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ON NEGATIVE SPECTRUM OF TWO-DIMENSIONAL SCHRÖDINGER OPERATOR IN ELECTROMAGNETIC FIELD

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In the paper we study singularities of a resolvent equation, asymptotic behavior and smoothness of solutions of a homogeneous resolvent equation, and also the negative part of the spectrum of two-dimensional magnetic Schrödinger operator in the space $L_2(\mathbb{R}_2)$.

1. Introduction

In the two-dimensional space $\mathbb{R}_2$ consider the Schrödinger magnetic differential expression

$$H_{a,V} = \sum_{k=1}^{2} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + a_k(x) \right)^2 + V(x), \tag{1}$$

where $i = \sqrt{-1}$ is an imaginary unit, $x = (x_1, x_2) \in \mathbb{R}_2$, $a(x) = (a_1(x), a_2(x))$ and $V(x)$ are magnetic and electric potentials, respectively.

It is well known that two-dimensional Schrödinger operators (without magnetic potential) have the following peculiarities that call significant problems in their analysis. Firstly, the classic Hardy inequality is not fulfilled and the Cwikel–Lieb–Rosenblum inequality doesn't hold.

There is a deep relation between the Hardy inequality and the threshold of the essential spectrum of the Schrödinger operator (see e.g. [15]). If the Hardy inequality is not fulfilled, the threshold of the essential spectrum is very sensitive to the smallest changes of the electrical potential, i.e. in this case the Schrödinger operator becomes a virtual operator. Recall that an operator is said to be virtual if to the smallest change of the electric potential there arises even one eigenvalue left from the threshold of the substantial spectrum. After introducing the magnetic field, by virtue of diamagnetic inequality (that is the consequence of the Kato inequality (see [8] or [2])) one can expect that the above described situations may be improved. Indeed in the paper [9], Laptev and Weidl showed that if one replaces an ordinary gradient by a magnetic gradient, then under certain conditions on the magnetic field the Hardy inequality becomes possible. In particular, the Aharonov–Bohm magnetic field being the Dirac $\delta-$function is

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contained in the Laptev-Weidl class. Using the results of Laptev-Weidl, in [3] Balinsky, Evans and Lewis proved that the Cwikel–Lieb–Rosenblum inequality is valid for the Aharonov–Bohm magnetic field. Note that if the magnetic field is an integer, then the ordinary and magnetic Schrödinger operators and unitary equivalent, therefore in this case if the ordinary Schrödinger operator is virtual, the magnetic Schrödinger operator is also virtual.

In the present paper, in the space $L_2(\mathbb{R}_2)$ we study the singularities of the resolvent equation, asymptotic behavior and smoothness of the solutions of a homogeneous resolvent equation and the negative part of the spectrum of two-dimensional magnetic Schrödinger operator generated by the differential expression $H_{a,V}$, where the real magnetic and electric potentials $a(x)$ and $V(x)$ satisfy the following conditions:

1) $\int_{\mathbb{R}_2} |a(x)|^{\nu}\, dx < +\infty$, where $\nu > 2$, $|a(x)| = \sqrt{a_1^2(x_1,x_2) + a_2^2(x_1,x_2)}$;

2) $\int_{\mathbb{R}_2} |\Phi(x)|^{\mu}\, dx < +\infty$, where $\mu > 1$,

$$\Phi(x) \equiv \Phi(x_1,x_2) = a^2(x_1,x_2) \, + V(x_1,x_2) + i\, div\, a(x_1,x_2),$$

$$a^2(x) \equiv a^2(x_1,x_2) = a_1^2(x_1,x_2) + a_2^2(x_1,x_2),$$

$$div\, a(x_1,x_2) \; = \frac{\partial a_1(x_1,x_2)}{\partial x_1} + \frac{\partial a_2(x_1,x_2)}{\partial x_2}.$$

Note that the similar issues in one-dimensional case were investigated in [1], in three-dimensional case in [10] and [11]. It is known that estimation of the number of negative eigen values plays an important role both in quantum mechanics and in spectral theory of differential operators. A lot of papers have been devoted to the investigation of the negative part of the spectrum of the Schrödinger operator. In the first turn we indicate the papers [4], [7], [18], [19] and references therein.

