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ON KOSTYUCHENKO PROBLEM

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In memory of M. G. Gasymov on his 75th birthday

Abstract. This work is dedicated to the Kostyuchenko problem, well-known in the spectral theory of differential operators. This is a review paper that covers the main results on this problem. The authors reveal the essence of Kostyuchenko problem and discuss the difficulties that arise when treating it. Unsolved matters concerning this problem are also mentioned.

1. Introduction

Kostyuchenko problem (hereinafter referred to as Problem $\mathscr{K}$) consists in studying the completeness of the system $\left\{e^{i\alpha nt}\sin nt\right\}_{n\in\mathbb{N}}$ in $L_2(0,\pi)$ by purely functional and theoretical methods, where $\alpha\in\mathbb{C}$ is in general some complex parameter. Sometimes this system is referred to as Kostyuchenko system. It is a part (some authors prefer to say "a half") of root elements of the quadratic pencil

$$\begin{gathered} f''+2a\lambda f'+b\lambda^2 f=0; \\ f(0)=f(\pi)=0, \end{gathered} \tag{1.1}$$

where $\alpha=-a\sqrt{b-a^2}$. Such pencils appear when solving some partial differential equations by the Fourier method. Let's illustrate the scheme of this method with a following model problem (see [31]).

Let $\mathscr{P}_\theta=\{(x;y)\in\mathbb{C}: x+iy=\tilde{x}+te^{i\theta}, \tilde{x}\in(0,\pi), t>0\}$ be an inclined half-string (see Fig. 1), where $\theta\in(0,\pi)$ is some number.

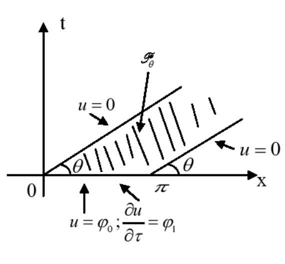

Fig. 1.

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Consider in $\mathscr{P}_\theta$ the Laplace equation

$$\Delta u(x;y)=0,\ (x;y)\in\mathscr{P}_\theta, \tag{1.2}$$

with the boundary conditions

$$u\left(t\cos\theta;t\sin\theta\right)=u\left(\pi+t\cos\theta;t\sin\theta\right)=0,\ \ t>0;$$

$$u\left(x;0\right)=\varphi_1\left(x\right)\ ;\ \left.\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)\right|_{(x;0)}=\varphi_2\left(x\right).$$

To apply the Fourier method to this problem, we make a change of variables $(x;y)\to(z;\tau):z=x-y\,ctg\,\theta\,;\ \tau=\frac{y}{\sin\theta}$. Then the half-string $\mathscr{P}_\theta$ in the plane $(x;y)$ becomes a half-string $\mathscr{P}$ in the plane $(z;\tau)$, where

$$\mathscr{P}\equiv\left\{(z;\tau):z\in(0,\pi),\tau>0\right\}.$$

Simple calculations show that the following relations hold

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial^2u}{\partial z^2},$$

$$\frac{\partial^2u}{\partial y^2}=ctg^2\theta\frac{\partial^2u}{\partial z^2}-2\frac{\cos\,\theta}{\sin^2\theta}\frac{\partial^2u}{\partial z\partial\tau}+\frac{1}{\sin^2\theta}\frac{\partial^2u}{\partial\tau^2}.$$

Substituting these relations in (1.2), we get the equation

$$\frac{\partial^2u}{\partial z^2}-2\cos\,\theta\frac{\partial^2u}{\partial z\partial\tau}+\frac{\partial^2u}{\partial\tau^2}=0,\ (z;\tau)\in\mathscr{P}, \tag{1.3}$$

with the boundary conditions

$$u\left(0;\tau\right)=u\left(\pi;\tau\right)=0,\ \ \forall\tau>0;$$

$$u\left(z;0\right)=\varphi_1\left(z\right)\ \ ;\ \left.\frac{\partial u}{\partial\tau}\right|_{(z;0)}=\varphi_2\left(z\right).$$

We will seek for the elementary solutions of this equation in the form $u\left(z;\tau\right)=f\left(z\right)e^{\lambda\tau}$, where $\lambda\in\mathbb{C}$ is some parameter. Taking into account this expression, from (1.3) we obtain a quadratic pencil (1.1) with the coefficients $a=-\cos\,\theta$ and $b=1$. Problem (1.1) has non-trivial solutions only for $\lambda_n=\frac{n}{\sin\,\theta}$, $n=\pm1,\pm2,...;$ and the corresponding system of solutions has the following form

$$f_n\left(z\right)=e^{\alpha nz}\sin\,nz,\ \ \ n\neq0,$$

where $\alpha=-\frac{1}{\sin\,\theta}$. Elementary solutions of (1.3) are the functions $u_n\left(z;\tau\right)=f_n\left(z\right)e^{\frac{n}{\sin\,\theta}n\tau}$, $n\neq0$. The general solution has the following form

$$u\left(z;\tau\right)=\sum_{n\neq0}a_nu_n\left(z;\tau\right).$$

Omitting formalities, we have

$$\frac{\partial u}{\partial\tau}=\frac{1}{\sin\theta}\sum_{n\neq0}na_nu_n\left(z;\tau\right).$$

For boundary values we obtain the expressions

$$\varphi_1\left(z\right)=\sum_{n\neq0}a_nf_n\left(z\right),$$

$$\varphi_2(z) = \frac{1}{\sin\theta} \sum_{n \neq 0} n a_n f_n(z),$$

or in vector form

$$\vec{\varphi}(z) = \sum_{n \neq 0} a_n \vec{f_n}(z),$$

where $\vec{\varphi}(z) = (\varphi_1(z); \varphi_2(z))$ and $\vec{f_n}(z) = \left(f_n(z); \frac{1}{\sin\theta} n f_n(z)\right)$, $n \neq 0$.

To transform the above relations into the abstract form, we proceed as follows. Assume that $\varphi_k \in X_k$, where $X_k, k = 1, 2$, are some Banach spaces furnished with the norms $\|\cdot\|_{(k)}$, respectively. Consider the direct sum $X = X_1 \dot{+} X_2$ with the corresponding norm $\|\cdot\|$ (for example, we can take $\|\vec{x}\| = \left(\|x_1\|_{(1)}^p + \|x_2\|_{(2)}^p\right)^{1/p}$, where $\vec{x} = (x_1; x_2) \in X$ and $p \in [1, +\infty]$ is some number). Thus, the formulation of the problem (1.2) or (1.3) requires the study of approximation properties of the system $\left\{\vec{f_n}\right\}_{n \neq 0}$ in the space X. So there appeared the concepts of twofold completeness, minimality and basicity. If the system $\left\{\vec{f_n}\right\}_{n \neq 0}$ is complete, minimal in X and forms a basis for it, then the system $\{f_n\}_{n \neq 0}$ is said to be *double complete*, *double minimal* in X and to form a *double basis* for it, respectively. If we impose on the solution of the problem (1.2) the condition of vanishing at infinity, then we should consider only those elementary solutions $u_n(z; \tau)$ which correspond to the negative values of $n : n < 0$. In this case, the condition $u(z; 0) = \varphi_1(z)$ is sufficient for the unique solvability (i.e. the condition $\frac{\partial u}{\partial \tau}\big|_{(z;0)} = \varphi_2(z)$ is unnecessary). And this in turn requires the study of approximation properties of the system $\{f_n\}_{n<0}$ in X_1 (i.e. those of the "half" of the system $\{f_n\}_{n \neq 0}$). In the simpler case $\alpha = 0$, we obtain the system of sines $\{\sin nz\}_{n \neq 0}$. As X_k's, we take in this case the Sobolev space $X_1 \equiv W_{1,0}^2(0, \pi) = \left\{\varphi \in W_1^2(0, \pi) : \varphi(0) = \varphi(\pi) = 0\right\}$ and the Lebesgue space $X_2 \equiv L_2(0, \pi)$. The corresponding system $\{(\sin nx; n \sin nx)\}_{n \neq 0}$ forms a basis for $H \equiv W_{1,0}^2(0, \pi) \dot{+} L_2(0, \pi)$, i.e. the system $\{\sin nx\}_{n \neq 0}$ forms a double basis for H. In turn, the "half" $\{\sin nx\}_{n<0}$ forms a usual basis for $W_{1,0}^2(0, \pi)$. The question naturally arises as to whether it is possible to preserve these properties for $\alpha \neq 0$. That was exactly the reason of Problem $\mathscr{K}$.

A similar situation occurs when considering the pencil

$$f'' - 2\alpha\lambda f' + \left(\alpha^2 + 1\right)\lambda^2 f = 0,$$

where the spectral parameter λ is included in the boundary conditions

$$\left(\alpha\lambda f - f'\right)\big|_{x=0} = \left(\alpha\lambda f - f'\right)\big|_{x=\pi} = 0.$$

System of solutions for this problem is $\{e^{\alpha n x} \cos nx\}_{n \in \mathbb{Z}}$. Note that such kind of problem has been earlier considered by M.G. Dzavadov [15]:

$$\begin{gathered} y'' + 2b\lambda y' + c\lambda^2 y = 0, \\ y'(0) + a\lambda y(0) = y'(1) + a\lambda y(1) = 0, \end{gathered} \tag{1.4}$$

where b and c are constant numbers with $b^2 - c < 0$.

These problems are complicated in many aspects, and the study of basis properties (such as completeness, minimality, basicity) of the systems of corresponding root elements by the methods of spectral theory of linear operators either fails or cannot give conclusive answers to the questions posed. That is why in 1969 A.G. Kostyuchenko suggested to use purely functional methods for the study of the completeness of the system $\mathscr{K}_\alpha^s \equiv \{e^{\alpha n x} \sin nx\}_{n\in\mathbb{N}}$ in $L_2(0,\pi)$.

