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SPECTRAL STABILITY ESTIMATES FOR THE EIGENVALUES OF ELLIPTIC OPERATORS

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In memory of M. G. Gasymov on his 75th birthday

Abstract. This is a survey of recent results obtained by the author and his co-authors G. Barbatis, E. B. Davies, E. Feleqi, V. Goldshtein, P. D. Lamberti, M. Lanza de Cristoforis, A. Ukhlov on the problem of estimates for the variation of the eigenvalues of elliptic operators upon variation of the open sets on which they are defined. These estimates are expressed in terms of various geometric characteristics of vicinity of the open sets.

1. Introduction

We shall mostly consider a non-negative self-adjoint operator

$$Hu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u \right), \quad x \in \Omega, \tag{1.1}$$

of order $2m$ subject to homogeneous Dirichlet or Neumann boundary conditions on an open set Ω in $\mathbb{R}^N$. Here $m \in \mathbb{N}$ is arbitrary and the coefficients $A_{\alpha\beta}$ are bounded measurable real-valued functions defined on Ω and the uniform ellipticity condition is satisfied: for some $\theta > 0$

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geq \theta |\xi|^2 \tag{1.2}$$

for all $x \in \Omega$, $\xi = (\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^{\hat{m}}$, where $\hat{m}$ is the number of all multi-indices $\alpha \in \mathbb{N}_0^N$, with $|\alpha| = \alpha_1 + \cdots + \alpha_N = m$.

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If Ω is sufficiently regular open set, then H has compact resolvent and its spectrum consists of a sequence of eigenvalues

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_n[\Omega] \leq \ldots$$

of finite multiplicity such that $\lim_{n\to\infty} \lambda_n[\Omega] = \infty$. (Here each eigenvalue is repeated as many times as its multiplicity.)

In this survey, for fixed coefficients $A_{\alpha\beta}$, we shall present sharp stability estimates for the variation of $\lambda_n[\Omega]$ upon variation of Ω. Some of them were discussed in [24, 31, 29, 30, 13, 15, 19]. Here we focus on the results obtained by the author and his co-authors in the last decade.

Certain, less complete results of such type are obtained for the variation of of the corresponding eigenfunctions and, more generally, for solutions of elliptic equations. We shall not discuss them in this survey. We only mention the following papers dedicated to this topic: [42, 45, 3, 43, 4, 7, 8, 9].

A wide class of open sets will be under consideration. For this reason first we state what is the meaning of the homogeneous Dirichlet or Neumann boundary conditions for an arbitrary open set Ω.

For $1 \leq p \leq \infty$, by $W^{m,p}(\Omega)$ we denote the Sobolev space of all complex-valued functions u in $L^p(\Omega)$, which have all weak derivatives $D^\alpha u$ up to order m in $L^p(\Omega)$, endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}.$$

By $W_0^{m,p}(\Omega)$ we denote the closure in $W^{m,p}(\Omega)$ of the space of the C^∞-functions with compact support in Ω.

We shall also use the semi-normed Sobolev spaces $L^{m,p}(\Omega)$ of all complex-valued functions u in $(L^p)^{\mathrm{loc}}(\Omega)$, which have all weak derivatives $D^\alpha u$ of order m in $L^p(\Omega)$, endowed with the semi-norm

$$\|u\|_{L^{m,p}(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)}.$$

Let $V(\Omega)$ be a closed subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$. We consider the following eigenvalue problem

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta} D^\alpha u D^\beta \bar{v}\, dx = \lambda \int_\Omega u\bar{v}\, dx, \tag{1.3}$$

for all test functions $v \in V(\Omega)$, in the unknowns $u \in V(\Omega), u \neq 0$, (the eigenfunctions) and $\lambda \in \mathbb{R}$ (the eigenvalues).

As is well known, problem (1.3) is the weak formulation of the eigenvalue problem for the operator H in (1.1) subject to suitable homogeneous boundary conditions: the choice of $V(\Omega)$ corresponds to the choice of boundary conditions (see *e.g.,* [41]).

We set

$$Q_\Omega(u, v) = \int_\Omega \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta} D^\alpha u D^\beta \bar{v} dx, \qquad Q_\Omega(u) = Q_\Omega(u, u),$$

for all $u, v \in W^{m,2}(\Omega)$.

If the embedding $V(\Omega) \subset W^{m-1,2}(\Omega)$ is compact, then the eigenvalues of equation (1.3) coincide with the eigenvalues of a suitable operator $H_{V(\Omega)}$ canonically associated with the restriction of the quadratic form Q_Ω to $V(\Omega)$. In fact, we have the following theorem (see [26] and [16, Theorem 2.8] for a detailed proof).

Theorem 1.4. *Let Ω be an open set in $\mathbb{R}^N$. Let $m \in \mathbb{N}$, $\theta > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ such that $|\alpha| = |\beta| = m$, let $A_{\alpha\beta}$ be bounded measurable real-valued functions defined on Ω, satisfying the condition $A_{\alpha\beta} = A_{\beta\alpha}$ and condition* (1.2).

Let $V(\Omega)$ be a closed subspace of $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$ and such that the embedding $V(\Omega) \subset W^{m-1,2}(\Omega)$ is compact.

