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REGULARIZED TRACE FORMULA FOR SOME SINGULAR SCHRÖDINGER TYPE OPERATOR

ZAKI F.A. EL-RAHEEM

In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper we prove and calculate a regularized trace formula for the spectra of schrödinger type operator with explosive factor on the half line. The methodology used is carried out by applying contour integration over a complex plane, the result obtained contains both the sum of eigenvalues and integration of scattering function, because of the nature of spectra is discreet and continuous.

The sum of the eigenvalues $\{\lambda_n\}$ of an operator is usually called its trace. For the eigenvalues $\lambda_n$of an differential operator, the series $\sum_n \lambda_n$, roughly speaking, diverges, however, it can be regularized by subtracting from λ_n the first term of the asymptotic expansion, which interfere with the convergence of the series. The sum of such a regularized series is called the regularized trace. The theory of regularized traces of differential operators dates back to Gelfand and Levitan [12], who considered the Sturm-Liouville operator

$$-y'' + q(x)y = \mu y, \quad y'(0) = 0, \ \ y'(\pi) = 0, \tag{1}$$

where $q(x) \in C^1[0, \pi]$ and $\int_0^\pi q(x)dx = 0$, they obtained the formula

$$\sum_{n=0}^{\infty}(\mu_n - \lambda_n) = \frac{1}{4}[q(0) + q(\pi)], \tag{2}$$

where μ_n and λ_n are the eigenvalues of operator (1) in cases of $q(x) \neq 0$ and $q(x) = 0$ respectively.

This formula gave rise to a large and very important theory, which started from the investigation of specific operators and further embraced the analysis of regularized traces of discrete operators in general form. In a short time, a number of authors turned their attention to trace theory and obtained interesting results. Gelfand [11] demonstrated a technique of using the trace of a resolvent for finding traces. Dikii provided a proof of the Gelfand-Levitan formula in [1] on the basis of direct methods of perturbation theory, and in [2], he derived trace formulas of all orders for the Sturm-Liouville operator by constructing the fractional powers of the operator in closed form and by computing an analytic extension for its zeta function. Later, Levitan [13] suggested one more method for computing

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the traces of the Sturm-Liouville operator: by matching the expressions for the characteristic determinant via the solution of an appropriate Cauchy problem and via the corresponding infinite product, he found and compared the coefficients of the asymptotic expansions of these expressions, thus obtaining trace formulas. The investigation carried out in 1957 by Faddeev [7] linked the trace theory to a substantially new class of problems. Afterwards these investigations were continued in many directions, such as Dirac operators, differential operators with abstract operator-valued coefficients, and the case of matrix-valued.

The trace formulae can be used for approximate calculation of the first eigenvalue of an operator [14], and in order to establish necessary and sufficient conditions for a set of complex numbers to be the spectrum of an operator [15].

In [8] Gasymov was the first one who tackled a singular differential operator with discrete spectrum. The later case is continued by Gasymov and his disciples [9]-[10] and others.

The present paper is organized as follows, in the introduction we demonstrate, briefly, some historical and scientific survey to regularized trace formula. From [6] we present the basic definitions and results that are needed in the subsequent investigation, then we prove Lemma 0.1 and the main theorem.

Consider the initial value problem

$$-y'' + q(x)y = \lambda^2\rho(x)y \ ,0 \le x < \infty, \tag{3}$$

$$y'(0) - hy(0) = 0 \ ,h > 0, \tag{4}$$

where

$$\rho(x) = \begin{cases} -1 \ ; \ 0 \le x \le 1, \\ 1 \ ; \ 1 < x < \infty, \end{cases} \tag{5}$$

$q(x)$ is a finite real valued twice differentiable function such that

$$\int_0^\infty (1+x)|q^{(j)}(x)|dx < \infty, \ j = 0, 1, 2,$$

and λ is a complex spectral parameter. It should be noted that the introduction of $\rho(x)$ as discontinuous function, specially ± 1, gives rise to a lot of analytical difficulties, rather that the classical case, indeed, it splits the spectra into two parts discreet and continuous , the later is treated by scattering function.

Following [6], the characteristic equation of the eigenvalues of (3)-(4) is given by $f'(0, \lambda) - hf(0, \lambda) = 0$, further,

$$f'(0, \lambda) - hf(0, \lambda) = f(1, \lambda)\varphi'(1, \lambda) - f'(1, \lambda)\varphi(1, \lambda), \tag{6}$$

where

$$\begin{aligned} f(x, \lambda) &= e^{i\lambda x} + \textstyle\int_x^\infty K(x, t)e^{i\lambda t}dt, \\ K(x, x) &= \tfrac{1}{2}\textstyle\int_0^x q(t)dt, \ 1 < x < \infty, \end{aligned} \tag{7}$$

$$\begin{aligned} \varphi(x, \lambda) &= \cosh \lambda x + \textstyle\int_0^x A(x, t) \cosh \lambda t dt, \\ A(x, x) &= h + \tfrac{1}{2}\textstyle\int_0^x q(t)dt, A(x, 0) = 0, A(0, 0) = h. \end{aligned} \tag{8}$$

