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INITIAL ABSTRACT BOUNDARY VALUE PROBLEMS FOR PARABOLIC DIFFERENTIAL-OPERATOR EQUATIONS IN *UMD* BANACH SPACES

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In memory of M. G. Gasymov on his 75th birthday

Abstract. We consider initial abstract boundary value problems for parabolic differential-operator equations in *UMD* Banach spaces settings on the rectangle $[0,T]\times[0,1]$. We use our previous results on norm-estimates of solutions of boundary value problems for abstract elliptic equations with a parameter on $[0,1]$ in a *UMD* Banach space. Unique solvability of the problems is proved in the spaces of vector-valued continuous functions. The corresponding estimates of the solution are also established. Then, completeness of a system of root functions of abstract elliptic boundary value problems and completeness of elementary solutions of initial abstract parabolic boundary value problems are obtained. All abstract results are provided by a relevant application to parabolic or elliptic PDEs. We also treat, in applications, integro-differential equations and boundary conditions.

1. Introduction and basic notations

In this paper, we take into account some previous results in [8] and [13] on resolvent estimates of boundary value problems for abstract elliptic equations with a parameter on [0,1] in a *UMD* Banach space to consider initial abstract boundary value problems for parabolic differential-operator equations in *UMD* Banach spaces settings on the rectangle $[0,T]\times[0,1]$. To this end, we apply the uniqueness, existence, and regularity results concerning the Cauchy problem $u'(t)=Lu(t)+f(t)$, $u(0)=u_0$ in a complex Banach space X in [13] and [7] (see the Appendix, Theorems 5.1 and 5.2).

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Let us briefly describe the main results of the paper. Of concern, is the abstract initial boundary value problem for the parabolic differential-operator equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) + A_1(x)u(t,x) = f(t,x),$$
$$(t,x) \in (0,T) \times (0,1),$$
$$\alpha_k \frac{\partial^{m_k} u(t,0)}{\partial x^{m_k}} + \beta_k \frac{\partial^{m_k} u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks} u(t,x_{ks}) = 0, \quad t \in (0,T),\ k = 1,2,$$
$$u(0,x) = u_0(x), \quad x \in (0,1),$$

where $m_k \in \{0,1\}$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x \in [0,1]$, and A, T_{ks} are, generally speaking, unbounded operators in E. Unique solvability of the problem is proved in the space of $L_p((0,1);E)$-valued continuous in time functions, i.e., in $C([0,T];L_p((0,1);E))$, while the right-hand side of the equation is from the space of some kind of Hölder continuous functions, $C_\mu^\gamma((0,T];L_p((0,1);E))$, $\gamma \in (0,1]$, $\mu \in [0,1)$, or $C_0^\gamma([0,T];L_p((0,1);E))$, $\gamma \in (0,1)$. We furnish here a concrete example of application to partial differential equations. In this case, the operator A means an elliptic differential operator in y-variable from some bounded domain of $\mathbb{R}^n$ with smooth boundary, $B(x)$ means a multiplication operator perturbed by some integral operator and $A_1(x)$ means some integro-differential operator, for each $x \in [0,1]$. Moreover, due to the abstract operators T_{ks} in the boundary conditions, we can also treat integro-differential boundary conditions. All the above considerations are in section 2 of the paper.

Then, we consider completeness of a system of root functions of abstract elliptic boundary value problems with a parameter (section 3) and completeness of elementary solutions of initial abstract parabolic boundary value problems (section 4). The corresponding applications to partial differential equations are also shown. Generally, we would like to note that the main purpose of the paper was to obtain results in abstract settings but we tried to illustrate some of abstract results of the paper by relevant applications to parabolic or elliptic PDEs, even to some of integro-differential problems.

The last section 5 is an appendix which collects the main abstract results we used in the paper.

We bring here some relevant papers on the subjects. Up to our best knowledge, there are only a few papers in the literature on the subject of section 2. In our previous paper joint with D. Guidetti [6], we consider a similar abstract initial boundary value problem for parabolic differential-operator equations to that in section 2 but with rather simple separated boundary conditions. In this case, we have succeeded to find necessary and sufficient conditions for the unique solvability of the problem (in fact, for L_q-maximal regularity) in the space $L_q((0,T);L_p((0,1);E))$. In [5], the authors also consider separated boundary conditions but the main part of the equation contains $2m$-order differential operator in x-variable in contrast to our second order. They also do not claim that the underlying Banach space is *UMD*. Necessary and sufficient conditions are found in order the corresponding problem will have a strict solution. So, our section 2, seems to be the first attempt to consider non-separated boundary conditions in such abstract settings. The papers concerned to the subject of section 3 are [1] and [2]. Both papers consider a situation of $B(x) = 0$ in the equation and treat other boundary conditions. Boundary conditions in [1] may contain the spectral parameter λ at the same first order that the equation and boundary conditions in [2] may contain an unbounded operator in the main part of the boundary condition. The reader can find some other papers on the subject of section 3 in the references in [1] and [2]. It seems to us that there are no other studies in the literature on the subject of section 4 in such abstract settings. So, the results of section 4 are completely new.