Then there exists a non-negative self-adjoint linear operator $H_{V(\Omega)}$ on $L^2(\Omega)$ with compact resolvent, such that

$$\mathrm{Dom}(H_{V(\Omega)}^{1/2}) = V(\Omega)$$

and

$$< H_{V(\Omega)}^{1/2} u, H_{V(\Omega)}^{1/2} v >_{L^2(\Omega)} = Q_\Omega(u,v) \quad \text{for all } u, v \in V(\Omega).$$

Moreover, the eigenvalues of equation (1.3) *coincide with the eigenvalues $\lambda_n[H_{V(\Omega)}]$ of $H_{V(\Omega)}$ and*

$$\lambda_n[H_{V(\Omega)}] = \inf_{\substack{\mathcal{L} \subset V(\Omega) \\ \dim \mathcal{L} = n}} \sup_{\substack{u \in \mathcal{L} \\ u \neq 0}} \frac{Q_\Omega(u)}{\|u\|_{L^2(\Omega)}^2},$$

where the infimum is taken with respect to all subspaces $\mathcal{L}$ of $V(\Omega)$ of dimension n (Min-Max Principle).

We pay particular attention to the cases $V(\Omega) = W_0^{m,2}(\Omega)$ and $V(\Omega) = W^{m,2}(\Omega)$ which correspond to Dirichlet and Neumann boundary conditions respectively.

Definition 1.5. *Let Ω be an open set in $\mathbb{R}^N$. Let $m \in \mathbb{N}$, $\theta > 0$ and, for all $\alpha, \beta \in \mathbb{N}_0^N$ such that $|\alpha| = |\beta| = m$, let $A_{\alpha\beta}$ be bounded measurable real-valued functions defined on Ω, satisfying the equality $A_{\alpha\beta} = A_{\beta\alpha}$ and condition* (1.2).

If the embedding $W_0^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$ is compact, we set

$$\lambda_{n,\mathcal{D}}[\Omega] = \lambda_n[H_{W_0^{m,2}(\Omega)}].$$

If the embedding $W^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$ is compact, we set

$$\lambda_{n,\mathcal{N}}[\Omega] = \lambda_n[H_{W^{m,2}(\Omega)}].$$

The numbers $\lambda_{n,\mathcal{D}}[\Omega]$, $\lambda_{n,\mathcal{N}}[\Omega]$ are called the Dirichlet eigenvalues, Neumann eigenvalues respectively, of operator (1.1).

If a result holds for both the Dirichlet and Neumann eigenvalues, we shall write just $\lambda_n[\Omega]$.

Remark 1.6. *If Ω is such that the embedding $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ is compact (for instance, if Ω is an arbitrary open set with finite Lebesgue measure), then also the embedding $W_0^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$ is compact and the Dirichlet eigenvalues are well-defined.*

If Ω is such that the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact (for instance, if Ω has a continuous boundary, see Definition 4.1*), then the embedding $W^{m,2}(\Omega) \subset W^{m-1,2}(\Omega)$ is compact and the Neumann eigenvalues are well-defined.*

In the next sections we shall study the variation of $\lambda_{n,\mathcal{D}}[\Omega]$ and $\lambda_{n,\mathcal{N}}[\Omega]$ upon variation of Ω in suitable classes of open sets defined below.

2. General spectral stability theorem

We start with stating the general stability theorem for non-negative self-adjoint operators, based on the notion of a transition operator [12].

Let H be a non-negative self-adjoint operator on a separable Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ with domain $\mathrm{Dom}(H)$ dense in $\mathcal{H}$. If H has a compact resolvent, its spectrum is discrete and consists of a sequence $\lambda_n[H]$, $n \in \mathbb{N}$, of non-negative eigenvalues of finite multiplicity satisfying $\lim_{n\to\infty} \lambda_n[H] = \infty$. We shall always assume that such eigenvalues are arranged in nondecreasing order and repeated as many times as their multiplicity. The corresponding eigenfunctions will be denoted by $\varphi_n[H]$ and it will always be assumed that they form an orthonormal set in $\mathcal{H}$. We shall also denote by $L_n[H]$ the linear span of $\varphi_1[H], \dots, \varphi_n[H]$ and set

$$\mathcal{L}[H] \equiv \bigcup_{n=1}^{\infty} L_n[H].$$

Definition 2.1. *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be two non-empty families of separable Hilbert spaces and $\mathcal{B}_1 = \{H_1(\mathcal{H}_1) : \mathcal{H}_1 \in \mathfrak{H}_1\}$, $\mathcal{B}_2 = \{H_2(\mathcal{H}_2) : \mathcal{H}_2 \in \mathfrak{H}_2\}$ where $H_1(\mathcal{H}_1)$ and $H_2(\mathcal{H}_2)$ are non-negative self-adjoint linear operators on $\mathcal{H}_1$, $\mathcal{H}_2$ respectively, with compact resolvents. Moreover, let $\delta : \mathcal{B}_1 \times \mathcal{B}_2 \to [0,\infty)$ (a* measure of vicinity *of $H_1 \in \mathcal{B}_1$ and $H_2 \in \mathcal{B}_2$).*

Given $H_1 \in \mathcal{B}_1$, $H_2 \in \mathcal{B}_2$ and $0 \le a_{mn}, b_{mn} < \infty$, $0 < \delta'_{mn}, \delta''_{mn} \le \infty$ for all $m,n \in \mathbb{N}$, we say that a linear operator $T_{12} : \mathcal{L}[H_1] \to \mathrm{Dom}(H_2^{1/2})$ is a transition operator from H_1 to H_2 with the measure of vicinity δ and the parameters a_{mn}, b_{mn}, δ'_{mn}, and δ''_{mn} *(briefly, a* transition operator from H_1 to H_2*), if the following conditions are satisfied:*

(i) $(T_{12}\varphi_n[H_1], T_{12}\varphi_n[H_1])_{\mathcal{H}_2} \ge 1 - a_{nn}\delta(H_1,H_2), \quad n \in \mathbb{N},$

if $\delta(H_1,H_2) < \delta'_{nn}$,

(ii) $|(T_{12}\varphi_m[H_1], T_{12}\varphi_n[H_1])_{\mathcal{H}_2}| \le a_{mn}\delta(H_1,H_2), \ m,n \in \mathbb{N}, \ m \ne n,$

if $\delta(H_1,H_2) < \delta'_{mn}$,

(iii) $(H_2^{1/2}T_{12}\varphi_n[H_1], H_2^{1/2}T_{12}\varphi_n[H_1])_{\mathcal{H}_2} \le \lambda_n[H_1] + b_{nn}\delta(H_1,H_2), \quad n \in \mathbb{N},$

if $\delta(H_1,H_2) < \delta''_{nn}$,

(iv) $|(H_2^{1/2}T_{12}\varphi_m[H_1], H_2^{1/2}T_{12}\varphi_n[H_1])_{\mathcal{H}_2}| \le b_{mn}\delta(H_1,H_2), \quad m,n \in \mathbb{N}, \ m \ne n,$

if $\delta(H_1,H_2) < \delta''_{mn}$.