It should be noted that the study of basis properties of the Kostyuchenko system is interesting from the point of view of optimal control theory, too. To illustrate this, let's consider the following control problem for distributed oscillatory system described by A.G. Butkovski [13]:

$$\frac{\partial^2 Q}{\partial t^2} = \frac{\partial^2 Q}{\partial x^2} + f(t)\,\delta(x - \vartheta(t)) \ , \ 0 < x < \pi \, , \, t > 0 \, ,$$

$$Q(x,0) = Q_0(x) \ , \ \left.\frac{\partial Q}{\partial t}\right|_{t=0} = Q_1(x) \ , \ 0 < x < \pi,$$

$$Q(0,t) = Q(\pi,t) = 0 \, , \ t > 0.$$

It is required to find a law of changes in the concentrated load $f(t)$ and in the speed $\vartheta(t)$ of its application such that the solution $Q(x,t)$ settles down at prescribed time T, i.e. $Q(x,t) \equiv 0$. Applying the Fourier method to this problem, we obtain the following system of integral equations of the first kind with regard to two unknown functions f and ϑ:

$$\int_0^T f(t)\, e^{ikt} \sin(k\vartheta(t))\, dt = d_k \, , \ k \in \mathbb{N}.$$

Suchlike problems are often encountered in the context of damped oscillations of big mechanical systems (see L.A. Muravey [38, 39]). In the linear case $\vartheta(t) = at$, change of variables $\tau = at$ leads us to the minimality of the Kostyuchenko system in $L_2(0,T)$.

In this work, we present a brief trip back to the history of basis properties of Kostyuchenko system. We try to reveal the difficulties associated with the study of these properties and state the results obtained in this field.

2. Notation and needful concepts

We will use the following notation. $\mathbb{N}$ will be a set of all positive integers; $\mathbb{Z}$ will denote a set of all integers; $\mathbb{Z}_+ \equiv \{0\} \cup \mathbb{N}$; $\mathbb{R}$ will stand for the real axis; $\mathbb{C}$ will be the complex plane; $\mathbb{C}^+ \equiv \{z \in \mathbb{C} : Im\, z > 0\}$; Re and Im will denote real and imaginary part, respectively; $\overline{(\cdot)}$ will stand for complex conjugation; $[x]$ will be used to denote the integer part of the number $x \in \mathbb{R}$; and I will be the identity operator.

We give some concepts needed for the statement of results.

Let B be a complex Banach space equipped with the norm $\|\cdot\|$, and $\{x_n^+; x_n^-\}_{n\in\mathbb{Z}_+} \subset B$ be a double system.

Definition 2.1. System $\{x_n^+; x_n^-\}_{n\in\mathbb{Z}_+} \subset B$ is called a basis for B if for every $x \in B$ there exists a unique double sequence $\{a_n^+; a_n^-\}_{n\in\mathbb{Z}_+} \subset \mathbb{C}$ such that

$$\left\| \sum_{n=0}^{N^+} a_n^+ x_n^+ + \sum_{n=0}^{N^-} a_n^- x_n^- - x \right\| \to 0 \text{ as } N^\pm \to \infty,$$

i.e.

$$x = \sum_{n=0}^{+\infty} a_n^+ x_n^+ + \sum_{n=0}^{-\infty} a_n^- x_n^- .$$

If we denote the closure of the linear span $\overline{L\left[\{x_n^\pm\}_{n\in\mathbb{Z}_+}\right]}$ by $B^\pm$, then we can say that B^+ and B^- are topologically complementable in B, and B can be represented as a direct sum $B = B^+ \dot{+} B^-$, with $\{x_n^\pm\}_{n\in\mathbb{Z}_+}$ forming a Schauder basis for $B^\pm$. If we represent the classical system of exponents $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ in the form of $\left\{e^{int}; e^{-(n+1)t}\right\}_{n\in\mathbb{Z}_+}$, then it will form a basis for $L_p\,, 1 < p < +\infty$, in the sense of this definition, with the Hardy classes $H^\pm$ playing the roles of the subspaces $B^\pm$ with accuracy to within the isomorphism. In the sequel, the basicity of a double system will be understood in the sense of the above definition.

Definition 2.2. System $\{x_n\}_{n\in\mathbb{N}} \subset B$ is called a basis sequence in B, if every element of the closure of the linear span $\overline{L\left[\{x_n\}_{n\in\mathbb{N}}\right]}$ can be expanded in this system.

It is absolutely clear that if the complete and minimal system $\{x_n\}_{n\in\mathbb{N}}$ in B is a basis sequence, then it forms a basis for B.

Let's recall the definition of uniformly minimal system in a Banach space B with the norm $\|\cdot\|$. Let $\{x_n\}_{n\in\mathbb{N}} \subset B$ be some system. Denote by B_k the closure of the linear span $\{x_n\}_{n\in\mathbb{N}; n\neq k}$.

Definition 2.3. If there exists $\delta > 0$ such that $\inf\limits_{y\in B_k} \|x_k - y\| \geq \delta \|x_k\|$ for every $k \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is called uniformly minimal in B.

We will also need some classes of continuous functions. $C[a,b]$ will denote a Banach space (over $\mathbb{R}$) of real functions continuous on $[a,b]$; $C_0^M[a,b] \equiv \{f \in C[a,b] : f(x) = 0, \forall x \in M\}$; and $CL_B[(\cdot)]$ will be the closure of the linear span of the system $(\cdot)$ in the topology of the space B.

3. On completeness of system $\mathscr{K}_\alpha^s$

3.1. A brief chronology. M.G. Dzavadov [15] was probably the first to pay attention to the completeness of the "half" of eigenfunctions of the quadratic pencil (1.4) in $L_2(0,1)$ in 1964. The whole system of solutions of problem (1.4) has the form

$$y_k(x) = Ae^{\alpha kx} + e^{\beta kx},\ k \in \mathbb{Z},$$

where

$$A = \frac{b - a + i\sqrt{c - b^2}}{a - b + i\sqrt{c - b^2}},$$

$$\alpha = \pi \left(\frac{-b}{\sqrt{c-b^2}} + i \right), \beta = -\pi \left(\frac{b}{\sqrt{c-b^2}} + i \right).$$

The following theorem was proved in [15].

Theorem [D]. *The half* $\{y_k^-\}_{k\in\mathbb{Z}_+}$ *of eigenfunctions of problem* (1.4) *is complete in* $L_2(0,1)$, *where* $y_k^-(x) = y_{-k}(x)$, $\forall k \in \mathbb{Z}_+$.

The proof of this theorem provided in [15] and based directly on the definition of the completeness of a system is quite smart. This theorem has in particular the following corollary.

Corollary [D]. *Kostyuchenko system* $\mathscr{K}_\alpha^s$ *is complete in* $L_2(0,1)$ *for* $\forall\alpha \in \mathbb{R}$.

It should be noted that the proof of Theorem [D] is based on the functional and theoretical methods. Therefore, the solution to the Problem $\mathscr{K}$ for real values of parameter α belongs to M.G. Dzavadov, not to B.Y. Levin as many sources claim. Another proof of this fact using the methods of the theory of entire functions was provided by B.Y. Levin [28] in 1971, who proved the following more general theorem.

Theorem [L]. *System* $\mathscr{K}_\alpha^s$ *is complete in* $L_p(0,\pi)$, $1 \le p < +\infty$, *for* $\forall\alpha \in \mathbb{R}$, *whereas it is not complete in* $C[0,\pi]$ *and has a defect equal to 2.*

We can't go without mentioning the results obtained by M.G. Gasymov [17, 18, 19] in 1971-1972. Though these results repeat those of [15, 28] concerning the system $\mathscr{K}_\alpha^s$, they were obtained by a different method using general facts about the k-fold completeness of root elements of operator pencils. To illustrate, consider the quadratic pencil

$$\begin{aligned} &y'' - d\lambda y' + \lambda^2 y = 0, \\ &y(0) = y(1) = 0, \end{aligned} \tag{3.1}$$

given in [17], where $d \in \mathbb{R}$ is some parameter. The results of [17] imply that for $|d| < 2$ the system of root elements of the pencil (3.1) corresponding to the eigenvalues in the left half-plane is complete in $L_2(0,1)$. Compared to problem (1.4), we have $a = \infty$, $c = 1$ and $b = -\frac{d}{2}$. Hence it follows the same result on the completeness of $\mathscr{K}_\alpha^s$ in L_2 for $\forall\alpha \in \mathbb{R}$. For comprehensive information about the n-fold completeness of root elements of operator pencils we refer the readers to the 1982 review article by G.V. Radzievski [40] (see also [37]).

Note that all the previous results cover the case $\alpha \in \mathbb{R}$. It is not accidental, and it is reasoned by the methods used in previous works (the methods of those works were functional and theoretical; besides, the ones of the spectral theory of operators were also used). The further research required the consideration of more general systems. In 1975, A.A. Shkalikov [42] considered the system

$$f_0(x) \equiv 1,$$

$$f_k(x) = \sum_{i=1}^{r} a_i [\varphi_i(x)]^k + b_i \left[\overline{\varphi_i(x)}\right]^k, \ k \in \mathbb{N},$$

where $0 \le x \le l$, and $a_i; b_i; i = \overline{1,r}$ are the complex numbers. Besides, at least one of the numbers a_i, b_i is nonzero. Conditions imposed on the functions $\varphi_i(x)$ are:

1. $\varphi_i(x)$, $i=\overline{1,r}$, are continuous (or with a finite number of discontinuities) complex functions of bounded variation on $[0,l]$;

2. for $\forall i,j=\overline{1,r}$ and for $\forall x_1,x_2\in[0,l]$, except maybe for a finite number of pairs, it holds

$$\varphi_i(x_1)\neq\varphi_j(x_2)\,(\text{if } i=j,\,\text{then}\, x_1\neq x_2),$$

and also $\varphi_i(x_1)\neq\overline{\varphi_j(x_2)}$;

3. Denote by Γ the union of all γ_i and $\overline{\gamma}_i$, where γ_i are the curves formed by the values of the functions $\varphi_i(x)$. Contour Γ satisfies one of the following conditions:

a) contour Γ or some part of it is not the boundary of a bounded domain;

b) the inside of any bounded domain, whose boundary is contour Γ or some part of it, does not contain any other point of Γ.