Let us now give necessary definitions and notations.

If E and F are Banach spaces, $B(E, F)$ denotes the Banach space of all bounded, linear operators from E into F with the norm equal to the operator norm; moreover, $B(E) := B(E, E)$. The spectrum of a linear operator A in E is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. The domain and range of an operator A is denoted by $D(A)$ and $R(A)$, respectively. The resolvent of an operator A is denoted by $R(\lambda, A) := (\lambda I - A)^{-1}$.

A Banach space E is said to be of **class** *HT*, if the Hilbert transform is bounded on $L_p(\mathbb{R}; E)$ for some (and then all) $p > 1$. Here the Hilbert transform H of a function $f \in S(\mathbb{R}; E)$, the Schwartz space of rapidly decreasing E-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f,$$

i.e., $(Hf)(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau} d\tau$. These spaces are often also called *UMD* Banach spaces, where the *UMD* stands for the property of *unconditional martingale differences.*

Definition 1.1. Let E be a complex Banach space, and A is a closed linear operator in E. The operator A is called **sectorial** if the following conditions are satisfied:

(1) $\overline{D(A)} = E$, $\overline{R(A)} = E$, $(-\infty, 0) \subset \rho(A)$;
(2) $\|\lambda(\lambda I + A)^{-1}\| \le M$ for all $\lambda > 0$, and some $M < \infty$.

Definition 1.2. Let E and F be Banach spaces. A family of operators $\mathcal{T} \subset B(E, F)$ is called **$\mathcal{R}$-bounded**, if there is a constant $C > 0$ and $p \ge 1$ such that for each natural number n, $T_j \in \mathcal{T}$, $u_j \in E$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_j on $[0, 1]$ (e.g., the Rademacher functions $\varepsilon_j(t) = \text{sign} \sin(2^j \pi t)$) the inequality

$$\Big\| \sum_{j=1}^{n} \varepsilon_j T_j u_j \Big\|_{L_p((0,1);F)} \le C \Big\| \sum_{j=1}^{n} \varepsilon_j u_j \Big\|_{L_p((0,1);E)}$$

is valid. The smallest such C is called **$\mathcal{R}$-bound** of $\mathcal{T}$ and is denoted by $\mathcal{R}\{\mathcal{T}\}$.

Definition 1.3. A sectorial operator A is called **$\mathcal{R}$-sectorial** if

$$\mathcal{R}_A(0) := \mathcal{R}\{\lambda(\lambda I + A)^{-1} \ : \ \lambda > 0\} < \infty.$$

The number

$$\phi_A^{\mathcal{R}} := \inf\{\theta \in (0, \pi) \ : \ \mathcal{R}_A(\pi - \theta) < \infty\},$$

where $\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda I + A)^{-1} \ : \ |\arg\lambda| \leq \theta\}$, is called an **$\mathcal{R}$-angle** of the operator A.

For the operator A closed in E, the domain of definition $D(A^n)$ of the operator A^n is turned into a Banach space $E(A^n)$ with respect to the norm

$$\|u\|_{E(A^n)} := \Big(\sum_{k=0}^{n} \|A^k u\|^2\Big)^{\frac{1}{2}}.$$

The operator A^n from $E(A^n)$ into E is bounded.

For the Banach spaces F and E, introduce the Banach space $W_p^n((0,1);F,E)$, $1 < p < \infty$, a natural number $n \geq 1$, of vector-valued functions with the finite norm

$$\|u\|_{W_p^n((0.1);F,E)} := \Big(\int_0^1 \|u(x)\|_F^p dx + \int_0^1 \|u^{(n)}(x)\|_E^p dx\Big)^{\frac{1}{p}}.$$

We write $W_p^n((0,1);E) := W_p^n((0,1);E,E)$.