Integrating the right hand side of (7) and (8) by parts suitable number of times, we have the following asymptotic formulas, for $Im\lambda \ge 0, |\lambda| \to \infty$

$$f(1, \lambda) = e^{i\lambda}\left[1 - \frac{K(1, 1)}{i\lambda} + \frac{\alpha}{(i\lambda)^2} + \bigcirc\left(\frac{1}{\lambda^3}\right)\right], \tag{9}$$

$$f'(1,\lambda) = e^{i\lambda}\left[i\lambda - K(1,1) - \frac{\beta}{i\lambda} + \bigcirc\left(\frac{1}{\lambda^2}\right)\right], \tag{10}$$

$$\varphi(1,\lambda) = \cosh\lambda + \frac{A(1,1)}{\lambda}\sinh\lambda - \frac{a}{\lambda^2}\cosh\lambda + \bigcirc\left(\frac{e^{|Re\lambda|}}{\lambda^3}\right), \tag{11}$$

$$\varphi'(1,\lambda) = \lambda\sinh\lambda + A(1,1)\cosh\lambda - \frac{b}{\lambda}\sinh\lambda + \bigcirc\left(\frac{e^{|Re\lambda|}}{\lambda^2}\right), \tag{12}$$

where

$$\alpha = \left.\frac{\partial K(1,t)}{\partial t}\right|_{t=1}, \beta = \left.\frac{\partial K(1,t)}{\partial x}\right|_{t=1}, a = \left.\frac{\partial A(1,t)}{\partial t}\right|_{t=1} \text{ and } b = \left.\frac{\partial A(1,t)}{\partial x}\right|_{t=1}. \tag{13}$$

Substituting from (9)-(12) into (6), we get

$$f'(0,\lambda) - hf(0,\lambda) = -i\lambda e^{i\lambda}v_o(\lambda)\left[1 + \frac{v_1(\lambda)}{\lambda\, v_o(\lambda)} + \frac{v_2(\lambda)}{\lambda^2 v_o(\lambda)} + \bigcirc\left(\frac{e^{|Re\lambda|}}{\lambda^3 v_o(\lambda)}\right)\right], \tag{14}$$

where

$$\begin{aligned} v_o(\lambda) &= \cosh\lambda + i\sinh\lambda, \\ v_1(\lambda) &= [A(1,1)+K(1,1)]\cosh\lambda + i[K(1,1)-A(1,1)]\sinh\lambda, \\ v_2(\lambda) &= [a+\beta-k(1,1)A(1,1)]\cosh\lambda + i[K(1,1)A(1,1)-\alpha-b]\sinh\lambda. \end{aligned} \tag{15}$$

From [6], equation (2.11), $f_o(0,\lambda) = -i\lambda e^{i\lambda}v_o(\lambda)$, so that equation (14) takes the form

$$f'(0,\lambda) - hf(0,\lambda) = f_o(0,\lambda)\left[1 + r(\lambda)\right], \tag{16}$$

where

$$r(\lambda) = \frac{v_1(\lambda)}{\lambda\, v_o(\lambda)} + \frac{v_2(\lambda)}{\lambda^2 v_o(\lambda)} + \bigcirc\left(\frac{e^{|Re\lambda|}}{\lambda^3 v_o(\lambda)}\right). \tag{17}$$

Let Ω_n be the rectangular contour

$$\Omega_n = \left\{|Re\ \lambda| \le \pi\left(n + \frac{3}{4}\right)\ ,\ 0 \le Im\ \lambda \le \pi\left(n+\frac{3}{4}\right)\right\}, \tag{18}$$

by using the inequality, $|v_o(\lambda)| \ge Ce^{|Re\lambda|}, \quad \lambda \in \Omega_n, \forall n$, of Lemma 2.2 [4], $r(\lambda)$ takes the form

$$r(\lambda) = \frac{v_1(\lambda)}{\lambda\, v_o(\lambda)} + \frac{v_2(\lambda)}{\lambda^2\, v_o(\lambda)} + \bigcirc\left(\frac{e^{|Re\lambda|}}{\lambda^3}\right). \tag{19}$$

The last discussion leads to the Lemma 0.1.

Lemma 0.1. *For all $Re\lambda \neq 0$ the characteristic function $f_h(0,\lambda) \stackrel{def}{=} f'(0,\lambda) - hf(0,\lambda)$, of the eigenvalues of problem (3)-(5) admits the asymptotic formula, for $\lambda \in \Omega_n, Re\lambda \neq 0$*

$$f_h(0,\lambda) = f_o(0,\lambda)\left[1 + \frac{v_1(\lambda)}{\lambda\, v_o(\lambda)} + \frac{v_2(\lambda)}{\lambda^2\, v_o(\lambda)} + \bigcirc\left(\frac{e^{|Re\lambda|}}{\lambda^3}\right)\right], \tag{20}$$

where $f_0(0,\lambda)$ is the characteristic equation of the eigenvalues of the boundary value (3)-(5), when $q(x) \equiv 0$ and $h = 0$, $v_0(\lambda), v_1(\lambda)$ and $v_2(\lambda)$ are given by (15).