We assume, without loss of generality, that $a_{mn} = a_{nm}, b_{mn} = b_{nm}, m,n \in \mathbb{N}$.

Theorem 2.2. *Let $\mathcal{B}_1$, $\mathcal{B}_2$ and δ be as in Definition* 2.1.

1. Assume that for each $n \in \mathbb{N}$ $\sup_{H_1\in\mathcal{B}_1} \lambda_n[H_1] < \infty$. Then the following statements are equivalent:

(s_1) For each $n \in \mathbb{N}$ there exist $c_n \in [0,\infty[$ and $\varepsilon_n \in]0,\infty[$ such that the

inequality

$$\lambda_n[H_2] \le \lambda_n[H_1] + c_n\delta(H_1, H_2), \tag{2.3}$$

holds for all $H_1 \in \mathcal{B}_1$ *and* $H_2 \in \mathcal{B}_2$ *satisfying* $\delta(H_1, H_2) < \varepsilon_n$;

(s_2) *For each* $m, n \in \mathbb{N}$ *there exist* $a_{mn}, b_{mn} \in [0, \infty[$, δ'_{mn}, $\delta''_{mn} \in]0, \infty]$ *such that for each* $H_1 \in \mathcal{B}_1$ *and* $H_2 \in \mathcal{B}_2$ *there exists a transition operator* T_{12} *from* H_1 *to* H_2 *with the measure of vicinity* δ *and the parameters* a_{mn}, b_{mn}, δ'_{mn}, δ''_{mn}.

2. If T_{12} *is a transition operator from* $H_1 \in \mathcal{B}_1$ *to* $H_2 \in \mathcal{B}_2$ *with the measure of vicinity* δ *and the parameters* a_{mn}, b_{mn}, δ'_{mn}, *and* δ''_{mn} *then inequality (2.3) holds for all* $H_1 \in \mathcal{B}_1$ *and* $H_2 \in \mathcal{B}_2$ *satisfying* $\delta(H_1, H_2) < \varepsilon_n$ *with*

$$c_n = 2(a_n\lambda_n[H_1] + b_n) \quad \text{and} \quad \varepsilon_n = \min\{\delta'_n, \delta''_n, (2a_n)^{-1}\},$$

where

$$\delta'_n = \min_{1\le k,l\le n} \delta''_{kl}, \quad \delta''_n = \min_{1\le k,l\le n} \delta''_{kl}$$

and a_n, b_n *are the operator norms of the matrices* $(a_{kl})_{k,l=1}^n$, $(b_{kl})_{k,l=1}^n$ *respectively.*

Remark 2.4. *Recall that, since the matrices* $(a_{kl})_{k,l=1}^n$ *and* $(b_{kl})_{k,l=1}^n$ *are symmetric*

$$a_n = \max_{1\le k\le n} |\mu_k|, \quad b_n = \max_{1\le k\le n} |\nu_k|,$$

where μ_k, ν_k *are all eigenvalues of the matrices* $(a_{kl})_{k,l=1}^n$, $(b_{kl})_{k,l=1}^n$ *respectively (repeated as many times as their multiplicities), and the following simple estimates hold:*

$$a_n \le \left(\sum_{k,l=1}^n a_{kl}^2\right)^{1/2}, \quad b_n \le \left(\sum_{k,l=1}^n b_{kl}^2\right)^{1/2}.$$

Remark 2.5. *Conditions (i)-(ii) in Definition* 2.1 *can replaced by the following condition: for all functions* $f \in L_n[H_1]$ *such that* $\|f\|_{\mathcal{H}_1} = 1$

$$(T_{12}f, T_{12}f)_{\mathcal{H}_2} \ge 1 - \hat{a}_n\delta(H_1, H_2)$$

if $\delta(H_1, H_2) < \hat{\delta}'_n$. *Conditions (iii)-(iv) in Definition* 2.1 *can be replaced by the following condition: for all functions* $f \in L_n[H_1]$ *such that* $\|f\|_{\mathcal{H}_1} = 1$

$$(H_2^{1/2}T_{12}f, H_2^{1/2}T_{12}f)_{\mathcal{H}_2} \le \lambda_n[H_1] + \hat{b}_n\delta(H_1, H_2)$$

if $\delta(H_1, H_2) < \hat{\delta}''_n$.

The statement of Theorem 2.2 *is true also for so amended Definition* 2.1, *mutatis mutandis. In particular, if these conditions are satisfied, then inequality* (2.3) *holds with* c_n *and* ε_n *defined by* (2.2) *where now*

$$a_n = \hat{a}_n, \ b_n = \hat{b}_n, \ \delta'_n = \hat{\delta}'_n, \ \delta''_n = \hat{\delta}''_n.$$

3. Estimates via vicinity of transformations

3.1. Bi-Lipschitz mappings. If φ_1 and φ_2 are Lipschitz mapping such that $\Omega_1 = \varphi_1(\mathbb{B})$ and $\Omega_2 = \varphi_2(\mathbb{B})$, where $\mathbb{B} \subset \mathbb{R}^N$ is the unit ball, the dependence of $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$ on the vicinity of the mappings φ_1 and φ_2 was investigated in [34, 37]. See also [16, 18] and survey paper [19], where one can find references to other related results.