Let E_0 and E_1 be two Banach spaces continuously embedded into the Banach space $E: \ E_0 \subset E$, $E_1 \subset E$. Two such spaces are called an **interpolation couple** $\{E_0, E_1\}$. Then, a standard **real interpolation space** $(E_0, E_1)_{\theta,p}$, $0 < \theta < 1$, $p \geq 1$, and a standard **complex interpolation space** $[E_0, E_1]_\theta$ are defined (for the exact definitions we refer the reader, e.g., to the book by H. Triebel [11]).

2. Initial abstract parabolic boundary value problems and application to parabolic initial boundary value problems

Let X be a Banach space and let A be a linear, closed operator in X. Consider Banach spaces

1) $C_\mu(I;X) := \Big\{f \mid f \in C(I;X), \ \|f\|_{C_\mu(I;X)} := \sup_{t\in I} \|t^\mu f(t)\| < \infty\Big\}, \ \ \mu \geq 0;$

2) $C_\mu^\gamma(I;X) := \Big\{f \mid f \in C(I;X), \ \|f\|_{C_\mu^\gamma(I;X)} := \sup_{t\in I} \|t^\mu f(t)\|$
$+ \sup_{\substack{t<t+h \\ t,t+h\in I}} \|f(t+h) - f(t)\| h^{-\gamma} t^\mu < \infty\Big\}, \ \ \gamma \in (0,1], \ \mu \geq 0;$

and the linear space

3) $C^1(I;X(A),X) := \Big\{f \mid f \in C(I;X(A)) \cap C^1(I;X)\Big\},$

where I denotes an interval containing into $[0,\infty)$.

Consider now, in a Banach space E, the following abstract initial boundary value problem for a parabolic differential-operator equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) + A_1(x)u(t,x) = f(t,x), \quad (t,x) \in (0,T) \times (0,1), \tag{2.1}$$

$$\alpha_k \frac{\partial^{m_k} u(t,0)}{\partial x^{m_k}} + \beta_k \frac{\partial^{m_k} u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks} u(t, x_{ks}) = 0, \quad t \in (0,T),\ k = 1,2, \tag{2.2}$$

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{2.3}$$

where $m_k \in \{0,1\}$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x \in [0,1]$, and A, T_{ks} are, generally speaking, unbounded operators in E.

Theorem 2.1. *Let the following conditions be satisfied:*

(1) *an operator A is closed, densely defined and invertible in a UMD Banach space E;*
(2) $\mathcal{R}\{\lambda R(\lambda, A) \ :\ |\arg \lambda| \geq \beta\} < \infty$, *for some* $0 < \beta < \frac{\pi}{2}$; [1]
(3) *the embedding $E(A) \subset E$ is compact*;
(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;
(5) *for any $\varepsilon > 0$ and for almost all $x \in [0,1]$,*

$$\|B(x)u\| \leq \varepsilon \|A^{\frac{1}{2}} u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}}),$$
$$\|A_1(x)u\| \leq \varepsilon \|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A);$$

for $u \in D(A^{\frac{1}{2}})$ the function $B(x)u$ and for $u \in D(A)$ the function $A_1(x)u$ are measurable on $[0,1]$ in E;
(6) *if $m_k = 0$, then $T_{ks} = 0$; if $m_k = 1$, then for $\varepsilon > 0$ and $u \in (E(A),E)_{\frac{1}{2p},p}$, where $p \in (1,\infty)$,*

$$\|T_{ks}u\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \leq \varepsilon \|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\| \leq \varepsilon \|u\|_{(E(A),E)_{\frac{1}{2},p}} + C(\varepsilon)\|u\|;$$

(7) $f \in C_\mu^\gamma((0,T]; L_p((0,1);E))$, *for some* $\gamma \in (0,1]$, $\mu \in [0,1)$;
(8) $u_0 \in W_p^2\big((0,1); E(A), E; L_k u = 0,\ k=1,2\big) := \big\{u \in W_p^2((0,1);E(A),E) \mid L_k u = 0,\ k = 1,2\}$, *where* $L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks} u(x_{ks})$, $k = 1,2$.