Denote by $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Im\lambda > 0\}$ ($\mathbb{C}$ is a complex plane) an upper half-plane. Let H_0 be an operator (it is called a free Hamiltonian) $-\Delta$ in $L_2(\mathbb{R}_2)$ with domain of definition $D(H_0) = W_2^2(\mathbb{R}_2)$ (second order Sobolev space). As the spectrum of the self-adjoint operator H_0 coincides with the positive semi-axis $[0,+\infty)$, then for any complex number from $\mathbb{C}_+$ the operator $H_0 - \lambda^2$ is the bijection of $W_2^2(\mathbb{R}_2)$ on $L_2(\mathbb{R}_2)$ with the bounded inverse $R_0(\lambda^2) = \left(H_0 - \lambda^2\right)^{-1}$. The operator $R_0(\lambda^2)$ is an integral operator i.e. for any $f(x) \in L_2(\mathbb{R}_2)$

$$R_0(\lambda^2)f(x) = \int_{\mathbb{R}_2} G_0(x,y,\lambda)f(y)dy.$$

Here the generalized function

$$G_0(x,y,\lambda) = \frac{i}{4} H_0^{(1)}(\lambda\, |x-y|)$$

is a fundamental (see e.g. [16, p. 204]) solution of the operator $-\Delta - \lambda^2$, i.e.

$$(-\Delta - \lambda^2)G_0(x,y,\lambda) = \delta(x-y),$$

where Δ two-dimensional Laplace operator, $\delta(x-y)$ δ is the Dirac function, $H_0^{(1)}(\lambda\, |x-y|)$ is the first order Hankel function.

Subject to conditions 1) and 2) we can write differential expression (1) in the form

$$\Delta_{a,V} = -\Delta + W,$$

where

$$W = -2i\, div\, a(x) + \Phi(x). \tag{2}$$

Consider in $L_2(\mathbb{R}_2)$ the quadratic forms

$$h_0(\varphi) = \int_{-\infty}^{+\infty} |\nabla\varphi|^2\, dx,$$

$$h_{a,V}(\varphi) = h_0(\varphi) + (W\varphi, \varphi)\,,$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ is a symbolic Hamilton vector, W is an operator acting by formula (2). Obviously, $h_0(\varphi)$ corresponds to the self-adjoint operator $H_0 := -\Delta$ with domain of definition $W_2^2(\mathbb{R}_2)$. It is known that $Q(h_0) = W_2^1(\mathbb{R}_2) = D(H_0^{1/2})$ (first order Sobolev space) and for any $\varphi \in Q(h_0)$, $h_0(\varphi) = (H_0^{1/2}\varphi, H_0^{1/2}\varphi)$.

Note that if $a(x)$ and $V(x)$ are sufficiently smooth bounded functions, then minimal (in this case they are maximal) operators H_0 and $H = H_0 + W$, respectively, that correspond to differential expression $-\Delta$ and, $H_{a,V}$ are self-adjoint operator in $L_2(\mathbb{R}_2)$ with the same domains of definition $W_2^2(\mathbb{R}_2)$ (second order Sobolev space). Generally speaking, under conditions 1) and 2) the differential expression $H_{a,V}$ doesn't define the minimal operator on the linear manifold $C_0^\infty(\mathbb{R}_2)$, therefore for constructing a self-adjoint operator by means of this expression, in the paper [5] the method of quadratic forms is used and the following theorem is proved.

Theorem 1. *Let conditions* 1) *and* 2) *be fulfilled. Then there exists a unique lower bounded self-adjoint operator* $H = H_0 + W$*, responding to the form* $h_{a,V}(\varphi) = h_0(\varphi) + (W\varphi, \varphi)$ *with* $Q(H_0) = Q(H)$ *such that any essential domain of the operator* H_0 *is a essential domain for the operator* H *as well. In particular, the space of the governing functions* $C_0^\infty(\mathbb{R}_2)$ *is the essential domain of definition of the operator* H.