Then, with the above assumptions, the following theorem is true.

Theorem [Sh]. *System of functions* $\{f_k(x)\}_{k\in\mathbb{Z}_+}$ *is complete in* $L_p(0,l)$, $\forall p\geq 1$.

This theorem has the following

Corollary [Sh]. *System* $\mathscr{K}_\alpha^s$ *is complete in* $L_p(0,\pi)$ *for* $\forall p\geq 1$ *and* $\forall\alpha\in\mathbb{R}$.

These results have been enhanced by some authors by taking Banach space of continuous functions in the role of Lebesgue space. The first enhanced results were obtained by Y.A. Kazmin in 1977-1980 [23, 24, 25]. We now state his results. Let $W(t)$ be a complex valued function with the following properties:

a) $W=W(t)$ maps injectively the segment $[a,b]$ onto $\mathbb{C}^+\cup\mathbb{R}$ with $Im\,W(t)>0$, $\forall t\in(a,b)$;

b) $W=W(t)$ is continuous and of bounded variation on $[a,b]$ (in other words, $W(t)$ is such that the W-image of the segment $[a,b]$ is a simple rectifiable unclosed curve Γ^+ lying in the upper half-plane $\mathbb{C}^+\cup\mathbb{R}$ with the endpoints $W(a)$ and $W(b)$, which satisfy one of the following four conditions:

1) $W(a)\in\mathbb{R}, W(b)\in\mathbb{R}$; 2) $W(a)\in\mathbb{R}, W(b)\in\mathbb{C}^+$;

3) $W(a)\in\mathbb{C}^+, W(b)\in\mathbb{R}$; 4) $W(a)\in\mathbb{C}^+; W(b)\in\mathbb{C}^+$.

The following systems are considered:

$$\{Im\,[W(t)]^n\}_{n\in\mathbb{N}}, \tag{3.2}$$

$$\{Re\,[W(t)]^n\}_{n\in\mathbb{Z}_+}. \tag{3.3}$$

The following theorem was proved in [23].

Theorem [K]. *If the function* $W(t)$ *satisfies the conditions a) and b), then the systems* (3.2) *and* (3.3) *possess the following properties:*

ia) $CL_{C[a;b]}[(3.2)]=C_0^{\{a;b\}}[a,b]$, *if* $ImW(a)=ImW(b)=0$;

ib) $CL_{C[a;b]}[(3.2)]=C_0^{\{a\}}[a,b]$, *if* $ImW(a)=0,\ ImW(b)\neq0$;

ic) $CL_{C[a;b]}[(3.2)]=C_0^{\{b\}}[a,b]$, *if* $ImW(a)\neq0,\ ImW(b)=0$;

id) $CL_{C[a;b]}[(3.2)]=C[a,b]$, *if* $ImW(a)\,ImW(b)\neq0$;

ii) $CL_{C[a;b]}[(3.3)]=C[a;b]$;

iii) $CL_{L_p(a,b)}[(3.3)]=CL_{L_p(a,b)}[(3.3)]=L_p(a,b)$, $\forall p\geq1$.

Denote by $\mathscr{K}_\alpha^c$ the following system:

$$\mathscr{K}_\alpha^c\equiv\left\{e^{\alpha nt}\cos nt\right\}_{n\in\mathbb{Z}_+}.$$

Theorem [K] has the following interesting corollary.

Corollary [K]. *Let* $\alpha\in\mathbb{R}$. *Then*

1) $CL_{C[0,\pi]}\left[\mathscr{K}_\alpha^s\right] = C_0^{\{0,\pi\}}[0,\pi]$;
2) $CL_{L_p(0,\pi)}\left[\mathscr{K}_\alpha^s\right] = L_p(0,\pi)$, $\forall p \geq 1$;
3) $CL_{C[0,\pi]}\left[\mathscr{K}_\alpha^c\right] = C[0,\pi]$;
4) $CL_{L_p(0,\pi)}\left[\mathscr{K}_\alpha^c\right] = L_p(0,\pi)$, $\forall p \geq 1$.

Note that the results on the n-fold completeness of root elements of the operator pencils do not cover the case of system $\mathscr{K}_\alpha^c$. In [23, 24], Y.A. Kazmin extended those results on a double system of the form

$$\left\{Re\,[W(t)]^n ; Im\,[W(t)]^n\right\}_{n\in\mathbb{Z}_+}, \tag{3.4}$$

which imply in particular the completeness of the system

$$\left\{e^{\alpha n t}\cos nt; e^{\alpha n t}\sin nt\right\}_{n\in\mathbb{Z}_+}$$

in $L_p(-\pi,\pi)$ for $\forall p \geq 1$, where $\alpha \in \mathbb{R}$ is an arbitrary parameter. Kazmin proved these results by a smart method which has never been used before.

In 1982-1984, A.G. Tumarkin [47, 48, 49] obtained similar results for the system of functions (3.2)-(3.4) using a different method. He found the completeness criterion for these systems in the space of continuous functions and sufficient conditions for minimality in the same space. Tumarkin proved that in cases when these systems are complete and minimal, they don't form a basis. Concerning systems $\mathscr{K}_\alpha^s$ and $\mathscr{K}_\alpha^c$, it turned out that they are complete and minimal in the spaces $C_0^{\{0;\pi\}}[0,\pi]$ and $C[0,\pi]$ for $\forall\alpha \in \mathbb{R}$, respectively.

In 1981-1983, extending Kazmin's method further, A.N. Barmenkov [1, 3, 2] considered more general system of the form

$$\left\{A(t)\,W^n(t) + B(t)\,\overline{W}^n(t)\right\}_{n\in\mathbb{Z}_+}, \tag{3.5}$$

where $A; B : [a,b] \to \mathbb{C}$ are complex coefficients. Under some conditions on the functions A, B and W, he established a completeness criterion for the system (3.5) in $L_p(a,b)$, $1 < p < \infty$. The obtained result was applied to the system

$$\left\{e^{\alpha n t}\sin(nt + v(t))\right\}_{n\in\mathbb{Z}_+}, \tag{3.6}$$

where $\nu : [0,\pi] \to \mathbb{C}$ is some complex valued function and $\alpha \in \mathbb{R}$. In particular, the following statement is true.

Statement [B]. *Functional sequence*

$$\left\{e^{\alpha n t}\sin(n+a)\,t\right\}_{n\in\mathbb{Z}_+}$$

is complete in $L_p(0,\pi)$, $1 < p < \infty$, *if and only if* $Re a \leq 1 + \frac{arctg\alpha}{\pi p}$.

It is absolutely clear that the similar statement can be made for the system

$$\left\{e^{\alpha n t}\cos(n+a)\,t\right\}_{n\in\mathbb{Z}_+},$$

too, where $\alpha \in \mathbb{R}$, $a \in \mathbb{C}$ are some parameters.

3.2. A few words about the methods of proof. Now we will talk about some details of the methods used in previous works. M.G. Dzavadov proceeded directly from the completeness criterion for the system, reduced the completeness of the "half" of the system to the completeness of the system itself, and then used the results on the twofold completeness of the whole system. B.Y. Levin used special technique of continuability of some analytic functions generated by the system $\mathscr{K}_\alpha^s$. M.G. Gasymov used the methods of spectral theory of operator pencils. In [17, 19, 18], M.G. Gasymov introduced a new method associating the completeness of a part of eigenvectors with the solvability of an operator differential equation on a half-axis. Y.A. Kazmin used the following classical fact. Let X be a Banach space, X^* be its conjugate and $M \subset X$ be some set. Then for the closure of M it holds $\overline{M} \equiv^{\perp}\left(M^{\perp}\right)$, where $M^{\perp} \equiv \{\vartheta \in X^* : \vartheta(x) = 0, \forall x \in M\}$ and $^{\perp}M_1 \equiv \{x \in X^* : \vartheta(x) = 0, \forall \vartheta \in M_1\}$ $(M_1 \subset X^*)$. In Kazmin's case we have $X \equiv C[a,b]$, and the role of M is played by the linear span of the system under consideration. For more information about this fact we refer the readers to the monograph by U. Rudin [41]. A.G. Tumarkin also proceeded from the definition of completeness and largely used Walsh's classical theorem on the completeness of the real and imaginary parts of polynomials on a curve. A.N. Barmenkov reduced the completeness of systems in L_p, $1 < p < +\infty$, to the trivial solvability of the corresponding Riemann boundary value problems in the theory of analytic functions in the Smirnov classes E_q, $\frac{1}{p}+\frac{1}{q}=1$. This reduction is realized with the help of classical Privalov theorem on the criterion for the belonging of a function to the Smirnov class E_1. For this theorem, we refer the readers to the monograph by G.M. Goluzin [20]. Theory of Riemann boundary value problems in classes E_q is well studied in the monograph by I.I. Daniliuk [14].