Let, for $\tau > 0$, F_τ be the set of all mappings φ of the unit ball $\mathbb{B}$ of the Sobolev class $L^{1,\infty}(\mathbb{B})$ such that

$$\|\nabla\varphi\|_{L^{1,\infty}(\mathbb{B})} \leq \tau\,, \qquad \operatorname*{ess\,inf}_{\mathbb{B}} \ |\det \nabla\varphi| \geq \frac{1}{\tau}\,.$$

Theorem 3.1. *For any $\tau > 0$ there exists $A_\tau > 0$ such that for any $\varphi_1, \varphi_2 \in F_\tau$ and for any $n \in \mathbb{N}$*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n A_\tau \|\varphi_1 - \varphi_2\|_{L^{1,\infty}(\mathbb{B})}\,, \tag{3.2}$$

where $\lambda_n[\Omega_1]$ and $\lambda_n[\Omega_2]$ are the eigenvalues of the Dirichlet or Neumann Laplacian on $\Omega_1 = \varphi_1(\mathbb{B})$, $\Omega_2 = \varphi_2(\mathbb{B})$ respectively, and

$$c_n = \max\{\lambda_n^2[\Omega_1], \lambda_n^2[\Omega_2]\}\,. \tag{3.3}$$

In the case of the Dirichlet Laplacian this theorem also holds if the ball $\mathbb{B}$ is replaced by any open set $\Omega \subset \mathbb{R}^N$ such that the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact [34]. In this case A_τ depends also on the Poincaré constant of Ω of the form

$$c_{\mathcal{D}}[\Omega] = \sup\nolimits^{1/2} \left\{ \frac{\int_\Omega |u|^2 dx}{\int_\Omega |\nabla u|^2 dx} \ : \ u \in W_0^{1,2}(\Omega) \setminus \{0\} \right\}\,.$$

In the case of the Neumann Laplacian this theorem also holds if the ball $\mathbb{B}$ is replaced by any open set $\Omega \subset \mathbb{R}^N$ such that the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact [37]. In this case A_τ depends also on the Poincaré constant of Ω of the form

$$c_{\mathcal{N}}[\Omega] = \sup\nolimits^{1/2} \left\{ \frac{\int_\Omega |u|^2 dx}{\int_\Omega |\nabla u|^2 dx} \ : \ u \in W^{1,2}(\Omega) \setminus \{0\},\ \int_\Omega u dx = 0 \right\}\,.$$

Further results on behaviour of $\lambda_n[\varphi(\Omega)]$ in dependence on the transformation φ, including analyticity results and persistence of multiplicity of the eigenvalues can be found in [32, 33, 34, 35, 36, 37].

3.2. Conformal mappings. In the two-dimensional case application of conformal mappings allows improving estimate (3.2) for the Dirichlet Laplacian for the class of *conformal regular* plane domains $\Omega \subset \mathbb{C}$. In [10] a bounded simply connected plane domain $\Omega \subset \mathbb{C}$ is called a conformal regular domain if there exists a conformal mapping $\varphi : \mathbb{D} \to \Omega$ of the Sobolev class $L^{1,p}(\mathbb{D})$ for some $p > 2$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

Let, for $2 < p \leq \infty, \tau > 0$, $G_{p,\tau}$ be the set of all conformal mappings φ of the unit disc $\mathbb{D}$ of the Sobolev class $L^{1,p}(\mathbb{D})$ such that

$$\|\nabla\varphi\|_{L^{1,p}(\mathbb{D})} \leq \tau\,.$$

Theorem 3.4. ([10]) *For any $2 < p \leq \infty$ there exists $B_{p,\tau} > 0$ such that for any $\varphi_1, \varphi_2 \in G_{p,\tau}$ and for any $n \in \mathbb{N}$*

$$|\lambda_{n,\mathcal{D}}[\Omega_1] - \lambda_{n,\mathcal{D}}[\Omega_2]| \leq c_n B_{p,\tau} \|\varphi_1 - \varphi_2\|_{L^{1,2}(\mathbb{D})}\,,$$

where $\lambda_{n,\mathcal{D}}[\Omega_1]$ and $\lambda_{n,\mathcal{D}}[\Omega_2]$ are the eigenvalues of the Dirichlet Laplacian on $\Omega_1 = \varphi_1(\mathbb{D})$, $\Omega_2 = \varphi_2(\mathbb{D})$ respectively, and c_n is defined by equality (3.3).

Now we describe a rather wide class of plane domains Ω for which there exist conformal mappings $\varphi : \mathbb{D} \to \Omega$ of class $G_{p,\tau}$ for some $2 < p \leq \infty$.

Definition 3.5. *A homeomorphism $\varphi : \mathcal{W}_1 \to \mathcal{W}_2$ between planar domains is called K-quasiconformal if it preserves orientation, belongs to the Sobolev class $W^{1,2}_{loc}(\mathcal{W}_1)$ and its directional derivatives ∂_a satisfy the distortion inequality*

$$\max_a |\partial_a \varphi| \leq K \min_a |\partial_a \varphi| \ \ a.e. \ in \ \mathcal{W}_1 .$$

Infinitesimally, quasiconformal homeomorphisms transform circles to ellipses with eccentricity uniformly bounded by K. If $K = 1$ we recover conformal homeomorphisms, while for $K > 1$ plane quasiconformal mappings need not be smooth.