Note that the sum $H_0 + W$ is understood in the sense of forms and it may differ from the operator sum.

Let $C(\mathbb{R}_2)$ be a Banach space of all bounded continuous functions on $\mathbb{R}_2$ with the nom $\sup\limits_{x\in\mathbb{R}_2} |f(x)| = \|f\|_{C(\mathbb{R}_2)} < +\infty$.

Let $h(x) \in C_0^\infty(\mathbb{R}_2)$ and $z = \lambda^2$, $Im\lambda > 0$. Suppose

$$u_0(\lambda) \equiv u_0(x, \lambda) = R_0(\lambda^2)h(x), \;\; u(\lambda) \equiv u(x, \lambda) = R(\lambda^2)h(x),$$

where $R_0(\lambda^2) = \left(H_0 - \lambda^2\right)^{-1}$ and $R(\lambda^2) = \left(H - \lambda^2\right)^{-1}$ are the resolvents of the operators H_0 and H, respectively. Taking into account that the operators $-\Delta$ and $R_0(\lambda^2)$ are permutational, and according to theorem 1 the space of governing functions $C_0^\infty(\mathbb{R}_2)$ is a essential domain of both operators H_0 and H from the equation of perturbation theory

$$R(\lambda^2) + R_0(\lambda^2)WR(\lambda^2) = \; R_0(\lambda^2),$$

for $u(\lambda)$ we get the inhomogeneous equation

$$u(\lambda)+K(\lambda)u(\lambda)= u_0(\lambda), \tag{3}$$

where $K(\lambda)$ is an integral operator with the kernel

$$K(x,y,\lambda)=$$

$$G_0(x,y,\lambda)\Phi(y)-2i\frac{\partial G_0(x,y,\lambda)}{\partial x_1}a_1(y)-2i\frac{\partial G_0(x,y,\lambda)}{\partial x_2}a_2(y).$$

In the paper [6] it is proved that the operator $K(\lambda)$ is compact in $C(\mathbb{R}_2)$ for all λ from $\mathbb{C}_+$, is continuous with respect to λ in the uniform operator topology and is analytic with respect to λ in $\mathbb{C}_+$ in the same topology. These results allow to apply to the equation

$$f+K(\lambda)f= 0 \tag{4}$$

the Fredholm analytic theorem [13, p. 224, theorem VI.14]. According to Fredholm's theory, inhomogeneous equation (3) for $Im\lambda>0$ has a unique solution in $C(\mathbb{R}_2)$ if the respective homogeneous equation (4) has only a zero solution.

2. Main Results

It is known (see e.g. [12], [14]) that study of peculiarities with respect to parameter λ of the solution of the scattering theory problem is reduced to investigation of the set of those λ at which homogeneous equation (4) has a non-trivial solution, and also asymptotic behavior and smoothness of the solutions $f(x)$ themselves. Now investigate equation (4).

The following theorem is valid

Theorem 2. *Let $\lambda\in\mathbb{C}_+$, and $f(x)$ be the solution of the equation*

$$f(x)=-\int_{\mathbb{R}_2}K(x,y,\lambda)f(y)dy. \tag{5}$$

Then if conditions 1) *and* 2) *are fulfilled, then for any $\delta\in(0,Im\lambda)$ there exists a number $C_\delta>0$ such that for any x from $\mathbb{R}_2$*

$$|f(x)|\le\frac{C_\delta}{\sqrt{1+|x|}}e^{-\delta|x|}. \tag{6}$$

Proof. According to general theory of compact operators (see. [12, p.41] or [11]) there exists a sequence of numbers $\{\gamma_n\}$ and sequence of functions $\{f_n(x)\}\subset C(\mathbb{R}_2)$ such that for each n

$$f_n(x)=-\gamma_n\int_{|y|\le n}K(x,y,\lambda)f_n(y)dy, \tag{7}$$

and $\lim\limits_{n\to\infty}\gamma_n=1$, $f_n(x)\to f(x)$ uniformly as $n\to\infty$.