Important results on the completeness and minimality of the system $\mathscr{K}_\alpha^s$ have been obtained by Y.I. Lyubarski, V.A. Tkachenko [35, 36] and Y.I. Lyubarski [31, 32]. Previous works which treated $\mathscr{K}_\alpha^s$ covered the case $\alpha \in \mathbb{R}$ only. It is not accidental, it is reasoned by the methods used in those works. It is not difficult to see that for $\alpha \notin \mathbb{R}$ the system $\mathscr{K}_\alpha^s$ is impossible to represent in the form of (3.5). In this case, the classical methods of the theory of entire functions are inapplicable to the study of basis properties of $\mathscr{K}_\alpha^s$. On the other hand, the completeness and minimality are easily reduced to the boundary value problems with shift in the Smirnov classes (again with the help of the Privalov theorem). This, in turn, allows considering more general system of the form

$$\left\{\vartheta_n^{\pm}\right\}_{n\geq m} = \left\{a(t)\,\varphi^n(t) \pm b(t)\,\psi^n(t)\right\}_{n\geq m}, \tag{3.7}$$

where $a; b; \phi; \psi : [a,b] \to \mathbb{C}$ are complex valued functions. System $\mathscr{K}_\alpha^s$ can be represented in the form (3.7) for $\forall\alpha \in \mathbb{C}$. The study of basis properties of the system (3.7) is different from that of system (3.5) in many aspects. And this gave birth to the new field in the approximation theory aimed at studying the basis properties of the systems like (3.7). The most widely studied case here is $\psi(t) \equiv \overline{\varphi(t)}$. This case does not cover the Kostyuchenko system for nonreal values of the parameter α. The system (3.7) proved to be hard to explore, because the study of its basis properties is reduced to the study of solvability of Riemann boundary value problems with shift in the theory of analytic functions in the Hardy $H_p^{\pm}$ or Smirnov spaces $E_p(D)$. And those boundary value problems are significantly more difficult that the others (see, e.g., [30, 16]). That's why the basis properties of the systems like (3.7) have not yet been fully studied. The authors used different methods to treat those problems, and therefore the conditions they imposed on the functions in (3.7) were also different.

3.3. Special features of the system $\mathscr{K}_\alpha^s$. For many reasons, the study of basis properties of the Kostyuchenko system $\mathscr{K}_\alpha^s$ is a hard problem. First, it leads to the boundary value problems with a Carleman shift which have specific features and are mostly not fully studied, especially in the Hardy classes $H_p^{\pm}$. Second, due to $\psi(t) \neq \overline{\varphi(t)}$, these problems are in a certain sense nonsymmetric with respect to the real axis. This creates another big difficulty for studying the Noetherness of these problems in corresponding spaces. Moreover, the corresponding quadratic pencil is not self-adjoint when $\alpha \neq 0$. For these reasons, the suggested methods are inapplicable for studying basis properties in case $\psi(t) \equiv \overline{\varphi(t)}$. Under severe restrictions to the functions $a(t)$, $b(t)$, $\varphi(t)$ and $\psi(t)$, the authors in [31, 35, 36, 32] investigated the Noetherness of corresponding problem and found its index. This allowed them to find $m \in \mathbb{Z}$ such that the system $\{\vartheta_n^-\}_{n \geq m}$ is complete and minimal in L_2. Actually, under allowable conditions, the latter provides a completeness and minimality criterion for the system (3.7) (and also for the system $\mathscr{K}_\alpha^s$) in L_2. We can also mention the results of [33, 34] in this context.

In 1993-2012, B.T. Bilalov [4, 5, 6, 7, 8, 9, 11, 10] suggested a new method for studying basis properties of single systems like (3.7). Namely, those properties are derived from the similar properties of a specially defined double system of exponents with shift. The advantage of new method is that the study of basis properties of double systems is reduced to the use of methods of the theory of boundary value problems with Gaseman shift in Hardy classes, which are already studied more elaborately (see, e.g.,[29, 30, 22]). This, in turn, allows obtaining some necessary and sufficient condition for the Riesz basicity of the system $\mathscr{K}_\alpha^s$ in $L_2(0,\pi)$. The concept of double Kostyuchenko system $\mathscr{K}_\alpha$ is introduced and the Riesz basicity criterion for this system in $L_2(-\pi,\pi)$ is provided.

4. Completeness and minimality criterion. Lyubarski's results

Here we discuss the main result of [31]. Assume that the following conditions hold:

$$\begin{gathered} a,\, b \in C[0,\,\pi]\,;\ \phi,\, \psi \in C^3[0,\,\pi]\,; \\ a(t)\, b(t)\, \phi(t)\, \psi(t)\, \phi'(t)\, \psi'(t) \neq 0, \quad \forall t \in [0,\,\pi]\,. \end{gathered} \tag{4.1}$$

Basis properties of the system (3.7) essentially depend on the curves

$$\Gamma_\varphi \equiv \{z \in \mathbb{C} :\ z = \varphi(t)\,,\ t \in [0,\,\pi]\}\,,$$

$$\Gamma_\psi \equiv \{z \in \mathbb{C} :\ z = \psi(t)\,,\ t \in [0,\,\pi]\}\,.$$

Let $\Gamma = \Gamma_\varphi \cup \Gamma_\psi$. It follows from the results of [8] that if Γ does not divide the plane, then the system (3.7) is complete in $L_2(0,\,\pi)$ with infinite defect. Therefore we will consider the case when $\mathbb{C}\backslash\Gamma$ is disjoint. Namely, let

$$\varphi(0) = \psi(0)\,,\ \varphi(1) = \psi(1)\,,\ \text{and}\,\Gamma\ \text{ be a simple curve.} \tag{4.2}$$

Besides, Γ is a piecewise smooth contour with possible breakpoints only at $\xi_0 = \varphi(0)$ and $\xi_1 = \varphi(\pi)$ (see Fig.2).

Let $\pi\theta_0$ and $\pi\theta_1$ be the inside angles of the contour Γ at these points. Assume

$$\theta_0 > 0;\ \ \theta_1 > 0. \tag{4.3}$$

Define the orientation of Γ_φ as the one from ξ_0 to ξ_1, and the orientation of Γ_ψ as the one from ξ_1 to ξ_0. We will assume that this orientation coincides with the positive orientation on Γ. By D_Γ^+ and D_Γ^- we denote the bounded and unbounded

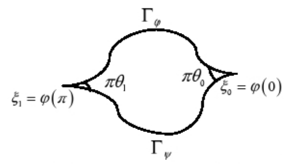

Fig. 2.

components of the set $\mathbb{C}\backslash\Gamma$, respectively. For simplicity, we assume that $0 \in D_\Gamma^+$. Let

$$E_m^2 = z^{-m} E^2\left(D_\Gamma^+\right) =$$
$$\left\{f : f \text{ is holomorphic in } D_\Gamma^+ \backslash \{0\} \text{ and } z^m f(z) \in E^2\left(D_\Gamma^+\right)\right\}.$$

Homeomorphism $\omega : \Gamma \to \Gamma$, defined by the relations

$$\omega : \xi \to \psi\left(\varphi^{-1}(\xi)\right), \ \xi \in \Gamma_\varphi;$$

$$\omega : \xi \to \varphi\left(\psi^{-1}(\xi)\right), \ \xi \in \Gamma_\psi,$$

is associated with the functions φ and ψ. Define the following functions on Γ:

$$G(\xi) = \frac{b(t)\varphi'(t)}{a(t)\psi'(t)}, \ \xi = \varphi(t) \in \Gamma_\varphi;$$
$$G(\xi) = \left[G(\omega(\xi))\right]^{-1}, \ \xi \in \Gamma_\psi,$$
$$g_k(\xi) = \frac{1}{2\pi i}\left[G(\xi)\,\xi^{-k-1} - \omega^{-k-1}(\xi)\right], \ \xi \in \Gamma, \ k = \overline{m, \infty}.$$

Completeness and minimality of the system $\{\vartheta_n^-\}_{n \ge m}$ in $L_2(0, \pi)$ are provided by the following main lemma.

Lemma [L]. *Let the conditions* (4.1) *and* (4.2) *be satisfied. Then: a) in order for the system* $\{\vartheta_n^-\}_{n \ge m}$ *to be complete in* $L_2(0, \pi)$, *it is necessary and sufficient that the homogeneous problem with shift*

$$\Phi^+(\omega(\xi)) = G(\xi)\,\Phi^+(\xi), \ \xi \in \Gamma, \tag{4.4}$$

has only trivial solution in E_m^2; *b) in order for the system* $\{\vartheta_n^-\}_{n \ge m}$ *to be minimal in* $L_2(0, \pi)$, *it is necessary and sufficient that the non-homogeneous boundary value problem with shift*

$$\Phi_k^+(\omega(\xi)) = G(\xi)\,\Phi_k^+(\xi) + g_k(\xi), \ \xi \in \Gamma, \tag{4.5}$$

has a solution $\Phi_k \in E_m^2$ *for* $\forall k \ge m$.

Functions G and g_k here satisfy the (Carleman) conditions

$$G(\xi)\,G(\omega(\xi)) = 1, \ \forall \xi \in \Gamma,$$

$$G(\omega(\xi))\,g_k(\xi) + g_k(\omega(\xi)) = 0, \ \forall \xi \in \Gamma,$$

without which the problems (4.4) and (4.5) are obviously unsolvable (see the monograph by G.S. Litvinchuk [29, 30]). It should be noted that the classical formulations of the problems (4.4) and (4.5), i.e. the case when Γ is a Lyapunov contour, the derivative ω' exists and satisfies the Hölder condition on Γ, have been well studied in [30]. In the case we consider here these conditions are not satisfied. In classical formulation, using theorem on conformal gluing, the problems (4.4) and (4.5) are reduced to the corresponding Riemann problems (without shift). It proved out that the similar theorem on conformal gluing exists in the case we consider here, too. Using the concept of expanding and shrinking logarithmic spiral, the authors in [31, 35, 36, 32] proved the existence of a needed conformal gluing. Then, using Lemma [L], they established the criterion of completeness and minimality for the system $\{\vartheta_n^-\}_{n\in\mathbb{N}}$ in $L_2(0,\pi)$ (more precisely, they found a number $m\in\mathbb{Z}$ such that the system $\{\vartheta_n^-\}_{n\geq m}$ is complete and minimal in $L_2(0,\pi)$, and this, in turn, is equivalent to the criterion of completeness and minimality of the system $\{\vartheta_n^-\}_{n\in\mathbb{N}}$ in $L_2(0,\pi)$). Let's state the obtained result.