We prove, now, the main theorem of the present paper.

Theorem 0.1. *Under the conditions of Lemma 0.1, the following regularized trace formula with respect to the eigenvalues of problem (3)-(5), takes place*

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi i}\int_0^{\tau_n}\lambda^2 dln\frac{S(\lambda)}{S_o(\lambda)}+\sum_{k=1}^{n}\left({\lambda_k}^2-\lambda_k^{o2}-\omega\right)\right\}=Q, \tag{21}$$

where

$$\begin{aligned}
&\tau_n=\pi\left(n+\tfrac{3}{4}\right),\ \lambda_k^{o2}=-\pi^2\left(n+\tfrac{1}{4}\right)^2, S_o(\lambda)=e^{-2i\lambda}\,\tfrac{\cosh\lambda-i\sinh\lambda}{\cosh\lambda+i\sinh\lambda},\\
&Q=\left(\tfrac{ln2}{\pi}-\tfrac{3}{4}\right)(b+\beta)-\tfrac{3\alpha}{2}-\tfrac{2}{\pi}KA-\tfrac{1}{2}(K^2+A^2)+\tfrac{3}{4}(i-1)A+A_1,\\
&A=A(1,1), K=K(1,1), A_1=\tfrac{3A(1,1)}{4}-\tfrac{A(1,1)K(1,1)}{\pi}+\tfrac{a+b+\alpha+\beta}{2\pi},\ \omega=niA(1,1),\\
&\alpha=\left.\tfrac{\partial K(1,t)}{\partial t}\right|_{t=1},\beta=\left.\tfrac{\partial K(1,t)}{\partial x}\right|_{t=1},a=\left.\tfrac{\partial A(1,t)}{\partial t}\right|_{t=1}\ \textit{and}\ b=\left.\tfrac{\partial A(1,t)}{\partial x}\right|_{t=1}.
\end{aligned} \tag{22}$$

Proof. From the theory of functions of a complex variable we have the well known formula

$$\frac{1}{2\pi i}\oint_{\Omega_n}\lambda^2 d\,ln\,[f_h(0,\lambda)]=\sum_{k=1}^{n}\lambda_k^2, \tag{23}$$

where the contour Ω_n is defined by (18), the characteristic function $f_h(0,\lambda)$ is defined in Lemma 0.1 and λ_k is the eigenvalue inside Ω_n. But since

$$\frac{1}{2\pi i}\oint_{\Omega_n}\lambda^2 d\,ln\,[f_h(0,\lambda)]=\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\,ln\,[f_h(0,\lambda)]+\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_h(0,\lambda)], \tag{24}$$

where Ω_n^+ is the upper part of the contour Ω_n, so that, from (23) and (24), we have

$$\sum_{k=1}^{n}\lambda_k^2=\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\,ln\,[f_h(0,\lambda)]+\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_h(0,\lambda)]. \tag{25}$$

Similarly, we carry out the same construction for $f_o(0,\lambda)$:

$$\sum_{k=1}^{n}\lambda_k^{o2}=\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\,ln\,[f_o(0,\lambda)]+\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_o(0,\lambda)]. \tag{26}$$

Subtracting (26) from (25) we have

$$\begin{aligned}\sum_{k=1}^{n}\left(\lambda_k^2-\lambda_k^{o2}\right)&=\tfrac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\,ln\,[f_h(0,\lambda)]-\tfrac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\,ln\,[f_o(0,\lambda)]\\ &+\tfrac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_h(0,\lambda)]-\tfrac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_o(0,\lambda)].\end{aligned} \tag{27}$$

By using the asymptotic relations (19) and (20), one can write

$$\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_n(0,\lambda)]=\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[f_n(0,\lambda)]+\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 d\,ln\,[1+r(\lambda)], \tag{28}$$

where $r(\lambda)$ is defined by (19). By virtue of (28) equation (27) becomes

$$\sum_{k=1}^{n}\left(\lambda_k^2-\lambda_k^{o2}\right)=\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 dln[f_h(0,\lambda)]$$

$$-\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 dln[f_o(0,\lambda)]+\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2 dln[1+r(\lambda)]. \tag{29}$$

First, we calculate the 3rd. term of (29). Integrating by parts we have

$$\frac{1}{2\pi i}\int_{\Omega_n^+}\lambda^2\, d\, ln[1+r(\lambda)]=\frac{1}{2\pi i}\lambda^2\, ln[1+r(\lambda)]\Big|_{\Omega_n^+}-\frac{1}{2\pi i}\int_{\Omega_n^+}2\lambda\, d\, ln[1+r(\lambda)]. \tag{30}$$