Assume

$$g_n(x)=\sqrt{1+|x|}e^{\delta|x|}f_n(x).$$

According to (7) we have

$$g_n(x) = -\gamma_n \int_{|y|\le n} K_\delta(x,y,\lambda) g_n(y)dy, \tag{8}$$

where

$$K_\delta(x,y,\lambda) = \frac{\sqrt{1+|x|}}{\sqrt{1+|y|}} e^{\delta(|x|-|y|)} K(x,y,\lambda).$$

From the conjecture for $|y| \le n, \quad |x| \ge n+1$ we have

$$|K(x,y,\lambda)| \le \frac{C}{\sqrt{1+|x|}} e^{-Im\lambda|x|}(|\Phi(y)| + |a(y)|), \tag{9}$$

where $C > 0$ is a constant number. From estimation (9) and conditions 1), 2), by virtue of compactness of the operator $K(\lambda)$ (see [6]) we get that the operator $K_\delta(\lambda)$ with the kernel $K_\delta(x,y,\lambda)$ is a completely continuous operator in $C(\mathbb{R}_2)$. It is clear that

$$\lim_{n\to\infty} \|K_{\delta,n}(\lambda) - K_\delta(\lambda)\|_{C(\mathbb{R}_2)} = 0,$$

where $K_{\delta,n}(\lambda) = K_\delta(\lambda)\chi_n$, χ_n is an operator of multiplication by the characteristic function of the sphere $S_n(0) = \{x \in \mathbb{R}_2 : |x| \le n\}$. It follows from equation (8) and compactness of the operators $K_{\delta,n}(\lambda)$ that there exist $g(x) \in C(\mathbb{R}_2)$ and the subsequence $\{g_{n_k}(x)\}$ such that $g_{n_k}(x) \to g(x)$ uniformly as $k \to \infty$. The uniform boundedness of the sequence $\{g_{n_k}(x)\}$ yields that there exists a number $C_\delta > 0$ such that for any natural k it is valid the inequality

$$\sup_{x\in\mathbb{R}_2} |g_{n_k}(x)| \le C_\delta.$$

Passing in this inequality to limit as $k \to \infty$, we get

$$\sup_{x\in\mathbb{R}_2} |g(x)| \le C_\delta,$$

whence taking into account the representation $g(x) = \sqrt{1+|x|}e^{\delta|x|}f(x)$ it follows that inequality (6) is valid. The theorem is proved.

Theorem 3. *Let $\lambda \in \mathbb{C}_+$, and $f(x)$ be the solution of equation (5). If in addition to conditions* 1) *and* 2) *for rather small positive number δ the condition*

3) $\int_{|x-y|<\delta} \frac{|a(y)|}{|x-y|^2} dy \in L_2(\mathbb{R}_2)$,

is fulfilled, then $f(x) \in W_2^1(\mathbb{R}_2)$.

Proof. It is known that (see [6]) if $x \ne y (x,y \in \mathbb{R}_2)$ and $0 \le \arg\lambda \le \pi$, then as $\lambda\,|x-y| \to 0$ it is valid the asymptotic formula

$$\frac{\partial}{\partial x_j} G_0(x,y,\lambda) \sim \; -\frac{1}{2\pi} \frac{x_j - y_j}{|x-y|^2}, \; j = 1,2.$$

From this property and lemma 2 from the book [16] (see [16, p. 281]) we get

$$\left| \int_{\mathbb{R}_2} \frac{\partial G_0(x,y,\lambda)}{\partial x_j} \overline{\frac{\partial G_0(x,y,\lambda)}{\partial x_j}} dy \right| \le A_j \left| \ln \frac{1}{|x-y|} \right|, \; j = 1,2 \;, \tag{10}$$

where A_j are some constants. By condition 2), from estimation (10) and the equality (see [5] or [6])

$$\lim_{0<\delta\to 0}\left\{\sup_{x\in\mathbb{R}_2}\int_{|x-y\leq\delta|}\ln\frac{1}{|x-y|}\,|\Phi(y)|\,dy\right\}=0\,,$$

with regard to theorem 2 we get

$$\int_{\mathbb{R}_2}\frac{\partial G_0(x,y,\lambda)}{\partial x_j}\Phi(y)f(y)dy\in L_2(\mathbb{R}_2),\ j=1,2. \tag{11}$$