Theorem [L]. *Let the conditions* (4.1)-(4.3) *be satisfied. Choose a continuous branch of the function* $L(t)=\frac{1}{2\pi i}\log\frac{a(t)\psi'(t)}{b(t)\varphi'(t)}$ *on* $[0,\pi]$ *and assume*

$$\begin{aligned}
&a_+=b_++ic_+=L(\pi);\ a_-=b_-+ic_-=L(0);\\
&\sigma_+=\frac{1}{\pi\theta_1}\log\left|\frac{\varphi'(\pi-0)}{\psi'(\pi-0)}\right|;\ \sigma_-=\frac{1}{\pi\theta_0}\log\left|\frac{\psi'(+0)}{\varphi'(+0)}\right|;\\
&\alpha_+=\frac{\theta_1}{2}\left(\sigma_+^2+1\right)-1;\ \alpha_-=\frac{\theta_0}{2}\left(\sigma_-^2+1\right)-1;\\
&\beta_\pm=\alpha_\pm\pm2\left(b_\pm-c_\pm\sigma_\pm\right);\ m_\pm=\left[\frac{1-\beta_\pm}{2}\right],\quad m=m_++m_--1.
\end{aligned}\tag{4.6}$$

Then the system $\{\vartheta_n^-\}_{n\geq m}$ *is complete and minimal in* $L_2(0,\pi)$.

Apply Theorem [L] to the system $\mathscr{K}_\alpha^s$. We have

$$\varphi(t)=e^{(\alpha+i)t},\ \psi(t)=e^{(\alpha-i)t};\ a=b=\frac{1}{2i}.$$

For $\alpha\in\mathbb{C}\setminus\{(-i\infty,-i]\bigcup[i,i\infty)\}$, these functions satisfy the conditions (4.1)-(4.3). For certainty we assume that $Im\alpha\geq0$. Let $\frac{\alpha+i}{\alpha-i}=e^{(r+i\omega\pi)}$, $\omega\in[0,1]$, $r\geq0$. By (4.6), we have $\beta_+=\beta_+(r;\omega)=-\frac{1}{2}\frac{r^2}{\pi^2\omega}-\omega-3$; $\beta_-=\beta_-(r;\omega)=\frac{-r^2}{2\pi^2(2-\omega)}+\frac{\omega}{2}$. Next, we calculate the numbers $m_\pm$ and m. For $\alpha\in\mathbb{R}$ we obtain $r=0$, and hence $m_+=2$, $m_-=0\Rightarrow m=1$. As a result, the system $\mathscr{K}_\alpha^s$ is complete and minimal in $L_2(0,\pi)$. In general case we have $m_+\geq2$ and $m_-\geq0$. For $\alpha\in[i,i\infty)$ the system $\left\{e^{\alpha nt}\sin nt\right\}_{n\in\mathbb{Z}}$ is not twofold complete in $L_2(0,\pi)$.

Note that the above-discussed research is technically very complicated, and this of course is reasoned by the chosen study method. Except for the case $\pm\alpha\in[i,i\infty)$, the matter of completeness and minimality of the system $\mathscr{K}_\alpha^s$ in $L_2(0,\pi)$ can be considered fully solved.

5. On the basicity of the system $\mathscr{K}_\alpha^s$. Shkalikov's results

5.1. Basicity of the system $\mathscr{K}_\alpha^s$. The only work so far which treated the basicity of the system $\mathscr{K}_\alpha^s$ in $L_2(0,\pi)$ is A.A. Shkalikov's [43] of 1988. He considered the quadratic pencil

$$P(\lambda)=T+\lambda G-\lambda^2F,$$

of unbounded operators acting in a Hilbert space H with the scalar product $(\cdot;\cdot)$. $P(\lambda)$ is called hyperbolic if T is a positive definite self-adjoint operator, i.e. $T>>0$. F is self-adjoint and positive ($F\geq0$) with $D(F)\supset D(T)$; G is symmetric, $D(G)\supset D(T)$ and

$$(Gy,y)\neq0\text{ for }0\neq y\in KerF\cap D\left(T^{\frac{1}{2}}\right).$$

By $H_\theta,\theta\in\mathbb{R}$ we denote the scale of Hilbert spaces generated by the operator $T^{\frac{1}{2}}>0$. System $\{y_k\}$ consisting of the eigenelements of the pencil $P(\lambda)$, which correspond to its positive (negative) eigenvalues, will be denoted by $E^+ \, (E^-)$. Shkalikov proved in [43] the following theorem.

Theorem [Sh1]. *If the spectrum of hyperbolic pencil $P(\lambda)$ on the half-axis $\mathbb{R}^- \, (\mathbb{R}^+)$ is discrete, then the system of elements $\left\{F^{\frac{1}{2}} y_k\right\}$, $y_k \in E^- \, (E^+)$ is complete in the subspace $H_0 \subset H$. If the restriction of the operator G on $D(T)$ admits a representation $G = G_+ - G_-$, where $G_+ \, (G_-) \geq 0$ and*

$$\left|\left(G_{(\mp)} x; y\right)\right|^2 \leq 4\,(Tx; x)\,(Fy; y) \;,\; x, y \in D(T)\,,$$

then the system $E^- \, (E^+)$ is complete in the space H_1.

The following theorem was also proved by Shkalikov in [43].

Theorem [Sh2]. *If $F = I$, $T^* = T >> 0\,, T^{-1}$ is completely continuous operator, and the symmetric operator G satisfies the condition*

$$|(Gx; y)|^2 \leq (4 - \varepsilon)^2\,(x; x)\,(Ty; y) \;,\; \forall x, y \in D(T)\,,$$

then both systems E^+, E^- of the pencil $P(\lambda)$ form Riesz bases for the spaces H_θ, with $\forall \theta \in [0, 1]$.

The obtained results are applied to the system $\mathscr{K}_\alpha^s$. Namely, the following spectral problem is considered:

$$\begin{gathered} -y'' + 2aiy' - \lambda^2 y = 0\,, \\ y(0) = y(\pi) = 0, \end{gathered} \tag{5.1}$$

where $a \in \mathbb{R}$ is some parameter. Eigenfunctions of this problem coincide with the system $\mathscr{K}_\alpha^s$ when $\alpha = \frac{ia}{\sqrt{1+a^2}}$. Define in $L_2(0,\, \pi)$ the following operators:

$$Hy = -y'', \;\; D(H) = W_2^2\,[0,\, \pi] \cap \overset{0}{W}{}_2^1\,[0,\, \pi]\,,$$

$$Gy = 2aiy', \; D(G) = \overset{0}{W}{}_2^1\,.$$

Then the problem (5.1) can be restated in the following operator form:

$$\left(H + \lambda G - \lambda^2 I\right) y = 0,$$

where obviously $H = H^* >> 0$. Using the above-mentioned general result, we obtain the following theorem on the basicity of the system $\mathscr{K}_\alpha^s$.

Theorem [Sh3]. *For $\alpha \in (-i,\, i)$, the system $\mathscr{K}_\alpha^s$ is complete in $L_2(0,\, \pi)$ and minimal in $\overset{0}{W}{}_2^1\,[0,\, \pi]$. For $\alpha \in \left(-\frac{i}{\sqrt{2}},\, \frac{i}{\sqrt{2}}\right)$, it forms (with accuracy to within the norm) a Riesz basis for $\overset{0}{W}{}_2^1\,[0,\, \pi]$ and $L_2(0,\, \pi)$.*

Obviously, with regard to the system $\mathscr{K}_\alpha^s$ Shkalikov's method is neither functional nor theoretical. It just uses a general result on quadratic pencils. And this does not give a conclusive answer to the question of the basicity of $\mathscr{K}_\alpha^s$ for $\alpha \in (-i,\, i)$ (when $\alpha \in i\mathbb{R} \backslash (-i,\, i)$, the full set of eigenfunctions is not twofold complete in $L_2(-\pi,\, \pi)$ as mentioned above).

5.2. Results on nonbasicity of the system $\mathscr{K}_\alpha^s$. In 1996, L.V. Kritskov [27] obtained a significant result on the basicity of the system $\mathscr{K}_\alpha^s$ in $L_2(0,\, \pi)$. He proved the following interesting

Theorem [Kr]. *If $\alpha \notin i\mathbb{R}$, then the system $\mathscr{K}_\alpha^s$ is not uniformly minimal in $L_2(0,\, \pi)$ even after removal of any finite number of its functions.*

In view of the fact that every basis is uniformly minimal, this theorem has the following

Corollary [Kr]. *When $\alpha \notin i\mathbb{R}$, the system $\mathscr{K}_{\alpha}^{s}$ does not form a basis for $L_2(0,\,\pi)$ even after removal of any finite number of its functions.*

The same result stays true for the system $\mathscr{K}_{\alpha}^{c}$. Note that the method of proof used in this work is smart and instructive. It uses the facts about the theory of continued fractions (see, e.g., [21]).