Definition 3.6. *A domain Ω is called a K-quasidisc if it is the image of the unit disc $\mathbb{D}$ under a K-quasiconformal homeomorphism of the plane onto itself.*

Theorem 3.7. ([10]) *For any quasidiscs Ω and conformal homeomorphisms $\varphi : \mathbb{D} \to \Omega$ there exist $p > 2$ and $M > 0$ such that $\varphi \in G_{p,\tau}$.*

Remark 3.8. *The estimated of this section are rather general. However, given, two open sets Ω_1 and Ω_2, in general, it is not easy to find an open set Ω and mappings φ_1 and φ_2 satisfying the conditions of Theorem* 3.1 *or Theorem* 3.4.

In the further sections direct estimates for $|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]|$ will be presented via various geometric characteristics of vicinity of Ω_1 and Ω_2.

4. Classes of open sets with continuous boundaries

We recall that for any set V in $\mathbb{R}^N$ and $\delta > 0$ we denote by V_δ the set $\{x \in V : \ d(x, \partial\Omega) > \delta\}$. Moreover, by a rotation in $\mathbb{R}^N$ we mean a $N \times N$-orthogonal matrix with real entries which we identify with the corresponding linear operator acting in $\mathbb{R}^N$.

Definition 4.1. *Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of rotations in $\mathbb{R}^N$.*

We say that that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with the parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$.

We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:

(i) $\Omega \subset \bigcup\limits_{j=1}^{s} (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ for $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ for $s' < j \leq s$;

(iii) for $j = 1, ..., s$

$$r_j(V_j) = \{\, x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij},\, i = 1, ..., N\}$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \ a_{Nj} < x_N < g_j(\bar{x}), \ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, ..., x_{N-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij},\, i = 1, ..., N-1\}$ and g_j is a continuous function defined on $\overline{W}_j$ (it is meant that if $s' < j \leq s$ then $g_j(\bar{x}) = b_{Nj}$ for all $\bar{x} \in \overline{W}_j$);

moreover for $j = 1, \dots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho,$$

for all $\bar{x} \in \overline{W}_j$.

We say that an open set Ω *in* $\mathbb{R}^N$ *is an open set with a continuous boundary if* Ω *is of class* $C(\mathcal{A})$ *for some atlas* $\mathcal{A}$.

Let $m \in \mathbb{N}, M > 0$. *We say that an open set* Ω *is of class* $C^m_M(\mathcal{A})$, $C^{m-1,1}_M(\mathcal{A})$ *if* Ω *is of class* $C(\mathcal{A})$ *and all the functions* g_j *in (iii) are of class* $C^m(\overline{W})$, $C^{m-1,1}(\overline{W})$ *and*

$$\sum_{1\le|\alpha|\le m} \|D^\alpha g_j\|_{L^\infty(\overline{W})} \le M,$$

$$\sum_{1\le|\alpha|\le m-1} \|D^\alpha g_j\|_{L^\infty(\overline{W})} + \sum_{|\alpha|=m-1} \sup_{\substack{\bar{x},\bar{y}\in\overline{W} \\ \bar{x}\neq\bar{y}}} \frac{|D^\alpha g_j(\bar{x}) - D^\alpha g_j(\bar{y})|}{|\bar{x}-\bar{y}|} \le M$$

respectively.

We say that an open set Ω *in* $\mathbb{R}^N$ *is an open set of class* C^m, $C^{m-1,1}$ *if* Ω *is of class* $C^m_M(\mathcal{A})$, $C^{m-1,1}_M(\mathcal{A})$ *respectively, for some atlas* $\mathcal{A}$ *and some* $M > 0$.

In the sequel we shall always assume that an atlas $\mathcal{A}$ is fixed and all open sets Ω under consideration belong to $C(\mathcal{A})$.

5. Estimates via the atlas distance

For all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ we define the *atlas distance* $d_{\mathcal{A}}$ by

$$d_{\mathcal{A}}(\Omega_1,\Omega_2) = \max_{j=1,\dots,s'} \sup_{(\bar{x},x_N)\in r_j(V_j)} |g_{1j}(\bar{x}) - g_{2j}(\bar{x})| \ .$$

Theorem 5.1. ([14, 16]) *Let* $\mathcal{A}$ *be an atlas in* $\mathbb{R}^N$. *Let* $m \in \mathbb{N}$, $L, \theta > 0$. *For all* $\alpha, \beta \in \mathbb{N}_0^N$ *with* $|\alpha| = |\beta| = m$, *let the coefficients* $A_{\alpha\beta} \in C^{0,1}(\cup_{j=1}^s V_j)$ *satisfy the equality* $A_{\alpha\beta} = A_{\beta\alpha}$, *the inequality*

$$\|A_{\alpha\beta}\|_{C^{0,1}(\cup_{j=1}^s V_j)} \le L,$$

and ellipticity condition (1.2).

Then for each $n \in \mathbb{N}$ *there exist* $c_n, \varepsilon_n > 0$ *depending only on* $n, N, \mathcal{A}, m, L, \theta$ *such that for both Dirichlet and Neumann boundary conditions*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \le c_n d_{\mathcal{A}}(\Omega_1,\Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ *satisfying* $d_{\mathcal{A}}(\Omega_1,\Omega_2) < \varepsilon_n$.

Theorem 5.2. ([14]) *Let* $\mathcal{A}$ *be an atlas in* $\mathbb{R}^N$, $m = 1$ *and let the assumptions of Theorem* 5.1 *on the coefficients* $A_{\alpha\beta}$ *with* $m = 1$ *be satisfied.*

Then there exist $c, E > 0$ *depending only on* $N, \mathcal{A}, L, \theta$ *such that for the Dirichlet boundary conditions for each* $n \in \mathbb{N}$

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \le c\lambda_n[\Omega_1 \cap \Omega_2]\, d_{\mathcal{A}}(\Omega_1,\Omega_2),$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ *satisfying* $d_{\mathcal{A}}(\Omega_1,\Omega_2) < E$.