Consider

$$\psi_{j,k,l}(x)=\int_{\mathbb{R}_2}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_j\partial x_k}a_l(y)f(y)dy,\ j,k,l=1,2.$$

Represent $\psi_{j,k,l}(x)$ in the form

$$\psi_{j,k,l}(x)=\int_{|x-y|<\delta}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_j\partial x_k}a_l(y)f(y)dy+$$
$$\int_{|x-y|\geq\delta}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_j\partial x_k}a_l(y)f(y)dy=$$
$$\psi^{(1)}_{j,k,l}(x)+\psi^{(2)}_{j,k,l}(x),\ j,k,l=1,2\ . \tag{12}$$

Using the asymptotic formula (see [6])

$$G_0(x,y,\lambda)=\frac{i}{4}\sqrt{\frac{1}{\pi\lambda\,|x-y|}}e^{i\left(\lambda|x-y|-\frac{\pi}{4}\right)}\left[1+O\left(\frac{1}{\lambda\,|x-y|}\right)\right]$$
$$(x\neq y\,(x,y\in\mathbb{R}_2),\ 0\leq\arg\lambda<\pi,\ \text{as}\ |\lambda|\,|x-y|\to+\infty)$$

and condition 1), with regard to theorem 2 we have

$$\psi^{(2)}_{j,k,l}(x)\in L_2(\mathbb{R}_2),\ j,k,l=1,2.$$

From the estimation

$$\left|\int_{|x-y|<\delta}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_j\partial x_k}a_l(y)f(y)dy\right|\leq$$
$$\|f(x)\|_{C(\mathbb{R}_2)}\int_{|x-y|<\delta}\frac{|a_l(y|}{|x-y|^2}dy,\ j,k,l=1,2,$$

and condition 3) it follows that $\psi^{(1)}_{j,k,l}(x)\in L_2(\mathbb{R}_2),\ j,k,l=1,2$. Thus, from representation (12) we have

$$\psi_{j,k,l}(x)=\int_{\mathbb{R}_2}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_j\partial x_k}a_l(y)f(y)dy\in L_2(\mathbb{R}_2),\ j,k,l=1,2. \tag{13}$$

Relations (11) and (13) show that the generalized derivatives

$$\frac{\partial f(x)}{\partial x_j}=\int_{\mathbb{R}_2}\frac{\partial G_0(x,y,\lambda)}{\partial x_j}\Phi(y)f(y)dy-$$

$$2i\int_{\mathbb{R}_2}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_1\partial x_j}a_1(y)f(y)dy-2i\int_{\mathbb{R}_2}\frac{\partial^2 G_0(x,y,\lambda)}{\partial x_2\partial x_j}a_2(y)f(y)dy,\ j=1,2,$$

of the function $f(x)$ belong to $L_2(\mathbb{R}_2)$. Hence and from theorem 2 it follows that $f(x)\in W_2^1(\mathbb{R}_2)$. The theorem is proved

Theorem 4. *Let $\lambda\in\mathbb{C}_+$, and $f(x)$ be the solution of equation (5). Then if conditions* 1), 2) *and* 3) *are fulfilled, then $\lambda=i\tau$, $\tau>0$. And $\lambda^2=-\tau^2$ is the eigenvalue of the operator H of finite multiplicity.*

Proof. Show that the number λ^2 is the eigenvalue of the operator H. From theorems 2 and 3 it follows that equation (5) has a non-trivial and descending solution from the space $W_2^1(\mathbb{R}_2)$. Show that it is valid