In 2009, using a completely different method, A.Sh.Shukurov [45, 46] proved nonbasicity of $\mathscr{K}_{\alpha}^{s}$ in $L_p(0,\,\pi)$, $1 \le p < +\infty$, for $\alpha \notin i\mathbb{R}$. This fact follows from Shukurov's earlier, more general result for the system of the form

$$\left\{\varphi^{n}\left(t\right)\sin nt\right\}_{n\in\mathbb{N}}, \tag{5.2}$$

where $\varphi : [a, b] \to \mathbb{C}$ is some measurable complex valued function on the segment $[a,\,b]$. He proved the following main theorem.

Theorem [Sh]. *If the system* (5.2) *forms a basis for $L_p(a,b)$, $1 \le p < +\infty$, then $|\varphi(t)| \equiv const$ a.e. on $[a,\,b]$.*

To prove this theorem, Shukurov largely uses the following

Lemma [Sh]. *Let $E \subset [a,b]$ be a Lebesgue measurable set. If $\exists \left\{n_k\right\}_{k\in\mathbb{N}} \subset \mathbb{N}$:*

$$\lim_{k\to\infty}\int_E \left|\sin n_k t\right|^p dt = 0 \ \Big(\lim_{k\to\infty}\int_E \left|\cos n_k t\right|^p dt = 0\Big),$$

for $1 \le p < +\infty$, then $mesE = 0$, where $mes(\cdot)$ denotes the Lebesgue measure.

Similar result is true for the system $\left\{\varphi^{n}\left(t\right)\cos nt\right\}_{n\in\mathbb{Z}_+}$. In particular, the following corollary holds:

Corollary [Sh]. *If $\alpha \notin i\mathbb{R}$, then the systems $\mathscr{K}_{\alpha}^{s}$ and $\mathscr{K}_{\alpha}^{c}$ don't form bases for $L_p(0,\,\pi)$, $1 \le p < +\infty$.*

6. Riesz basicity of double system

This section discusses the elimination of above-mentioned defect, or, in other words, the establishment of basicity criterion for the system $\mathscr{K}_{\alpha}^{s}$ in $L_2(0,\,\pi)$ when $\alpha \in (-i,\,i)$. For this purpose, B.T. Bilalov [4, 5, 6, 7, 8, 9, 11, 10] used a different method, which can be briefly described as follows. Single systems $\{\vartheta_n^{\pm}\}$ of the form (3.7) are considered in $L_p(0,a)$. Based on these systems, a double system of functions is constructed in $L_p(-a,a)$. Relationship between the basis properties of a double system in $L_p(-a,a)$ and those of single systems in $L_p(0,a)$ is established. The study of the basis properties of a double system is reduced to the study of corresponding Gaseman problem with shift. Compared to Carleman problem, the latter is well-studied. Under certain conditions on system data, this allows to establish basis properties of a double system. Then, using above-mentioned relationship, the basis properties of single systems are established. In some cases, criteria for basis properties are found. In special case, basicity criterion for the system $\mathscr{K}_{\alpha}^{s}$ is found when $|\alpha| < 1$.

Now we state the main results obtained in this context. Without loss of generality, we assume that the system (3.7) is defined on the segment $[0,\pi]$. General case can be reduced to this one by a linear change of variables, which does not affect the basis properties of the system (3.7) in L_p. Introduce the following functions:

$$A(t) \equiv \left\{ \begin{array}{l} a(t) \ , \ t \in [0,\pi] \ , \\ b(-t) \ , \ t \in [-\pi, 0] \ , \end{array} \right. \quad W(t) \equiv \left\{ \begin{array}{l} \varphi(t) \ , \ t \in [0,\pi] \ , \\ \psi(-t) \ , \ t \in [-\pi, 0] \ . \end{array} \right.$$

Let

$$W_{n;k}(t) \equiv \left[A(t) W^n(t) ; A(-t) W^k(-t) \right].$$

The following relationship is available between the basis properties of double system $\{W_{n;k}\}$ and those of system (3.7).

Lemma [B1]. *Double system $\{W_{n;k}\}_{n;\, k\geq m}$ $\left(1 \cup \{W_{n;k}\}_{n;\, k\geq m}\right)$ is complete, minimal in $L_p(-\pi,\pi)$ and forms a basis for it or an absolute basis for $L_2(-\pi,\pi)$ if and only if every one of the single systems $\{\vartheta_n^+\}_{n\geq m}$ and $\{\vartheta_n^-\}_{n\geq m}$ $(1\cup\{\vartheta_n^+\}_{n\geq m}$ and $\{\vartheta_n^-\}_{n\geq m})$ is complete, minimal in $L_p(0,\pi)$ and forms a basis for it or an absolute basis for $L_2(0,\pi)$, respectively, $1 \leq p < +\infty$.*

To obtain the main results, the following lemma is largely used.

Lemma [B2]. *(a) Let the system $\{W_{n;k}\}$ with $n \geq 0$, $k \geq 1$, form a basis for $L_p(-\pi,\pi)$ (absolute basis for $L_2(-\pi,\pi)$). If the system $\{\vartheta_n^-\}_{n\in\mathbb{N}}$ is complete in $L_p(0,\pi)$, then it forms a basis for $L_p(0,\pi)$ (absolute basis for $L_2(0,\pi)$).*

(b) Let the system $\{W_{n;k}\}$ with $n \geq 1$, $k \geq 2$, form a basis for $L_p(-\pi,\pi)$ (absolute basis for $L_2(-\pi,\pi)$). If the system $\{\vartheta_n^-\}_{n\in\mathbb{N}}$ is minimal in $L_p(0,\pi)$, then it forms a basis for $L_p(0,\pi)$ (absolute basis for $L_2(0,\pi)$).

Similar assertions are true for the system $\{\vartheta_n^+\}_{n\in\mathbb{N}}$.

The next lemma allows establishing Riesz basicity of the single systems.

Lemma [B3]. *Let the conditions $(a1)$, $(a2)$ be satisfied and*

$$\left\|\omega_n^-\right\|_2 \neq 0, \ \ \forall n \geq m, \quad \left(\left\|\omega_n^+\right\|_2 \neq 0, \ \forall n \geq m\right),$$

where $\|\cdot\|_2$ is the usual norm in $L_2(0,\pi)$. Then

$$\inf_{n\geq m} \left\|\omega_n^-\right\|_2 > 0, \ \sup_{n\geq m} \left\|\omega_n^-\right\|_2 < +\infty; \ \left(\inf_{n\geq m} \left\|\omega_n^+\right\|_2 > 0, \ \sup_{n\geq m} \left\|\omega_n^+\right\|_2 < +\infty\right).$$

To prove the main theorem, the results of [9] are largely used. To make our reasoning easier, we state here one of the results of [9] concerning Riesz basicity in $L_2(-\pi,\pi)$ of a double system of exponents

$$\left\{ A(t) e^{i\tilde{\nu}(t)n}; B(t) e^{-ikt} \right\}_{n\in\mathbb{Z}_+;k\in\mathbb{N}}, \tag{6.1}$$

where $\tilde{\nu}(t)$ is a shift of the interval $[-\pi,\pi]$, and the complex valued coefficients $A(t) \equiv |A(t)| e^{i\alpha(t)}$, $B(t) \equiv |B(t)| e^{i\beta(t)}$ satisfy the following conditions:

$(A1)$. $A(t)$, $B(t)$ are piecewise continuous functions on $[-\pi,\pi]$, $\{\tilde{t}_k\}_1^l$ is the set of discontinuity points of these functions on $(-\pi,\pi)$, and $A(t) B(t) \neq 0$, $\forall t \in [-\pi,\pi]$;

$(A2)$. $\tilde{\nu}(t) \in C[-\pi,\pi]$, $\tilde{\nu}(t)$ is piecewise differentiable and $\tilde{\nu}'(t)$ is piecewise Hölder on $[-\pi,\pi]$, $\{\tilde{\tau}_k\}_1^m$'s are its discontinuity points, $\tilde{\nu}(\pi) - \tilde{\nu}(-\pi) = 2\pi$ and $\tilde{\nu}'(t) > 0$ for $\forall t \in [-\pi,\pi]$;

$(A3)$. $\{\tilde{\varphi}_k\}_0^{\tilde{r}} \cap \mathbb{Z} = \emptyset$, where $\tilde{\varphi}_k = \frac{1}{8\pi^2}\left[\tilde{\nu}_k + 2\ln\left|\tilde{\lambda}_k\right|\right]\tilde{\nu}_k + \frac{\pi - \arg\tilde{\lambda}_k}{2\pi}$;

$$\tilde{\nu}_0 = \ln\left|\frac{\tilde{\nu}'(-\pi+0)}{\tilde{\nu}'(\pi-0)}\right|, \tilde{\nu}_k = \ln\left|\frac{\tilde{\nu}'(\tilde{s}_k+0)}{\tilde{\nu}'(\tilde{s}_k-0)}\right|, k = \overline{1,\tilde{r}};$$

$$\tilde{\lambda}_0 = \frac{G(-\pi+0)}{G(\pi-0)}; \tilde{\lambda}_k = \frac{G(\tilde{s}_k+0)}{G(\tilde{s}_k-0)}, k=\overline{1,\tilde{r}};$$

$$G(t) \equiv \frac{B(t)}{A(t)}; \{\tilde{t}_k\}_1^l \cup \{\tilde{\tau}_k\}_1^m \equiv \{\tilde{s}_k\}_1^{\tilde{r}} : -\pi < \tilde{s}_1 < ... < \tilde{s}_{\tilde{r}} < \pi.$$

Let $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ and $\tilde{h}_k = \tilde{\theta}(\tilde{s}_k+0) - \tilde{\theta}(\tilde{s}_k-0)$, $k=\overline{1,\tilde{r}}$. Take $\tilde{n}_0 = 0$ and define $\{\tilde{n}_i\}_1^{\tilde{r}} \subset \mathbb{Z}$ by the following relations:

$$-\frac{1}{2} < \frac{\tilde{h}_k}{2\pi} - \frac{1}{8\pi^2}\left[\tilde{\nu}_k + 2\ln\left|\tilde{\lambda}_k\right|\right]\tilde{\nu}_k + \tilde{n}_{k-1} - \tilde{n}_k < \frac{1}{2}, k=\overline{1,\tilde{r}}. \tag{6.2}$$

Denote

$$\tilde{\omega}_{\tilde{r}} = \frac{1}{2\pi}\left[\tilde{\theta}(-\pi+0) - \tilde{\theta}(\pi-0)\right] - \frac{1}{8\pi^2}\left[\tilde{\nu}_0 + 2\ln\left|\tilde{\lambda}_0\right|\right]\tilde{\nu}_0 + \tilde{n}_{\tilde{r}}. \tag{6.3}$$

The following main theorem on Riesz basicity of the system (6.1) was proved in [36].