Then, problem (2.1)–(2.3) has a unique solution $u(t,x)$ in

$$C\big([0,T]; L_p((0,1);E)\big) \cap C^1\big((0,T];$$

$$W_p^2\big((0,1); E(A), E; L_k u = 0,\ k = 1,2\big), L_p((0,1);E)\big)$$

and, for $t \in (0,T]$, the following estimates hold:

$$\|u(t,\cdot)\|_{L_p((0,1);E)} \leq C\Big(\|u_0(\cdot)\|_{W_p^2((0,1);E(A),E)} + \|f\|_{C_\mu((0,t];L_p((0,1);E))}\Big),$$
$$\Big\|\frac{\partial u(t,\cdot)}{\partial t}\Big\|_{L_p((0,1);E)} \leq C\Big(\|u_0(\cdot)\|_{W_p^2((0,1);E(A),E)} + t^{-\mu}\|f\|_{C_\mu^\gamma((0,t];L_p((0,1);E))}\Big), \tag{2.4}$$

where C does not depend on t.

[1] In fact, conditions (1) and (2) are equivalent to that A is invertible $\mathcal{R}$-sectorial operator in E with the $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} < \beta$.

Proof. In the Banach space $X = L_p((0,1);E)$, consider an operator L defined by the equalities

$$\begin{aligned} &D(L) := W_p^2\big((0,1);E(A),E;L_k u = 0,\ k=1,2\big),\\ &Lu := u''(x) - B(x)u'(x) - Au(x) - A_1(x)u(x) \end{aligned} \tag{2.5}$$

and rewrite problem (2.1)-(2.3) in the following form

$$\begin{aligned} &u'(t) = Lu(t) + f(t),\\ &u(0) = u_0, \end{aligned} \tag{2.6}$$

to which we want to apply [13, Theorem 7.2.2/1] (see Theorem 5.1 in the Appendix) with $\eta = 1$. To this end, the only thing is to check that, for some $\alpha > 0$,

$$\|R(\lambda, L)\|_{B(X)} \le M|\lambda|^{-1}, \quad |\arg\lambda| \le \frac{\pi}{2} + \alpha,\ |\lambda| \to \infty.$$

In turn, the latter inequality (with $\alpha = \frac{\pi}{2} - \beta$) follows from [8, Theorem 6] (with $\varphi = \pi - \beta$; see Theorem 5.3 in the Appendix). So, from Theorem 5.1, it follows that problem (2.1)–(2.3) has a unique solution in

$$C\big([0,T];L_p((0,1);E)\big)\cap$$
$$C^1\big((0,T];W_p^2\big((0,1);E(A),E;L_k u = 0,\ k=1,2\big),L_p((0,1);E)\big)$$

and, for $t \in (0,T]$, the following estimates hold:

$$\begin{aligned} \|u(t,\cdot)\|_{L_p((0,1);E)} &\le C\Big(\|u_0(\cdot)\|_{W_p^2((0,1);E(A),E)} + \|A_1(\cdot)u_0(\cdot)\|_{L_p((0,1);E)}\\ &\quad + \|B(\cdot)u_0'(\cdot)\|_{L_p((0,1);E)} + \|f\|_{C_\mu((0,t];L_p((0,1);E))}\Big),\\ \Big\|\frac{\partial u(t,\cdot)}{\partial t}\Big\|_{L_p((0,1);E)} &\le C\Big(\|u_0(\cdot)\|_{W_p^2((0,1);E(A),E)} + \|A_1(\cdot)u_0(\cdot)\|_{L_p((0,1);E)}\\ &\quad + \|B(\cdot)u_0'(\cdot)\|_{L_p((0,1);E)} + t^{-\mu}\|f\|_{C_\mu^\gamma((0,t];L_p((0,1);E))}\Big), \end{aligned}$$

where C does not depend on t. Taking into account condition (5) and that the operator $\frac{d}{dx}$ is a bounded operator from $W_p^2((0,1);E(A),E)$ into $W_p^1((0,1);E(A^{\frac{1}{2}}),E)$ (see, e.g., [9, Theorem 7 and Corollary 8]), we get, from the last inequalities, inequalities in (2.4). □

If one uses [7, Theorem 7.2] (see Theorem 5.2 in the Appendix) instead of [13, Theorem 7.2.2/1] (see Theorem 5.1 in the Appendix) then one gets the following result instead of Theorem 2.1.