From the asymptotic formula of $r(\lambda)$, we have $r(\lambda)\to 0$ as $n\to\infty$ on Ω_n^+, so that

$$-\frac{1}{2\pi i}\int_{\Omega_n^+}2\lambda d\, ln[1+r(\lambda)]=-\frac{1}{2\pi i}\int_{\Omega_n^+}2\lambda\left[r(\lambda)+\frac{1}{2}r^2(\lambda)+\bigcirc\left(\frac{1}{\lambda^3}\right)\right]d\lambda, \tag{31}$$

substituting from (19) into (31), we get

$$\begin{aligned}&-\tfrac{1}{2\pi i}\textstyle\int_{\Omega_n^+}2\lambda d\, ln[1+r(\lambda)]=\\&-\tfrac{1}{\pi i}\textstyle\int_{\Omega_n^+}\frac{v_1(\lambda)}{v_o(\lambda)}d\lambda-\frac{1}{\pi i}\int_{\Omega_n^+}\frac{v_2(\lambda)}{\lambda\, v_o(\lambda)}d\lambda+\frac{1}{2\pi i}\int_{\Omega_n^+}\frac{v_1^2(\lambda)}{\lambda\, v_o^2(\lambda)}d\lambda+\int_{\Omega_n^+}\bigcirc\left(\frac{1}{\lambda^3}\right)d\lambda.\end{aligned} \tag{32}$$

Further, by virtue of (32) and (30), equation (29) takes the form

$$\begin{aligned}\textstyle\sum_{k=1}^n\left(\lambda_k^2-\lambda_k^{o2}\right)&=\tfrac{1}{2\pi i}\textstyle\int_{-\tau_n}^{\tau_n}\lambda^2 dln[f_h(0,\lambda)]-\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 dln[f_o(0,\lambda)]\\&+\tfrac{1}{2\pi i}\lambda^2\, ln[1+r(\lambda)]\big|_{\Omega_n^+}-\tfrac{1}{\pi i}\textstyle\int_{\Omega_n^+}\frac{v_1(\lambda)}{v_o(\lambda)}d\lambda\\&-\tfrac{1}{\pi i}\textstyle\int_{\Omega_n^+}\frac{v_2(\lambda)}{\lambda\, v_o(\lambda)}d\lambda+\frac{1}{2\pi i}\int_{\Omega_n^+}\frac{v_1^2(\lambda)}{\lambda\, v_o^2(\lambda)}d\lambda+\int_{\Omega_n^+}\bigcirc\left(\frac{1}{\lambda^3}\right)d\lambda.\end{aligned} \tag{33}$$

For more convenient, we write equation (33) in the following abbreviated form

$$\sum_{k=1}^n\left(\lambda_k^2-\lambda_k^{o2}\right)=I_{f_h}-I_{f_o}+I_{r(\lambda)}-I_{v_1}-I_{v_2}+I_{v_1^2}+\int_{\Omega_n^+}\bigcirc\left(\frac{1}{\lambda^3}\right)d\lambda, \tag{34}$$

where

$$I_{f_h}=\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\, ln[f_h(0,\lambda)], \tag{35}$$

$$I_{f_o}=\frac{1}{2\pi i}\int_{-\tau_n}^{\tau_n}\lambda^2 d\, ln[f_o(0,\lambda)], \tag{36}$$

$$I_{r(\lambda)}=\frac{1}{2\pi i}\lambda^2\, ln[1+r(\lambda)]\Big|_{\Omega_n^+}, \tag{37}$$

$$I_{v_1}=\frac{1}{\pi i}\int_{\Omega_n^+}\frac{v_1(\lambda)}{v_o(\lambda)}d\lambda, \tag{38}$$

$$I_{v_2}=\frac{1}{\pi i}\int_{\Omega_n^+}\frac{v_2(\lambda)}{\lambda\, v_o(\lambda)}d\lambda, \tag{39}$$

$$I_{v_1^2}=\frac{1}{2\pi i}\int_{\Omega_n^+}\frac{v_1^2(\lambda)}{\lambda\, v_o^2(\lambda)}d\lambda. \tag{40}$$

Now we calculate each of the integrations (35)-(40).

$$I_{f_h}=\frac{1}{2\pi i}\int_{-\tau_n}^{0}\lambda^2 d\, ln[f_h(0,\lambda)]+\frac{1}{2\pi i}\int_{0}^{\tau_n}\lambda^2 d\, ln[f_h(0,\lambda)],$$

$$I_{f_h}=-\frac{1}{2\pi i}\int_{0}^{\tau_n}\lambda^2 d\, ln[f_h(0,-\lambda)]$$

$$+\frac{1}{2\pi i}\int_0^{\tau_n}\lambda^2 d\, ln[f_h(0,\lambda)] = \frac{-1}{2\pi i}\int_0^{\tau_n}\lambda^2 d\, ln\, S(\lambda). \tag{41}$$