Theorem [B1]. *Let the functions $A(t)$, $B(t)$ and $\nu(t)$ satisfy the conditions (A1)-(A3). Let $\tilde{\omega}_{\tilde{r}}$ be defined by the relations* (6.2), (6.3). *Then the system* (6.1) *forms a Riesz basis for $L_2(-\pi,\pi)$ only if $\tilde{\omega}_{\tilde{r}} \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. For $\tilde{\omega}_{\tilde{r}} < -\frac{1}{2}$, this system is complete, but not minimal. For $\tilde{\omega}_{\tilde{r}} > \frac{1}{2}$, it is minimal, but not complete.*

7. Riesz basicity of single systems

Here we will consider a special case of the system (3.7):

$$\omega_n^{\pm}(t) \equiv a(t)\, e^{i\sigma(t)n} \pm b(t)\, e^{-i\xi(t)n},\ n \in \mathbb{N}.$$

The following conditions are imposed on the functions in the above expression:

$(a1)$. $a(t)$, $b(t)$ are piecewise continuous functions on $[0,\pi]$, $\{t_k\}_1^m$ are discontinuity points of the function $\theta(t) \equiv \arg b(t) - \arg a(t)$ on $(0,\pi)$, and $a(t)\, b(t) \neq 0$, $\forall t \in [0,\pi]$;

$(a2)$. $\sigma(t)$, $\xi(t)$ are piecewise differentiable, continuous, increasing functions on $[0,\pi]$, $\sigma'(t)$ and $\xi'(t)$ are piecewise Hölder on $[-\pi,\pi]$, and $\sigma'(t)\xi'(t) \neq 0$, $\forall t \in [0,\pi]$, with $\sigma(0) = \xi(0)$, $\sigma(\pi) + \xi(\pi) = 2\pi$. Let $\{\tau_k\}_1^l$ be the discontinuity points of the functions $\sigma'(t)$ and $\xi'(t)$ on $(0,\pi)$. Denote $\{t_k\}_1^m \cup \{\tau_k\}_1^l = \{s_k\}_1^r$, where $0 < s_1 < ... < s_r < \pi$. Define

$$h_k = \theta(s_k+0) - \theta(s_k-0) = \arg b(s_k+0) - \arg b(s_k-0) -$$
$$-\arg a(s_k+0) + \arg a(s_k-0),\ k=\overline{1,r};$$
$$h_0 = \arg b(0) - \arg a(0);\ h_\pi = \arg b(\pi) - \arg a(\pi);$$

$$\nu_k = \ln\left|\frac{\sigma'(s_k+0)\,\xi'(s_k-0)}{\sigma'(s_k-0)\,\xi'(s_k+0)}\right|,\ k=\overline{1,r};\ \nu_0 = 2\ln\left|\frac{\sigma'(0)}{\xi'(0)}\right|;\quad \nu_\pi = 2\ln\left|\frac{\xi'(\pi)}{\sigma'(\pi)}\right|;$$

$$\lambda_k = \frac{b(s_k+0)\,a(s_k-0)}{b(s_k-0)\,a(s_k+0)}, k=\overline{1,r};\ \lambda_0 = \frac{b^2(0)}{a^2(0)};\quad \lambda_\pi = \frac{a^2(\pi)}{b^2(\pi)}.$$

Put

$$\omega_0 = \frac{h_0}{\pi} - \frac{1}{8\pi^2}\left[\nu_0 + 2\ln|\lambda_0|\right]\nu_0;$$

$$\omega_k = \frac{h_k}{2\pi} - \frac{1}{8\pi^2}\left[\nu_k + 2\ln|\lambda_k|\right]\nu_k,\quad k=\overline{1,r};\ \omega_\pi = \frac{h_\pi}{\pi} - \frac{1}{8\pi^2}\left[\nu_\pi + 2\ln|\lambda_\pi|\right]\nu_\pi.$$

Besides, we assume that the following condition is fulfilled:

$(a3)$. $\left\{\omega_0 - \frac{1}{2};\ \omega_\pi - \frac{1}{2};\ \omega_k - \frac{1}{2},\ k = \overline{1,r}\right\} \cap \mathbb{Z} = \emptyset$.

Define the numbers $\{n_i\}_0^r \subset \mathbb{Z}$ by the following relations

$$\left.\begin{array}{l} -\frac{3}{2} + 2n_0 < \omega_0 < \frac{1}{2} + 2n_0, \\ -\frac{1}{2} < \omega_k + n_{i-1} - n_i < \frac{1}{2},\ i = \overline{1,r}. \end{array}\right\} \tag{7.1}$$

Put

$$\omega_r = \omega_\pi + 2n_r. \tag{7.2}$$

The following main theorem was proved in [10].

Theorem [B2]. *Let the conditions* $(a1) - (a3)$ *be satisfied,* ω_r *be defined by* $(a2)$, (7.1), (7.2), *and one of the following conditions be fulfilled:*

$(b1)$. $-\frac{1}{2} < \omega_0 - 2n_0 < \frac{1}{2}, -\frac{3}{2} < \omega_r < -\frac{1}{2}$;

$(b2)$. $-\frac{3}{2} < \omega_0 - 2n_0 < -\frac{1}{2}, -\frac{1}{2} < \omega_r < \frac{1}{2}$;

$(b3)$. $-\frac{1}{2} < \omega_0 - 2n_0 < \frac{1}{2},\ -\frac{1}{2} < \omega_r < \frac{1}{2}$, *and the system* $\{\upsilon_n^-\}_{n\in\mathbb{N}}$ *is complete in* $L_2(0,\pi)$;

$(b4)$. $-\frac{3}{2} < \omega_0 - 2n_0 < -\frac{1}{2},\ -\frac{3}{2} < \omega_r < -\frac{1}{2}$, *and the system* $\{\upsilon_n^-\}_{n\in\mathbb{N}}$ *is minimal in* $L_2(0,\pi)$. *Then the system* $\{\upsilon_n^-\}_{n\in\mathbb{N}}$ *forms a Riesz basis for* $L_2(0,\pi)$.

Consider some special cases. Let the following conditions be satisfied:

$(c1)$. $a(t), b(t) \in C[0,\pi]; a(t)\,b(t) \neq 0, \forall t \in [0,\pi]$;

$(c2)$. $\sigma'(t), \xi'(t) \in C[0,\pi]; \sigma'(t) > 0, \xi'(t) > 0, \forall t \in [0,\pi]; \sigma(0) = \xi(0), \sigma(\pi) + \xi(\pi) = 2\pi$;

$(c3)$. $\left\{\omega_0 - \frac{1}{2}; \omega_\pi - \frac{1}{2}\right\} \cap \mathbb{Z} = \emptyset$.

We have

Corollary [B1]. *Let the conditions* $(c1) - (c3)$ *be satisfied. Define the number* $n_0 \in \mathbb{Z}$ *by the inequality*

$$-\frac{3}{2} + 2n_0 < \omega_0 < \frac{1}{2} + 2n_0.$$

Let $\omega_r = \omega_\pi + 2n_0$, *where* ω_π *is defined in* $(a2)$. *Then the assertion of Theorem [B2] is true for the system* $\{\omega_n^-\}_{n\in\mathbb{N}}$.

In fact, the relations (7.1) imply in this case that $n_r = n_0$. Consider the system

$$e_n(t) = e^{i\sigma(t)n} - e^{-i\xi(t)n},\ \ n \in \mathbb{N}.$$

Put $\alpha_0 = -\frac{\nu_0^2}{8\pi^2}$, $\alpha_\pi = -\frac{\nu_\pi^2}{8\pi^2}$, and suppose the following condition is true:

$(c4)$. $\left\{\alpha_0 - \frac{1}{2}; \alpha_\pi - \frac{1}{2}\right\} \cap \mathbb{Z} = \emptyset$.

Theorem [B2] has the following corollary.

Corollary [B2]. *Let the conditions* $(c2), (c4)$ *be satisfied. Find the number* $n_0 \in \mathbb{Z}$ *from the inequality*

$$-\frac{3}{2} + 2n_0 < \alpha_0 < \frac{1}{2} + 2n_0.$$

Put $\omega = \alpha_\pi + 2n_0$. *Let one of the following conditions be true:*

$(d1)$. $-\frac{1}{2} < \alpha_0 - 2n_0 < \frac{1}{2}, -\frac{3}{2} < \omega < -\frac{1}{2}$;

$(d2)$. $-\frac{3}{2} < \alpha_0 - 2n_0 < -\frac{1}{2}, -\frac{1}{2} < \omega < \frac{1}{2}$;

$(d3)$. $-\frac{1}{2} < \alpha_0 - 2n_0 < \frac{1}{2}, -\frac{1}{2} < \omega < \frac{1}{2}$;

$(d4)$. $-\frac{3}{2} < \alpha_0 - 2n_0 < -\frac{1}{2},\ -\frac{3}{2} < \omega < -\frac{1}{2}$ *and* $\{e_n\}_{n\in\mathbb{N}}$ *is minimal in* $L_2(0,\pi)$.

