3, Ch. 3] (see also [6]) on the continuous dependence of the solution of system of differential equations on the parameter we find that the eigenvalues $\lambda_n(\mu)$, $n \in \mathbb{Z}$, of the problem (3.13) depends continuously on the parameter $\mu \in [0, 1]$. In this $\lambda_n(0)$ and $\lambda_n(1)$, $n \in \mathbb{Z}$, coincide with the eigenvalues of the problems (3.1), (1.2), (1.3) and (1.1)-(1.3), respectively. Therefore we can assume that for the eigenvalues of problem (1.1)-(1.3) also true the relation (3.4).

The following oscillation theorem for Dirac systems (1.1)-(1.3) is valid.

Theorem 3.2. *There exist numbers $m_0 \in \mathbb{N} \cup \{0\}$ and $m_1 \in \mathbb{Z}_- \equiv -\mathbb{N}$ such that for eigenvector-functions of the problem* (1.1)-(1.3) *at $n \geq m_0$ are valid assertions $a_1) - e_1)$, when $n \leq m_1$ are valid assertions $a_2) - e_2)$ of Theorem 3.1.*

Proof. Let $\lambda \notin (m, M)$, and let $(u(x, \lambda), \vartheta(x, \lambda))$ be a solution of system (1.1) which satisfies the initial condition (2.6).

Suppose that $(\varphi(x, \lambda), \psi(x, \lambda))$ is a solution of the system of differential equations

$$\psi' - (\lambda - M)\, \varphi = 0, \;\; \varphi' + (\lambda - M)\, \psi = 0, \tag{3.14}$$

satisfying the initial condition

$$\varphi(0, \lambda) = \cos\alpha, \;\; \psi(0, \lambda) = -\sin\alpha, \tag{3.15}$$

and $(\gamma(x, \lambda), \chi(x, \lambda))$ is a solution of the system of differential equations

$$\chi' - (\lambda - m)\, \gamma = 0, \;\; \gamma + (\lambda - m)\, \chi = 0, \tag{3.16}$$

satisfying the initial condition

$$\gamma(0, \lambda) = \cos\alpha, \;\; \chi(0, \lambda) = -\sin\alpha. \tag{3.17}$$

It is easy to verify that in this cases

$$\varphi(x,\lambda) = \cos((\lambda - M)x - \alpha), \quad \psi(x,\lambda) = \sin((\lambda - M)\, x - \alpha), \tag{3.18}$$

$$\gamma(x,\lambda) = \cos((\lambda - m)\, x - \alpha), \quad \psi(x,\lambda) = \sin((\lambda - m)\, x - \alpha). \tag{3.19}$$

From (3.18) and (3.19) see that the number of zeros in $(0,\pi)$ of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ tends to $+\infty$ as $\lambda \to +\infty$, the number of zeros in $(0,\pi)$ of the functions $\gamma(x,\lambda)$ and $\chi(x,\lambda)$ tends to $+\infty$ as $\lambda \to -\infty$. We compare the boundary value problem (1.1), (2.6) with the problems (3.14)-(3.15) and (3.16)-(3.17). Then from Theorem 2.2 implies that the number of zeros in $(0,\pi)$ of the solutions $u(x,\lambda)$ and $\vartheta(x,\lambda)$ of the problem (1.1), (2.6) tends to $+\infty$ as $\lambda \to \pm\infty$.

Let us consider the equations $u(x,\lambda) = 0$ and $\vartheta(x,\lambda) = 0$ for $\lambda \in (-\infty, m] \bigcup [M, +\infty)$. By Lemma 3.1 the roots of these equations depend continuously on λ. On the other hand, by Corollary 2.1 with increasing $|\lambda|$ each zero of $u(x,\lambda)$ and $\vartheta(x,\lambda)$ moves to the left, and through the point 0 can not leave because the number of zeros does not decrease. Then, by Corollary 2.2 zeros of these functions entering the interval $(0,\pi)$ through point π.

Let $m_0 \geq 0$ and $m_1 < 0$ are numbers such that μ_{m_0-1} is first value of parameter $\lambda \geq M$ and μ_{m_1} is first value of parameter $\lambda \leq m$ for which $\vartheta(\pi,\lambda) = 0$. We recall that $\mu_n, n \in \mathbb{Z}$, are the eigenvalues of the problem (1.1)-(1.3) for $\beta = 0$. Assume that the function $\vartheta(x, \mu_{m_0-1})$ has $\tilde{m}_0$ zeros and $\vartheta(x, \mu_{m_1})$ has $\tilde{m}_1$ zeros in the interval $(0,\pi)$.