Similarly

$$I_{f_o} = \frac{-1}{2\pi i}\int_0^{\tau_n}\lambda^2 d\, ln\, S_o(\lambda), \tag{42}$$

so that,

$$I_{f_h} - I_{f_o} \;=\; \frac{-1}{2\pi i}\int_0^{\tau_n}\lambda^2 d\, ln\, \frac{S(\lambda)}{S_o(\lambda)}. \tag{43}$$

Further, we evaluate $I_{r(\lambda)}$, $\lambda = \tau + i\zeta$ on $\Omega_n^+, \tau_n = \zeta_n = \pi\left(n + \frac{3}{4}\right)$,

$$\begin{aligned} I_{r(\lambda)} &= \frac{1}{2\pi i}\lambda^2\, ln[1 + r(\lambda)]\Big|_{\Omega_n^+} = -\frac{1}{2\pi i}\lambda^2\, ln[1 + r(\lambda)]\Big|_{-\tau_n}^{\tau_n} \\ &= -\frac{1}{2\pi i}\left[\tau_n^2\, ln[1 + r(\tau_n)] - \tau_n^2\, ln[1 + r(-\tau_n)]\right] \\ &= \frac{-1}{2\pi i}\left[\tau_n\frac{v_1(\tau_n)}{v_o(\tau_n)} + \frac{v_2(\tau_n)}{v_o(\tau_n)} - \frac{1}{2}\frac{v_1^2(\tau_n)}{v_o^2(\tau_n)}\right. \\ &\left. -\tau_n\frac{v_1(-\tau_n)}{v_o(-\tau_n)} - \frac{v_2(-\tau_n)}{v_o(-\tau_n)} + \frac{1}{2}\frac{v_1^2(-\tau_n)}{v_o^2(-\tau_n)} + \bigcirc\left(\frac{1}{\tau_n}\right)\right]. \end{aligned} \tag{44}$$

Each term of (44) can be written as

$$\frac{v_1(\tau_n)}{v_o(\tau_n)} = \begin{cases} \frac{[A(1,1)+K(1,1)]+i\,[(K(1,1)-A(1,1)]}{1+i} + \bigcirc\left(e^{-2\tau_n}\right), & \tau_n \to \infty, \\ \frac{[A(1,1)+K(1,1)]-i\,[K(1,1)-A(1,1)]}{1-i} + \bigcirc\left(e^{2\tau_n}\right), & \tau_n \to -\infty, \end{cases} \tag{45}$$

$$\frac{v_2(\tau_n)}{v_o(\tau_n)} = \begin{cases} \frac{[a+\beta-K(1,1)A(1,1)]+i\,[K(1,1)A(1,1)-\alpha-b]}{1+i} + \bigcirc\left(e^{-2\tau_n}\right), & \tau_n \to \infty, \\ \frac{[a+\beta-K(1,1)A(1,1)]-i\,[K(1,1)A(1,1)-\alpha-b]}{1-i} + \bigcirc\left(e^{2\tau_n}\right), & \tau_n \to -\infty. \end{cases} \tag{46}$$

So that, by virtue of (45), (46) and (44) we have

$$\begin{aligned} I_{r(\lambda)} = \tfrac{-1}{2\pi i}\Big\{\tau_n\left[\tfrac{d_1+id_2}{1+i} - \tfrac{d_1-id_2}{1-i}\right] + \tfrac{c_1+ic_2}{1+i} - \tfrac{c_1-ic_2}{1-i} + \\ \tfrac{1}{2}\left[-\left(\tfrac{d_1+id_2}{1+i}\right)^2 + \left(\tfrac{d_1-id_2}{1-i}\right)^2\right]\Big\} + \mathsf{o}(1), \end{aligned} \tag{47}$$

where $c_1 = a + \beta - K(1,1)A(1,1),\ c_2 = K(1,1)A(1,1) - \alpha - b,\ d_1 = A(1,1) + K(1,1)$ and $d_2 = K(1,1) - A(1,1)$, after simplification, we have

$$I_{r(\lambda)} = nA(1,1) + A_1 + \mathsf{o}(1), \tag{48}$$

where $A_1 = \frac{3A(1,1)}{4} - \frac{A(1,1)K(1,1)}{\pi} + \frac{a+b+\alpha+\beta}{2\pi}$.