6. Estimates via the lower Hausdorff-Pompeiu deviation

If $C \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$ we denote by $d(x, C)$ the euclidean distance of x to C. Let $A, B \subset \mathbb{R}^N$. We define the lower Hausdorff-Pompeiu deviation of A from B by

$$d_{\mathcal{HP}}(A,B) = \min\left\{ \sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A) \right\}.$$

If the minimum is replaced by the maximum, then the right-hand side becomes the usual Hausdorff-Pompeiu distance $d^{\mathcal{HP}}(A,B)$ of A and B.

We now introduce a class of open sets for which we can estimate the atlas distance $d_{\mathcal{A}}$ via the lower Hausdorff-Pompeiu deviation of the boundaries.

Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$. Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous non-decreasing function such that $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \geq kt$ for all $0 \leq t \leq 1$.

Let $M > 0$. We denote by $C_M^{\omega(\cdot)}(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ belonging to $C(\mathcal{A})$ and such that all the functions g_j in the part (iii) of the definition of an open set of class $C(\mathcal{A})$ satisfy the condition

$$|g_j(\bar{x}) - g_j(\bar{y})| \leq M\omega(|\bar{x} - \bar{y}|),$$

for all $\bar{x}, \bar{y} \in \overline{W}_j$.

Theorem 6.1. ([14, 16]) *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let the assumptions of Theorem 5.1 on the coefficients $A_{\alpha\beta}$ be satisfied. Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous non-decreasing function satisfying $\omega(0) = 0$ and, for some $k > 0$, $\omega(t) \geq kt$ for all $0 \leq t \leq 1$.*

Then for each $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, M$, θ, ω such that for both Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2)),$$

for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying $d_{\mathcal{HP}}(\partial\Omega_1, \partial\Omega_2) < \varepsilon_n$.

Corollary 6.2. *Under the assumptions of Theorem 6.1 for each $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, N, \mathcal{A}, m, L, M$, θ, ω such that for both Dirichlet and Neumann boundary conditions*

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \omega(\varepsilon),$$

for all $0 < \varepsilon < \varepsilon_n$ and for all $\Omega_1, \Omega_2 \in C_M^{\omega(\cdot)}(\mathcal{A})$ satisfying the inclusions

$$(\Omega_1)_\varepsilon \subset \Omega_2 \subset (\Omega_1)^\varepsilon \text{ or } (\Omega_2)_\varepsilon \subset \Omega_1 \subset (\Omega_2)^\varepsilon ,$$

where, for $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$, Ω^ε denotes the ε-neighbourhood of Ω.

If $\omega(\varepsilon)\varepsilon^\gamma, 0 < \gamma \leq 1$, then $C_M^{\omega(\cdot)}(\mathcal{A}) = C_M^{0,\gamma}(\mathcal{A})$ and the above estimate takes the form

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \leq c_n \varepsilon^\gamma .$$

This estimate for the Dirichle Laplacian was obtained in [25, 27] and for the Neumann Laplacian in [6].

7. Estimates via the measure of the symmetric difference

Theorem 7.1. ([12, 14, 18]) *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let the assumptions of Theorem* 5.1 *on the coefficients $A_{\alpha\beta}$ be satisfied. Let $2 < p \le \infty$ and let $\mathfrak{A}$ be a family of open sets of class $C_M^{m-1,1}(\mathcal{A})$ such that for each $n \in \mathbb{N}$*

$$\sup_{\Omega\in\mathfrak{A}} \|\varphi_n[\Omega]\|_{W^{m,p}(\Omega)} < \infty$$

for all $n \in \mathbb{N}$.

Then for each $n \in \mathbb{N}$ there exists $c_n, \varepsilon_n > 0$ depending only on $n, \mathcal{A}, m, M, \theta$, p,

$$\sup_{\Omega\in\mathfrak{A}} \|\varphi_k[\Omega]\|_{W^{m,p}(\Omega)}\,, k = 1, \dots, n\,,$$

such that for both the Dirichlet and Neumann boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \le c_n|\Omega_1 \vartriangle \Omega_2|^{1-\frac{2}{p}},$$

where $|\Omega_1 \vartriangle \Omega_2|$ is the Lebesgue measure of the symmetric difference of Ω_1 and Ω_2, for all $\Omega_1, \Omega_2 \in \mathfrak{A}$ such that $|\Omega_1 \vartriangle \Omega_2| < \varepsilon_n$.

Moreover, the exponent $1 - \frac{2}{p}$ is sharp. It cannot be replaced by $1 - \frac{2}{p} + \delta$ where $\delta > 0$ is a constant independent of p.

Corollary 7.2. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let the assumptions of Theorem* 7.1 *on the coefficients $A_{\alpha\beta}$ be satisfied.*

Then for all $n \in \mathbb{N}$ there exist $c_n, \varepsilon_n > 0$ depending only on $n, \mathcal{A}, m, L, M, \theta$ such that for the Dirichlet boundary conditions

$$|\lambda_n[\Omega_1] - \lambda_n[\Omega_2]| \le c_n|\Omega_1 \vartriangle \Omega_2|,$$

for all $\Omega_1, \Omega_2 \in C_M^{2m}(\mathcal{A})$ such that $|\Omega_1 \vartriangle \Omega_2| < \varepsilon_n$.

8. Estimates for the p-Laplacian

Let Ω be a bounded open set in $\mathbb{R}^N$ and $1 < p < \infty$. Consider the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u$$

for $u \in W_0^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$, where

$$\Delta_p u = \operatorname{div} |\nabla u|^{p-2} \nabla u$$

is the p-Laplacian. Clearly Δ_2 is the usual Dirichlet Laplacian. The real numbers λ for which this equation has a nontrivial solution are by definition the eigenvalues of $-\Delta_p$.