Theorem 2.2. *Let the following conditions be satisfied:*

(1) *an operator A is closed, densely defined and invertible in a UMD Banach space E;*
(2) $\mathcal{R}\{\lambda R(\lambda, A) : |\arg\lambda| \ge \beta\} < \infty$*, for some* $0 < \beta < \frac{\pi}{2}$*;*
(3) *the embedding $E(A) \subset E$ is compact*;
(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \ne 0$;
(5) *for any $\varepsilon > 0$ and for almost all $x \in [0,1]$,*

$$\begin{aligned} &\|B(x)u\| \le \varepsilon\|A^{\frac{1}{2}}u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}}),\\ &\|A_1(x)u\| \le \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A); \end{aligned}$$

for $u \in D(A^{\frac{1}{2}})$ *the function* $B(x)u$ *and for* $u \in D(A)$ *the function* $A_1(x)u$ *are measurable on* $[0,1]$ *in* E;

(6) *if* $m_k = 0$, *then* $T_{ks} = 0$; *if* $m_k = 1$, *then for* $\varepsilon > 0$ *and* $u \in (E(A), E)_{\frac{1}{2p},p}$, *where* $p \in (1,\infty)$,

$$\|T_{ks}u\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \le \varepsilon \|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\| \le \varepsilon \|u\|_{(E(A),E)_{\frac{1}{2},p}} + C(\varepsilon)\|u\|;$$

(7) $f \in C_0^\gamma([0,T]; L_p((0,1);E))$ *with* $f(0,x) + u_0''(x) - B(x)u_0'(x) - Au_0(x) - A_1(x)u_0(x) \in W_\gamma$, *for some* $\gamma \in (0,1)$, *where* $W_\gamma := \{v \in L_p((0,1);E) \mid \exists \lambda_0 > 0$ *big enough such that* $\sup_{\lambda > \lambda_0} (1+\lambda)^\gamma \|L(\lambda I - L)^{-1} v\|_{L_p((0,1);E)} < \infty\}$ *and the operator* L *is defined by (2.5);*

(8) $u_0 \in W_p^2\big((0,1); E(A), E; L_k u = 0,\ k = 1,2\big)$.

Then, problem (2.1)–(2.3) has a unique strict solution $u(t,x)$ *in*

$$C^1\big([0,T]; W_p^2\big((0,1); E(A), E; L_k u = 0,\ k=1,2\big), L_p((0,1);E)\big)$$

with the regularity $\frac{\partial u(t,x)}{\partial t} \in C_0^\gamma([0,T]; L_p((0,1);E))$.

Show now the following application of Theorem 2.1. Let $\Omega := (0,1) \times G$, where $G \subset \mathbb{R}^r$, $r \ge 2$ be a bounded open domain with an $(r-1)$-dimensional boundary ∂G which locally admits rectification, and let us consider in the domain $(0,T) \times \Omega$ a very nonclassical parabolic initial boundary value problem (with integro-differential terms in the equation and unbounded operators and the values of the unknown function in intermediate points in boundary conditions)

$$\begin{aligned}
&D_t u(t,x,y) - D_x^2 u(t,x,y) + b(x,y) D_x u(t,x,y) + \int_G c(x,y,z) D_x u(t,x,z)\,dz - \\
&\sum_{s,j=1}^r a_{sj}(y) D_s D_j u(t,x,y) + \sum_{j=1}^r b_j(x,y) D_j u(t,x,y) + b_0(x,y) u(t,x,y) \\
&+ \sum_{\ell=0}^1 \sum_{j=1}^r \int_G c_{\ell j}(x,y,z) D_{z_j}^\ell u(t,x,z)\,dz = f(t,x,y), \\
&(t,x,y) \in (0,T) \times (0,1) \times G, \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
&L_k u := \alpha_k D_x^{m_k} u(t,0,y) + \beta_k D_x^{m_k} u(t,1,y) + \sum_{s=1}^{N_k} T_{ks} u(t, x_{ks}, \cdot) = 0, \\
&(t,y) \in (0,T) \times G, \quad k = 1,2, \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
&L_0 u := \sum_{j=1}^r c_j(y') D_j u(t,x,y') + c_0(y') u(t,x,y') = 0, \\
&(t,x,y') \in (0,T) \times (0,1) \times \partial G, \qquad (2.9)
\end{aligned}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in (0,1) \times G, \qquad (2.10)$$

where $D_t := \frac{\partial}{\partial t}$, $D_x := \frac{\partial}{\partial x}$, $D_{z_j} := \frac{\partial}{\partial z_j}$, $D_j := -i\frac{\partial}{\partial y_j}$, $D_y := (D_1, \ldots, D_r)$, $m_k \in \{0,1\}$, α_k, β_k are complex numbers, $y := (y_1, \ldots, y_r)$, $x_{ks} \in [0,1]$, T_{ks} are, generally speaking, unbounded operators in $L_q(G)$, $1 < q < \infty$. Let m be the order of the differential boundary operator L_0 in (2.9), i.e., $m = 0$ if all $c_j(y') \equiv 0$, $j = 1, \dots, r$ (and then $c_0(y') \neq 0$, $\forall y' \in \partial G$), and $m = 1$ if at least one of $c_j(y')$, $j = 1, \dots, r$, is not identically zero.