To calculate the integration I_{v_1}, we write

$$I_{v_1} = I_{v_{11}} + I_{v_{12}} + I_{v_{13}}, \tag{49}$$

where

$$I_{v_{11}} = \frac{-1}{\pi}\int_0^{\zeta_n}\frac{d_1\cosh(\tau_n + i\zeta) + i\, d_2\sinh(\tau_n + i\zeta)}{\cosh(\tau_n + i\zeta) + i\,\sinh(\tau_n + i\zeta)}\, d\zeta, \tag{50}$$

$$I_{v_{12}} = \frac{1}{\pi}\int_0^{\zeta_n}\frac{d_1\cosh(-\tau_n + i\zeta) + i\, d_2\sinh(-\tau_n + i\zeta)}{\cosh(-\tau_n + i\zeta) + i\,\sinh(-\tau_n + i\zeta)}\, d\zeta, \tag{51}$$

$$I_{v_{13}} = \frac{1}{\pi i}\int_{-\tau_n}^{\tau_n}\frac{d_1\cosh(\tau + i\zeta_n) + i\, d_2\sinh(\tau + i\zeta_n)}{\cosh(\tau + i\zeta_n) + i\,\sinh(\tau + i\zeta_n)}\, d\tau. \tag{52}$$

As for $I_{v_{11}}$, by the help of the asymptotic formula $\tanh(\tau_n + i\zeta) = 1 + \bigcirc(e^{-2\tau_n})$, we have

$$I_{v_{11}} = \frac{-1}{\pi} \int_0^{\zeta_n} \left[\frac{d_1 + i\, d_2}{1+i} + \bigcirc(e^{-2\tau_n})\right] d\zeta = \frac{-1}{\pi}\, \frac{d_1 + id_2}{1+i}\, \zeta_n + \mathsf{o}(1). \tag{53}$$

Similarly, by using, the asymptotic formula $\tanh(-\tau_n + i\zeta) = -1 + \bigcirc(e^{-2\tau_n})$, in $I_{v_{12}}$, we have

$$I_{v_{12}} = \frac{1}{\pi} \int_0^{\zeta_n} \left[\frac{d_1 - i\, d_2}{1-i} + \bigcirc(e^{-2\tau_n})\right] d\zeta = \frac{1}{\pi}\, \frac{d_1 - id_2}{1-i}\, \zeta_n + \mathsf{o}(1), \tag{54}$$

since $\zeta_n = \pi\left(n + \frac{3}{4}\right)$ we have

$$I_{v_{11}} + I_{v_{12}} = nA(1,1) + \frac{3A(1,1)}{4} + \mathsf{o}(1). \tag{55}$$

The asymptotic formulas used in $I_{v_{11}}$ and $I_{v_{12}}$ could not apply into $I_{v_{13}}$ (because the integration, in $I_{v_{13}}$ is with respect to τ), so that it is more convenient to write $\cosh(\tau + i\zeta_n) = \frac{(-1)^n}{\sqrt{2}}[i\ \sinh\tau - \cosh\tau]$ and $\sinh(\tau + i\zeta_n) = \frac{(-1)^n}{\sqrt{2}}[i\ \cosh\tau - \sinh\tau]$, so that, from(52), we have

$$\begin{aligned} I_{v_{13}} &= \frac{1}{\pi i} \int_{-\tau_n}^{\tau_n} \frac{d_1[i\ \sinh\tau - \cosh\tau] + i\ d_2[i\ \cosh\tau - \sinh\tau]}{[i\ \sinh\tau - \cosh\tau] + i\ [i\ \cosh\tau - \sinh\tau]}\ d\tau \\ &= \frac{1}{\pi i} \int_{-\tau_n}^{\tau_n} \frac{d_1 + d_2}{2} d\tau = -iA(1,1)n - \frac{3i}{4} A(1,1). \end{aligned} \tag{56}$$

From (49), (55) and(56) we obtain

$$I_{v_1} = n\ (1-i)A(1,1) + \frac{3}{4}(1-i)A(1,1) + \mathsf{o}(1). \tag{57}$$

To evaluate I_{v_2} let

$$I_{v_2} = I_{v_{21}} + I_{v_{22}} + I_{v_{23}}, \tag{58}$$

where

$$I_{v_{21}} = \frac{-1}{\pi} \int_0^{\zeta_n} \frac{c_1 \cosh(\tau_n + i\zeta) + i\ c_2 \sinh(\tau_n + i\zeta)}{(\tau_n + i\zeta)[\cosh(\tau_n + i\zeta) + i\ \sinh(\tau_n + i\zeta)]}\, d\zeta, \tag{59}$$

$$I_{v_{22}} = \frac{1}{\pi} \int_0^{\zeta_n} \frac{c_1 \cosh(-\tau_n + i\zeta) + i\ c_2 \sinh(-\tau_n + i\zeta)}{(-\tau_n + i\zeta)[\cosh(-\tau_n + i\zeta) + i\ \sinh(-\tau_n + i\zeta)]}\, d\zeta, \tag{60}$$

$$I_{v_{23}} = \frac{1}{\pi i} \int_{-\tau_n}^{\tau_n} \frac{c_1 \cosh(\tau + i\zeta_n) + i\ c_2 \sinh(\tau + i\zeta_n)}{(\tau + i\zeta_n)[\cosh(\tau + i\zeta_n) + i\ \sinh(\tau + i\zeta_n)]}\ d\tau, \tag{61}$$