Then the system $\{e_n\}_{n\in\mathbb{N}}$ *forms a Riesz basis for* $L_2(0,\pi)$.

With one of the conditions $(d1)$, $(d2)$ and $(d4)$ fulfilled, the assertion of this corollary follows immediately from Theorem [B2]. If the condition $(d3)$ is fulfilled, then, due to $A(t) \equiv 1$, we obtain in this case that the corresponding system $1 \bigcup \{W^n(t); W^n(-t)\}_{n\in\mathbb{N}}$ forms a basis for $L_2(-\pi,\pi)$. Consequently, by Lemma [B1], the system $\{e_n\}_{n\in\mathbb{N}}$ is complete in $L_2(0,\pi)$. The rest is obvious.

8. Basicity criterion for Kostyuchenko system

8.1. Single Kostyuchenko system. Consider the Kostyuchenko system

$$S_\alpha^+ \equiv \left\{e^{i\alpha nt}\sin nt\right\}_{n\in\mathbb{N}},$$

where $\alpha \in \mathbb{R}$ is a real number. Let's apply the obtained results to this system in $L_2(0,\pi)$. Represent it in the form $\{\omega_n^-\}_{n\in\mathbb{N}}$. Then $\sigma(t) = (\alpha+1)t$, $\xi(t) = (1-\alpha)t$. For $|\alpha| < 1$ the condition $(c2)$ is true. We find

$$\nu_0 = 2\ln\left|\frac{\sigma'(0)}{\xi'(0)}\right| = 2\ln\left|\frac{1+\alpha}{1-\alpha}\right|;\ \nu_\pi = 2\ln\left|\frac{\xi'(\pi)}{\sigma'(\pi)}\right| = 2\ln\left|\frac{1-\alpha}{1+\alpha}\right| = -\nu_0.$$

Following Corollary [B2], we obtain: $\alpha_0 = -\frac{\nu_0^2}{8\pi^2}$; $\alpha_\pi = -\frac{\nu_\pi^2}{8\pi^2} = -\frac{\nu_0^2}{8\pi^2} = \alpha_0$. Condition $(c4)$ takes the form $\alpha_0 - \frac{1}{2} \notin \mathbb{Z}$. In view of $\left|\ln\frac{1+\alpha}{1-\alpha}\right| = \ln\frac{1+|\alpha|}{1-|\alpha|}$, we hence get

$$|\alpha| \neq \frac{e^{\pi\sqrt{2n-1}}-1}{e^{\pi\sqrt{2n-1}}+1} = \pi_n,\quad \forall n \in \mathbb{N}.$$

The main inequalities for α_0 and ω ($\omega = \alpha_0 + 2n_0$) imply

$$-\frac{3}{2} < \alpha_0 - 2n_0 < \frac{1}{2};\ -\frac{3}{2} < \alpha_0 + 2n_0 < \frac{1}{2},$$

hence $n_0 = 0$. Therefore $\omega = \alpha_0$.

First we consider the case

$$-\frac{1}{2} < \alpha_0 < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < -\frac{\nu_0^2}{8\pi^2} < \frac{1}{2} \Leftrightarrow |\nu_0| < 2\pi \Leftrightarrow \left|\ln\frac{1+\alpha}{1-\alpha}\right| < \pi \Leftrightarrow |\alpha| < \frac{e^\pi-1}{e^\pi+1}.$$

In this case, the condition $(d3)$ of Corollary [B2] is fulfilled, therefore it is clear that the system S_α^+ forms a Riesz basis for $L_2(0,\pi)$.

Now consider the case when $-\frac{3}{2} < \alpha_0 < -\frac{1}{2}$, i.e.

$$\frac{e^\pi-1}{e^\pi+1} < |\alpha| < \frac{e^{\pi\sqrt{3}}-1}{e^{\pi\sqrt{3}}+1}. \tag{8.1}$$

In this case, the condition $(d4)$ of Corollary [B2] is fulfilled. It suffices to show that the system S_α^+ is minimal in $L_2(0,\pi)$. For this aim, we will use the results of [32]. Following [32], we find the numbers $r \geq 0$ and $\omega \in (0,1]$ from

$$\frac{\alpha+1}{\alpha-1} = \exp(r + i\omega\pi) \Rightarrow \omega = 1,\quad r = \ln\frac{1+\alpha}{1-\alpha}.$$

We assume without loss of generality that $0 \leq \alpha < 1$. Using the notation of [32], we have

$$\beta_+ = \beta_+(r,\omega) = -\frac{1}{2}\frac{r^2}{\pi^2} + \frac{1}{2} - 4,\ \beta_- = \beta_-(r,\omega) = -\frac{1}{2}\frac{r^2}{\pi^2} + \frac{1}{2},$$

and, as a result, $N_0 = N_+ + N_- - 1$, where $N_\pm = [(1-\beta_\pm)/2]$. Consequently,

$$N_0 = \left[\frac{r^2}{4\pi^2} + \frac{1}{4} + 2\right] + \left[\frac{r^2}{4\pi^2} + \frac{1}{4}\right] - 1 = 2\left[\frac{r^2}{4\pi^2} + \frac{1}{4}\right] + 1.$$

Taking into account that $\nu_0 = 2r$, from (8.1) we obtain $\left[\frac{r^2}{4\pi^2} + \frac{1}{4}\right] = 0$, i.e. $N_0 = 1$. Hence, by the results of [32], it follows that the system S_α^+ is minimal in $L_2(0,\pi)$. Then Corollary [B2] implies a Riesz basicity of S_α^+ in $L_2(0,\pi)$. Thus, we get the following statement for the system S_α^+.

Statement [B1]. *Let $\alpha \in (-1,1)$ and $|\alpha| \neq \frac{e^\pi - 1}{e^\pi + 1}$. Then the system S_α^+ forms a Riesz basis for $L_2(0,\pi)$ if and only if the following inequality holds:*

$$|\alpha| < \frac{e^{\pi\sqrt{3}} - 1}{e^{\pi\sqrt{3}} + 1}. \tag{8.2}$$

In fact, if the inequality (8.2) is true, then the system S_α^+ forms a Riesz basis for $L_2(0,\pi)$. If (8.2) is not true, then, by the results of [32], the system $\left\{e^{i\alpha nt}\sin nt\right\}_{n\geq m}^{\infty}$ is complete and minimal in $L_2(0,\pi)$ for $m > 1$.

8.2. Double Kostyuchenko system. Here we give a result on the basicity of the following double system

$$K_\alpha \equiv \{e^{i\alpha n|t|}e^{int}; e^{i\alpha k|t|}e^{-ikt}\}_{n\in\mathbb{Z}_+;k\in\mathbb{N}}$$

in $L_2(-\pi,\pi)$. This system is obtained from the system S_α^+ in the manner depicted above. Namely, we first represent the system S_α^+ in the form

$$S_\alpha^+ \equiv c\{\varphi^n(t) - \psi^n(t)\}_{n\in\mathbb{N}},$$

where $\varphi(t) \equiv e^{i(\alpha+1)t}$; $\psi(t) \equiv e^{i(\alpha-1)t}$, $c = \frac{1}{2i}$. Then we have

$$W(t) \equiv \left\{\begin{array}{l} \phi(t), t\in(0,\pi], \\ \psi(-t), t\in[-\pi,0] \end{array}\right. \equiv \left\{\begin{array}{l} e^{i\alpha t}e^{it}, t\in(0,\pi], \\ e^{-i\alpha t}e^{it}, t\in[-\pi,0] \end{array}\right. \equiv e^{it}e^{i\alpha|t|},\ t\in[-\pi,\pi].$$

Taking into account the relationship between the basis properties of single and double systems, we call K_α a double Kostyuchenko system. Direct use of a theorem proved in [9] yields the following

Statement [B2]. *Let $|\alpha| < 1$ $\alpha \neq \pi_n$, $\forall n \in \mathbb{N}$. Double Kostyuchenko system K_α forms a Riesz basis for $L_2(-\pi,\pi)$ if and only if $|\alpha| < \pi_1 = \frac{e^\pi - 1}{e^\pi + 1}$.*

8.3. Remark. *Similar results can be obtained for the system*

$$\left\{e^{i\alpha nt}\sin\left[(n+\beta)t+\gamma\right]\right\}_{n\in\mathbb{Z}_+},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are some parameters. For example, using Lemma [B1], from Statement [B2] we immediately obtain that if $\alpha \in (-\pi_1, \pi_1)$, then the system

$$\{e^{i\alpha nt}\cos nt\}_{n\in\mathbb{Z}_+}$$

forms a Riesz basis for $L_2(0,\pi)$.

9. Unsolved matters

In conclusion, we mention some unsolved matters concerning basis properties of the system $\mathscr{K}_\alpha^s$, which are interesting mostly from a theoretical point of view.

(1) Study the basis properties of the system $\mathscr{K}_\alpha^s$ in $L_2(0,\pi)$ when $|\alpha| > 1$;
(2) Study the basis properties of the system $\mathscr{K}_\alpha^s$ in the weighted space $L_{2,\rho}(0,\pi)$;
(3) Study the same matters in Lebesgue spaces $L_p(0,\pi)$ and in weighted Lebesgue spaces $L_{p,\rho}(0,\pi)$ for $p \neq 2$;
(4) Find out whether the system $\mathscr{K}_\alpha^s$ is uniformly minimal in $L_p(0,\pi)$ for $p \neq 2$.

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