By Corollary 2.1 the function $\vartheta(x, \mu_{m_0-1+k})$ has $\tilde{m}_0 + k$ zeros and $\vartheta(x, \mu_{m_1-k})$ has $\tilde{m}_1 + k$ zeros in the interval $(0,\pi)$. On the base Theorem 2.2 the sequences $\mu_{m_0-1}, \mu_{m_0}, ...$ and $..., \mu_{m_1-1}, \mu_{m_1}$ have the property that the function $\vartheta(x,\lambda)$ for $\mu_{k-1} < \lambda \leq \mu_k$, $k = m_0, m_0+1, m_0+2, ...$, and for $\mu_{k-1} \leq \lambda < \mu_k$, $k = ..., m_1 - 2, m_1 - 1, m_1$, has $\tilde{k}$ zeros in the interval $(0,\pi)$.

From (2.7) we see that the eigenvalues of problem (1.1)-(1.3) for $\beta \in (0,\pi)$ are the roots of the equation

$$G(\lambda) = -\cot\beta.$$

By virtue of Corollary 3.1 the function $G(\lambda)$ is continuous and strictly decreasing on each interval (μ_{k-1}, μ_k), $k \in \mathbb{Z}$. Since $\vartheta(\pi, \mu_k) = 0$, $k \in \mathbb{Z}$, then in the interval (μ_{k-1}, μ_k) the function $G(\lambda)$ should strictly decrease from $+\infty$ to $-\infty$. Hence there exists a unique $\lambda = \lambda_k^*$, $k \in \mathbb{Z}$, such that $G(\lambda) = -\cot\beta$, i.e. condition (1.3) is satisfied. Therefore, λ_k^* is an eigenvalue of the boundary value problem (1.1)-(1.3) and $(u(x,\lambda_k^*), \vartheta(x,\lambda_k^*))$ is the corresponding eigenvector-function. Moreover, the second component $\vartheta(x,\lambda_k^*)$ of this eigenvector-function for $k \geq m_0$ has as many zeros as the function $\vartheta(x,\mu_k)$, and for $k \leq m_1$ has as many zeros as the function $\vartheta(x,\mu_{k-1})$ in the interval $(0,\pi)$. By Remark 3.2 one can readily see that λ_k^* is the kth eigenvalue of the boundary value problem (1.1)-(1.3); i.e. $\lambda_k = \lambda_k^*$.

The eigenvalues ν_k, $k \in \mathbb{Z}$, of the boundary value problem (1.1)-(1.3) for $\beta = \pi/2$ are zeros of the function $G(\lambda)$. In a similar way one can show that the equation $G(\lambda) = 0$ has the unique solution ν_k in each interval (μ_{k-1}, μ_k), $k \in \mathbb{Z}$. Consequently,

$$\mu_{k-1} < \nu_k < \mu_k, \;\; k \in \mathbb{Z}.$$

Moreover, the following relations are valid for $k \in \mathbb{Z}$:

$$\mu_{k-1} < \nu_k < \lambda_k < \mu_k, \text{ if } \beta \in (0,\pi/2), \tag{3.20}$$

$$\mu_{k-1} < \lambda_k < \nu_k < \mu_k, \text{ if } \beta \in (\pi/2, \pi). \tag{3.21}$$

From the relations (3.20) and (3.21) it follows that if $u(x, \nu_k)$ has k^* zeros in the interval $(0, \pi)$, then $u(x, \lambda_k)$ has $k^* + 1$ zeros at $\beta \in (0, \pi/2)$, has k^* zeros at $\beta \in (\pi/2, \pi)$ for $k \geq m_0$, and has k^* zeros at $\beta \in (0, \pi/2)$, has $k^* + 1$ zeros at $\beta \in (\pi/2, \pi)$ for $k \leq m_1$ in the same interval $(0, \pi)$.