where c_1, c_2 are given by (47). For $I_{v_{21}}$, as we did befor , by using the asymptotic formula $\tanh(\tau_n + i\zeta) = 1 + \bigcirc(e^{-2\tau_n})$, after some calculation, we have

$$I_{v_{21}} = \frac{-1}{\pi} \frac{c_1 + i\ c_2}{1+i} \int_0^{\zeta_n} \frac{d\zeta}{(\tau_n + i\zeta)} + \mathsf{o}(1) = \frac{-1}{\pi} \frac{c_1 + i\ c_2}{1+i}\ \ ln(\tau_n + i\zeta)\Big|_0^{\zeta_n} + \mathsf{o}(1), \tag{62}$$

keeping in mined that $\tau_n = \zeta_n = \pi\left(n + \frac{3}{4}\right)$ on Ω_n^+, $I_{v_{21}}$ becomes

$$I_{v_{21}} = \frac{-1}{\pi i} \frac{c_1 + i\ c_2}{1+i}\ \ ln(1+i) + \mathsf{o}(1). \tag{63}$$

Using the asymptotic formula $\tanh(-\tau_n + i\zeta) = -1 + \bigcirc(e^{-2\tau_n})$, in $I_{v_{12}}$, we deduce, after some calculation, that

$$I_{v_{22}} = \frac{1}{\pi i}\frac{c_1 - i\,c_2}{1-i}\int_0^{\zeta_n}\frac{d\zeta}{(\tau_n + i\zeta)} = \frac{1}{\pi i}\;\frac{c_1 - ic_2}{1-i}\;\; ln(\tau_n + i\zeta)\Big|_0^{\zeta_n} + \mathsf{o}(1), \quad (64)$$

again $\zeta_n = \pi\left(n + \frac{3}{4}\right)$ on Ω_n^+, so that,

$$I_{v_{22}} = \frac{1}{\pi i}\;\frac{c_1 - ic_2}{1-i}\;\; ln(1-i) + \mathsf{o}(1). \quad (65)$$

From (63) and (65) we have

$$I_{v_{21}} + I_{v_{22}} = 2\;Re\left\{\frac{1}{\pi i}\;\frac{c_1 - ic_2}{1-i}\;\; ln(1-i)\right\} + \mathsf{o}(1)$$

$$= \frac{1}{\pi}\left[\;ln\sqrt{2}\;(c_1 - c_2)\; + \frac{3\pi}{4}(c_1 + c_2)\;\right] + \mathsf{o}(1). \quad (66)$$

Further, to evaluate $I_{v_{23}}$, we use the equalities

$$\cosh(\tau + i\zeta_n) = \frac{(-1)^n}{\sqrt{2}}[i\;\sinh\tau - \cosh\tau], \quad \sinh(\tau + i\zeta_n) = \frac{(-1)^n}{\sqrt{2}}[i\;\cosh\tau - \sinh\tau], \quad (67)$$

$$I_{v_{23}} = \frac{1}{\pi i}\int_{-\tau_n}^{\tau_n}\frac{c_1[\cosh\tau - i\;\sinh\tau] + i\;c_2[\sinh\tau - i\cosh\tau]}{(\tau + i\zeta_n)[\cosh\tau - i\;\sinh\tau + i(\sinh\tau - i\cosh\tau)]}\;d\tau,$$

$$= \frac{c_1 + c_2}{\pi i}\int_{-\tau_n}^{\tau_n}\frac{d\tau}{(\tau + i\zeta_n)} \;+\; \frac{c_2 - c_1}{\pi}\int_{-\tau_n}^{\tau_n}\frac{tanh(\tau + i\zeta_n)}{(\tau + i\zeta_n)}\;d\tau,$$

therefoe,

$$I_{v_{23}} = \frac{3(c_1 + c_2)}{4} \;+\; \frac{c_2 - c_1}{\pi}\;ln2 \;+\; \mathsf{o}(1). \quad (68)$$

From (58), (66) and (68) we have

$$I_{v_2} = \frac{3(c_1 + c_2)}{2} \;+\; \frac{c_2 - c_1}{\pi}\;ln\sqrt{2} \;+\; \mathsf{o}(1). \quad (69)$$

Let

$$I_{v_1^2} = I_{v_{11}^2} + I_{v_{12}^2} + I_{v_{13}^2}, \quad (70)$$

where

$$I_{v_{11}^2} = \frac{1}{2\pi}\int_0^{\zeta_n}\frac{1}{(\tau_n + i\zeta)}\left(\frac{d_1\cosh(\tau_n + i\zeta) + i\;d_2\sinh(\tau_n + i\zeta)}{\cosh(\tau_n + i\zeta) + i\;\sinh(\tau_n + i\zeta)}\right)^2 d\zeta, \quad (71)$$

$$I_{v_{12}^2} = \frac{-1}{2\pi}\int_0^{\zeta_n}\frac{1}{(-\tau_n + i\zeta)}\left(\frac{d_1\cosh(-\tau_n + i\zeta) + i\;d_2\sinh(-\tau_n + i\zeta)}{\cosh(-\tau_n + i\zeta) + i\;\sinh(-\tau_n + i\zeta)}\right)^2 d\zeta, \quad (72)$$

$$I_{v_{13}^2} = \frac{-1}{2\pi i}\int_{-\tau_n}^{\tau_n}\frac{1}{(\tau + i\zeta_n)}\left(\frac{d_1\cosh(\tau + i\zeta_n) + i\;d_2\sinh(\tau + i\zeta_n)}{\cosh(\tau + i\zeta_n) + i\;\sinh(\tau + i\zeta_n)}\right)^2 d\tau. \quad (73)$$