As is known, it is possible to produce a nondecreasing unbounded sequence of eigenvalues $\lambda_{p,n}[\Omega]$, $n \in \mathbb{N}$ by means of the following variant of the Min-Max Principle [28, 23, 40]

$$\lambda_{p,n}[\Omega] = \inf_{\mathcal{M}\in\mathfrak{M}_{p,n}(\Omega)} \sup_{u\in\mathcal{M}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

where $\mathfrak{M}_{p,n}(\Omega)$ is the family of those conic subsets $\mathcal{M}$ of $W_0^{1,p}(\Omega) \setminus \{0\}$, whose intersection with the unit sphere of $L^p(\Omega)$ is compact in $W_0^{1,p}(\Omega)$ and whose Krasnoselskii's genus $\gamma(\mathcal{M})$ is greater than or equal to n.

Theorem 8.1. ([17]) *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and let and $1 < p < \infty$.*
Then there exist $c, E > 0$ depending only on $N, \mathcal{A}$ and p such that for the Dirichlet boundary conditions for each $n \in \mathbb{N}$

$$|\lambda_{p,n}[\Omega_1] - \lambda_{p,n}[\Omega_2]| \leq c\lambda_{p,n}[\Omega_1 \cap \Omega_2]\, d_{\mathcal{A}}(\Omega_1, \Omega_2)$$

for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ satisfying $d_{\mathcal{A}}(\Omega_1, \Omega_2) < E$.

9. Estimates for the Robin Laplacian

Let Ω be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$, h be an essentially bounded non-negative measurable function on $\partial\Omega$, and the quadratic form on $L^2(\Omega)$ be defined by

$$Q_{\Omega,h}[f] \equiv \begin{cases} \int_\Omega |\nabla f|^2\, dx + \int_{\partial\Omega} h|\mathrm{tr}\, f|^2\, d\sigma & \text{if } f \in W^{1,2}(\Omega)\,, \\ +\infty & \text{if } f \in L^2(\Omega) \setminus W^{1,2}(\Omega)\,, \end{cases}$$

where $d\sigma$ denotes the usual surface measure on $\partial\Omega$, and $\mathrm{tr}\, f$ denotes the trace on $\partial\Omega$ of the function $f \in W^{1,2}(\Omega)$. The Robin Laplacian in Ω, corresponding to h is defined to be the non-negative selfadjoint operator $-\Delta_{\Omega,h}$ acting in $L^2(\Omega)$, and associated to the quadratic form $Q_{\Omega,h}$. We consider the eigenvalue problem

$$-\Delta_{\Omega,h}[u] = \lambda u\,.$$

Note that the classical formulation of this problem in a domain with a smooth boundary is

$$-\Delta u = \lambda u \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} + hu = 0 \text{ on } \partial\Omega\,,$$

where ν is the exterior unit normal to $\partial\Omega$. (For $h = 0$, one obtains the Neumann problem.) We denote by $\{\lambda_n[\Omega, h]\}_{n\in\mathbb{N}}$ the non-decreasing sequence of all eigenvalues.

Definition 9.1. *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and $M > 0$. Let Ω_1, Ω_2 be bounded domains in $\mathbb{R}^N$ of class $C_M^{0,1}$ with corresponding families of functions $\{g_{1,j}\}_{j=1}^{s'}$, $\{g_{2,j}\}_{j=1}^{s'}$ as in Definition 4.1 (iii). Then we set*

$$G(\partial\Omega_1, \partial\Omega_2) \equiv \sum_{j=1}^{s'} \int_{W_j} \Big| |\nabla\varphi_{1,j}(\overline{x})| - |\nabla\varphi_{2,j}(\overline{x})| \Big|\, d\overline{x}\,.$$

For $h_1 \in L^\infty(\partial\Omega_1)$, $h_2 \in L^\infty(\partial\Omega_2)$, we also set

$$a(h_1, h_2) \equiv \max\left\{\|h_1\|_{L^\infty(\partial\Omega_1)}, \|h_2\|_{L^\infty(\partial\Omega_2)}, \|h_1\|^2_{L^\infty(\partial\Omega_1)}, \|h_2\|^2_{L^\infty(\partial\Omega_2)}\right\}\,.$$

Theorem 9.2. ([20]) *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^N$ and $M > 0$. Then for each $n \in \mathbb{N}$ there exists $b_n > 0$ such that*

$$\lambda_n[\Omega_2, h_2] \leq \lambda_n[\Omega_1, h_1] + b_n\Big[|\Omega_1 \setminus \Omega_2| + a(h_1, h_2)\,(|\Omega_1 \setminus \Omega_2| + G(\partial\Omega_1, \partial\Omega_2)) + \|h_1 - h_2\|_{L^1(\partial\Omega)}\Big]\,,$$

for all bounded domains Ω_1, Ω_2 *in* $\mathbb{R}^N$ *which are of class* $C_M^{0,1}(\mathcal{A})$ *and satisfy the conditions*

$$\Omega_2 \subset \Omega_1, \quad |\Omega_1 \setminus \Omega_2| \le b_n^{-1},$$

and for all nonnegative $h_1 \in L^\infty(\partial\Omega_1)$, $h_2 \in L^\infty(\partial\Omega_2)$.