$$h_{a,V}(f)=h_0(f)+(Wf,f)=\lambda^2(f,f).$$

From the everywhere density of the space of governing functions $C_0^\infty(\mathbb{R}_2)$ in $W_2^1(\mathbb{R}_2)$ it follows that there exists a sequence $\{f_n(x)\}_{n=1}^\infty\subset C_0^\infty(\mathbb{R}_2)$ such that $\lim\limits_{n\to\infty}\|f_n(x)-f(x)\|_{W_2^1(\mathbb{R}_2)}=0$. Since $Im\lambda>0$, then the operator $-\Delta-\lambda^2$ one-to-one maps the spaces generalized functions of slower growth S' onto itself (S is the Schwartz space [17, p. 87]). It is known that $C_0^\infty(\mathbb{R}_2)\subset S'$ and $W_2^1(\mathbb{R}_2)\subset S'$. Hence it follows that the images of the elements of the spaces $C_0^\infty(\mathbb{R}_2)$ and $W_2^1(\mathbb{R}_2)$ at mapping $-\Delta-\lambda^2$ became the elements of the space of generalized functions of slower growth S'. In particular, the linear manifold $\left(-\Delta-\lambda^2\right)C_0^\infty(\mathbb{R}_2)$ is everywhere dense both in $L_2(\mathbb{R}_2)$ and $W_2^1(\mathbb{R}_2)$. The similar results hold for the operator $-\Delta-\overline{\lambda}^2$ as well. Now let $\psi\in\left(-\Delta-\overline{\lambda}^2\right)C_0^\infty(\mathbb{R}_2)$. Then there exists a unique element $\varphi\in C_0^\infty(\mathbb{R}_2)$ such that $\psi=\left(-\Delta-\overline{\lambda}^2\right)\varphi$. Taking into account (5) and the equality $K(\lambda)=R_0(\lambda^2)W$, we have

$$0=(f+K(\lambda)f,\psi)=\lim_{n\to\infty}(f_n+K(\lambda)f_n,\psi)=$$

$$\lim_{n\to\infty}\left(f_n+K(\lambda)f_n,\left(-\Delta-\overline{\lambda}^2\right)\varphi\right)=$$

$$\lim_{n\to\infty}\left(\left(-\Delta-\lambda^2\right)\left(f_n+K(\lambda)f_n\right),\varphi\right)=$$

$$\lim_{n\to\infty}\left(\left(-\Delta-\lambda^2\right)\left(f_n+R_0(\lambda^2)Wf_n\right),\varphi\right)=$$

$$\lim_{n\to\infty}\left(\left(-\Delta-\lambda^2\right)f_n+Wf_n,\varphi\right)=$$

$$\left(\left(-\Delta-\lambda^2\right)f+Wf,\varphi\right)=\left(Hf-\lambda^2f,\varphi\right).$$

By arbitrariness of ψ (together with it by arbitrariness of φ) we get

$$Hf=-\Delta f+Wf=\lambda^2 f. \tag{14}$$

For proving the equality $\lambda=i\tau$, $\tau>0$ it suffices to note that $\lambda=\sigma+i\tau\in\mathbb{C}_+$ and the eigenvalue λ^2 of the self-adjoint operator H should be real. Note that the finiteness of multiplicity of the eigenvalue $\lambda^2=-\tau^2$ of the operator H follows from the Fredholm analytic theorem. The theorem is proved.

Remark. *In equality (14) the sum* $-\Delta f + Wf$ *should be understood in the sense of quadratic forms but not as the sum of the operators. The matter is that though both functions* $f(x)$ *and* $(Hf)(x)$ *belong to the space* $L_2(\mathbb{R}_2)$*, but it may happen that none of the functions* $-\Delta f$ *and* Wf *belong to the space* $L_2(\mathbb{R}_2)$.

Theorem 5. *Let conditions* 1), 2) *and* 3) *be fulfilled. Then the negative part of the spectrum of* H *consists of eigenvalues of finite multiplicity, and only the number* $\lambda = 0$ *may be their limit point.*

Proof. Boundedness of the negative part of the spectrum of the operator H follows from theorem 1. On the basis of theorem 1 we deduce that each negative eigenvalue of the operator H may be only of finite multiplicity. For proving the last statement of the theorem, it suffices to note that from the analyticity of the operator valued function $K(\lambda)$ (see [6]) and Fredholm's analytic theorem it follows that each negative eigenvalue is an isolated point of the negative part of the spectrum of the operator H. The theorem is proved.

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