We evaluate, now, the right hand side of (70)

$$I_{v_{11}^2} = \frac{1}{2\pi}\int_0^{\zeta_n}\frac{1}{(\tau_n + i\zeta)}\left(\frac{d_1\cosh(\tau_n + i\zeta) + i\;d_2\sinh(\tau_n + i\zeta)}{\cosh(\tau_n + i\zeta) + i\;\sinh(\tau_n + i\zeta)}\right)^2 d\zeta$$

$$= \frac{1}{2\pi}\int_0^{\zeta_n}\frac{1}{(\tau_n + i\zeta)}\left(\frac{d_1 + i\;d_2\tanh(\tau_n + i\zeta)}{1 + i\;\tanh(\tau_n + i\zeta)}\right)^2 d\zeta,$$

by using the asymptotic formula, $tanh(\tau_n + i\zeta) = 1 + \bigcirc(e^{-2\tau_n})$, we have

$$I_{v_{11}^2} = \frac{1}{2\pi}\left(\frac{d_1 + id_2}{1+i}\right)^2 \int_0^{\zeta_n} \frac{d\zeta}{(\tau_n + i\zeta)} + \mathsf{o}(1)$$

$$= \frac{1}{2\pi i}\left(\frac{d_1 + id_2}{1+i}\right)^2 ln(1+i) \ + \ \mathsf{o}(1). \qquad (74)$$

By using the asymptotic formula $tanh(-\tau_n + i\zeta) = -1 + \bigcirc(e^{-2\tau_n})$ and similar technique, as in $I_{v_{11}^2}$, we obtain

$$I_{v_{12}^2} = \frac{-1}{2\pi i}\left(\frac{d_1 - id_2}{1-i}\right)^2 ln(1-i) \ + \ \mathsf{o}(1). \qquad (75)$$

From (74) and (75), we have

$$I_{v_{11}^2} + I_{v_{12}^2} = 2Re\left\{\frac{1}{2\pi i}\left(\frac{d_1 + id_2}{1-i}\right)^2 ln(1+i)\right\} + \mathsf{o}(1)$$

$$= \frac{(d_2^2 - d_1^2)ln2}{4\pi} + \frac{d_1\ d_2}{4} \ + \mathsf{o}(1). \qquad (76)$$

As for $I_{v_{13}^2}$, we use the equalities $cosh(\tau + i\zeta_n) = \frac{(-1)^n}{\sqrt{2}}[\ i\ sinh\tau - cosh\tau\ ]$, $sinh(\tau + i\zeta_n) = \frac{(-1)^n}{\sqrt{2}}[\ i\ cosh\tau - sinh\tau\ ]$, therefore, (73) becomes

$$I_{v_{13}^2} = \frac{-(d_1 + d_2)^2}{8\pi i}\int_{-\tau_n}^{\tau_n} \frac{d\tau}{(\tau + i\zeta_n)}$$

$$+\frac{d_1^2 - d_2^2}{4\pi}\int_{-\tau_n}^{\tau_n} \frac{\tanh\tau d\tau}{(\tau + i\zeta_n)} + \frac{(c_2 - c_1)^2}{8\pi i}\int_{-\tau_n}^{\tau_n} \frac{\tanh^2\tau d\tau}{(\tau + i\zeta_n)}. \qquad (77)$$

By using the asymptotic formula $tanh\ \tau = 1 + \bigcirc(e^{-2\tau_n})$ in the second and third terms of (77), we obtain

$$I_{v_{13}^2} = -\frac{-3(d_1 + d_2)^2}{16} + \frac{d_1^2 - d_2^2}{4\pi}\ ln2 - \frac{(d_2 - d_1)^2}{16} \ + \ \mathsf{o}(1). \qquad (78)$$

From (76) and (78), we have

$$I_{v_1^2} = -\frac{d_1^2 + d_2^2}{4} \ + \ \mathsf{o}(1). \qquad (79)$$

Substituting from (43),(48),(57),(69) and (79) into (34), and passing to limit as $n \to \infty$ we obtain the required formula (21), which complete the proof. □

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Zaki F.A. El-Raheem

Department of Mathematics, Faculty of Education, Alexandria University, Alexandria, Egypt.

E-mail address: `zaki55@hotmail.com`

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