Theorem 9.3. ([20]) *Let* $\mathcal{A}$ *be an atlas in* $\mathbb{R}^N$, $M > 0$, $\alpha > 0$, *and* $0 < \gamma \le 1$. *Then for each* $n \in \mathbb{N}$, *there exists* $c_n > 0$ *such that*

$$\lambda_n[\Omega_1, h_1] - c_n\varepsilon^\gamma \le \lambda_n[\Omega_2, h_2] \le \lambda_n[\Omega_1, h_1] + c_n\varepsilon,$$

for all $0 < \varepsilon < c_n^{-1}$, *for all bounded domains* Ω_1 *in* $\mathbb{R}^N$ *of class* $C_M^{1,\gamma}(\mathcal{A})$, *for all non-negative* $h_1 \in L^\infty(\partial\Omega_1)$ *such that* $\|h_1\|_{L^\infty(\partial\Omega_1)} \le \alpha$ *and* $\mathrm{Lip}_\gamma[h_1] \le \alpha$ *and for all bounded domains* Ω_2 *in* $\mathbb{R}^N$ *of class* $C_M^{0,1}(\mathcal{A})$ *satisfying the conditions*

$$(\Omega_1)_\varepsilon \subset \Omega_2 \subset \Omega_1, \qquad G(\partial\Omega_1, \partial\Omega_2) \le \varepsilon,$$

and for all nonnegative $h_2 \in L^\infty(\partial\Omega_2)$ *satisfying the conditions* $\|h_2\|_{L^\infty(\partial\Omega_2)} \le \alpha$, *and* $L(h_1, h_2) \le \varepsilon$.

10. Estimates of singular numbers for correct restrictions of elliptic operators

10.1. Correct restrictions. Let $m, N \in \mathbb{N}$ and $\mathcal{L}$ be an elliptic differential expression of the following form: for $u \in C^\infty(\mathbb{R}^N)$

$$(\mathcal{L}u)(x) = (-1)^m \sum_{|\alpha|,|\beta|\le m} D^\alpha \left(A_{\alpha\beta}(x) D^\beta u\right), \quad x \in \mathbb{R}^N,$$

where $A_{\alpha\beta} \in C^m(\mathbb{R}^N)$ are real-valued functions for all multi-indices α, β satisfying $|\alpha|, |\beta| \le m$.

Moreover, let, for a domain $\Omega \subset \mathbb{R}^N$,

$$L_\Omega : D(L_\Omega) \to L_2(\Omega)$$

be a linear operator closed in $L_2(\Omega)$ generated by the differential expression $\mathcal{L}$ on Ω.

A restriction

$$A : D(A) \to L_2(\Omega), \; D(A) \subset D(L_\Omega)$$

of L_Ω is *correct* if the equation $Au = f$ 1) has a unique solution $u \in D(A)$ for any $f \in L_2(\Omega)$, 2) the corresponding inverse operator $A^{-1} : L_2(\Omega) \to D(A)$ is bounded.

Note that, in general, the operator A is not selfadjoint. For this reason the singular numbers $s_n(A)$ are under consideration (the eigenvalues of $\sqrt{A^*A}$). As usual it is assumed that they are arranged in non-decreasing order:

$$s_1(B) \le s_2(B) \le \cdots \le s_n(B) \le \cdots.$$

Here each singular number is repeated as many times as its multiplicity.

10.2. Coinciding asymptotics.

Theorem 10.1. ([21, 22]) *Let $m, N \in \mathbb{N}$, $N \geq 2$,*

$$2m\Big(1-\frac{1}{N}\Big) < s \leq 2m$$

and Ω be a bounded domain in $\mathbb{R}^N$ of class C^{2m}.

Then there exists $b > 0$ such that, for the singular numbers $s_n(B)$ of each correct restriction B of the operator L_Ω satisfying the condition

$$D(B) \subset W_2^s(\Omega)$$

with the bounded inverse $B^{-1} : L_2(\Omega) \to W_2^s(\Omega)$, the following equality holds:

$$\lim_{n\to\infty} s_n(B) n^{-\frac{2m}{N}} = b\,.$$

10.3. Spectral stability estimates. Let now, for $u \in C^\infty(\mathbb{R}^N)$, $\mathcal{L}u$ be a second order elliptic differential expression without lower terms with symmetric $A_{\alpha\beta}$, namely

$$\mathcal{L}u = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big), \quad x \in \mathbb{R}^N,$$

where $a_{ij} \in C^1(\mathbb{R}^N)$ are real-valued functions satisfying $a_{ij} = a_{ji}$ for all $i, j = 1, ..., N$.

Theorem 10.2. ([21, 22]) *Let $N \in \mathbb{N}$, $N \geq 2$, $2 - \frac{2}{N} < s \leq 2$. Moreover, let $\mathcal{A}$ be a fixed atlas in $\mathbb{R}^n$, $M > 0$, $G_M(\mathcal{A})$ be a family of bounded domains $\Omega \subset \mathbb{R}^N$ of class $C_M^2(\mathcal{A})$ and*

$$\mathfrak{B}(\mathcal{A}) = \{B_\Omega\}_{\Omega \in G_M(\mathcal{A})}$$

be a family of correct restrictions B_Ω of the operator L_Ω such that

$$D(B_\Omega) \subset W_2^s(\Omega) \quad \text{and} \quad \sup_{B_\Omega \in \mathfrak{B}(\mathcal{A})} \|B_\Omega^{-1}\|_{L_2(\Omega)\to W_2^s(\Omega)} < \infty\,.$$

Then there exist $\delta, c > 0$ and for each $\varepsilon \in (0, \delta]$ there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$|s_n(B_{\Omega_1}) - s_n(B_{\Omega_2})| \leq c\, n^{\frac{2}{N}}\, \varepsilon$$

for all $n \geq k(\varepsilon)$ and for all $\Omega_1, \Omega_2 \in G_M(\mathcal{A})$ satisfying the inclusions

$$(\Omega_1)_\varepsilon \subset \Omega_2 \subset (\Omega_1)^\varepsilon \quad \text{or} \quad (\Omega_2)_\varepsilon \subset \Omega_1 \subset (\Omega_2)^\varepsilon\,.$$

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