Note that the formula (11.18) of [10, Ch. 1] has an error. This is due to the fact that in the formula (11.9) [10, Ch. 1] the expression for the function $\beta(x)$ to be a minus sign, whereby the formula (11.12) [10, Ch. 1] can be the following form

$$\xi(x, \lambda) = \lambda x + (1/2) \int_0^x \{p(t) + r(t)\} dt. \tag{3.22}$$

By Lemma 11.1 [10, Ch. 1], we have that the following estimates hold uniformly with respect to x, $0 \leq x \leq \pi$:

$$u(x, \lambda) = \cos(\xi(x, \lambda) - \alpha) + O(1/\lambda), \tag{3.23}$$

$$\vartheta(x, \lambda) = \sin(\xi(x, \lambda) - \alpha) + O(1/\lambda). \tag{3.24}$$

As was already mentioned above, the eigenvalues of problem (1.1)-(1.3) coincide with the roots of the equation (2.7). Inserting now the values of the functions $u(\pi, \lambda)$ and $\vartheta(\pi, \lambda)$ from the estimates (3.23) and (3.24), we obtain

$$\sin(\xi(\pi, \lambda) - \alpha + \beta) + O(1/\lambda) = 0,$$

whence, by (3.22), it follows that

$$\sin\left(\lambda\pi - \alpha + \beta + (1/2) \int_0^\pi \{p(t) + r(t)\} dt\right) + O(1/\lambda) = 0. \tag{3.25}$$

Further, following the corresponding arguments conducted in [10, p. 57], by (3.25), we obtain

$$\lambda_n = n + \frac{\alpha - \beta - (1/2) \int_0^\pi \{p(t) + r(t)\} dt}{\pi} + O\left(\frac{1}{n}\right). \tag{3.26}$$

Using the formula (3.26), we obtain an asymptotic formula for the eigenvector-functions, $u(x, \lambda_n) = u_n(x)$, $\vartheta(x, \lambda_n) = \vartheta_n(x)$, namely:

$$u_n(x) = \cos(\lambda_n x + (1/2) \int_0^x \{p(t) + r(t)\} dt - \alpha) + O(1/n), \tag{3.27}$$

$$\vartheta_n(x) = \sin(\lambda_n x + (1/2) \int_0^x \{p(t) + r(t)\} dt - \alpha) + O(1/n). \tag{3.28}$$

By (3.26) for sufficiently large $|n|, n \in \mathbb{Z}$, we have that for $x \in (0, \pi)$

$$(\lambda_n x + (1/2) \int_0^x \{p(t) + r(t)\} dt - \alpha) \in (-\alpha, n\pi - \beta + O(1/n)),$$

if $n \geq m_0$,

$$(\lambda_n x + (1/2) \int_0^x \{p(t) + r(t)\} dt - \alpha) \in (n\pi - \beta + O(1/n), -\alpha),$$

if $n \leq m_1$. Since for $x \in (0, \pi)$

$$(n + (\alpha - \beta)/\pi)x - \alpha) \in (-\alpha, n\pi - \beta), \text{ if } n \geq m_0$$

and

$$(n + (\alpha - \beta)/\pi)x - \alpha) \in (n\pi - \beta, \ -\alpha), \text{ if } n \leq m_1,$$

then following the corresponding arguments given in the proof of Theorem 3.1 in [9], we see that for sufficiently large $|n|, n \in \mathbb{Z}$, the number of zeros in the interval $(0, \pi)$ of the eigenvector- functions

$$\cos(\lambda_n x + (1/2)\int_0^x \{p(t) + r(t)\}dt - \alpha) + O(1/n)$$

and

$$\sin(\lambda_n x - (1/2)\int_0^x \{p(t) + r(t)\}dt - \alpha) + O(1/n))$$

(whence obtained from (3.27), (3.28)) of the problem (1.1)-(1.3) coincide with the number of zeros of the eigenvector-functions

$$\cos\left((n + (\alpha + \beta)/\pi)x - \alpha\right) \text{ and } \sin\left((n + (\alpha - \beta)/\pi)x - \alpha\right),$$

respectively. Therefore, given the above mentioned arguments, we obtain that for the problem (1.1)-(1.3) at $n \geq m_0$ are valid assertions $a_1) - e_1)$, when $n \leq m_1$ are valid assertions $a_2) - e_2)$ of the theorem 3.1. The proof of Theorem 3.2 is complete.

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