We will consider the space

$$B^s_{p,q}(G) := (W^{s_0}_p(G), W^{s_1}_p(G))_{\theta,q},$$

where $0 \leq s_0, s_1$ are integers, $W^n_p(G)$ stands for the Sobolev space, $0 < \theta < 1$, $1 < p < \infty$, $1 < q < \infty$ and $s = (1-\theta)s_0 + \theta s_1$.

Theorem 2.3. *Let the following conditions be satisfied:*

(1) *(smoothness conditions)* $|a_{sj}(y) - a_{sj}(z)| \leq C|y-z|^\delta$ *for some* $C > 0$ *and* $\delta \in (0,1)$, $\forall y, z \in \overline{G}$; $b, b_j, b_0 \in L_\infty(\overline{\Omega})$; $c, c_{\ell j} \in L_\infty(\overline{\Omega \times G})$; $c_j, c_0 \in C^{2-m}(\partial G)$; $\partial G \in C^2$;

(2) *(ellipticity condition for the below operator A) for* $y \in \overline{G}$, $\sigma \in \mathbb{R}^r$, $|\arg \lambda| \geq \beta$, *for some* $0 < \beta < \frac{\pi}{2}$, $|\sigma| + |\lambda| \neq 0$, *we have*

$$\lambda + \sum_{s,j=1}^{r} a_{sj}(y)\sigma_s\sigma_j \neq 0;$$

(3) *(Lopatinskii-Shapiro condition for the below operator A)* y' *is any point on* ∂G, *the vector* σ' *is tangent and* σ *is a normal vector to* ∂G *at the point* $y' \in \partial G$. *Consider the following ordinary differential problem, for* $|\arg\lambda| \geq \beta$ *with* β *from condition (2):*

$$\Big[\lambda + \sum_{s,j=1}^{r} a_{sj}(y')\Big(\sigma'_s - i\sigma_s\frac{d}{dt}\Big)\Big(\sigma'_j - i\sigma_j\frac{d}{dt}\Big)\Big]u(t) = 0, \quad t > 0, \tag{2.11}$$

$$\sum_{j=1}^{r} c_j(y')\Big(\sigma'_j - i\sigma_j\frac{d}{dt}\Big)u(t)\Big|_{t=0} = h, \quad for\ m = 1, \tag{2.12}$$

$$u(0) = h, \quad for\ m = 0; \tag{2.13}$$

it is required that for $m = 1$ *problem (2.11), (2.12) (for* $m = 0$ *problem (2.11), (2.13)) has one and only one solution, including all its derivatives, tending to zero as* $t \to \infty$ *for any numbers* $h \in \mathbb{C}$; [2]

(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;

(5) *if* $m_k = 0$ *then* $T_{ks} = 0$; *if* $m_k = 1$ *then, for* $\varepsilon > 0$ *and* $u \in B^{2-\frac{1}{p}}_{q,p}(G; L_0u = 0, m < 2 - \frac{1}{p} - \frac{1}{q})$,

$$\|T_{ks}u\|_{B^{1-\frac{1}{p}}_{q,p}(G)} \leq \varepsilon\|u\|_{B^{2-\frac{1}{p}}_{q,p}(G)} + C(\varepsilon)\|u\|_{L_q(G)},$$

$$\|T_{ks}u\|_{L_q(G)} \leq \varepsilon\|u\|_{B^1_{q,p}(G)} + C(\varepsilon)\|u\|_{L_q(G)},$$

where $p \neq \frac{q}{q-1}$ *and* $p, q \in (1,\infty)$, *or* $p = \frac{q}{q-1}$ *and* $m = 0$;[3]

---

[2]Remind that, in the case $m = 0$, boundary condition (2.9) is transformed into the Dirichlet boundary condition $u(t,x,y') = 0$, $(t,x,y') \in (0,T) \times (0,1) \times \partial G$.

[3]In the case when $p = \frac{q}{q-1} = 2$ and $m = 1$, $B^{\frac{3}{2}}_{2,2}(G; L_0u \in \widetilde{B}^{\frac{1}{2}}_{2,2}(G))$ (see [11, Theorem 4.3.3]) should be written instead of $B^{\frac{3}{2}}_{2,2}(G; L_0u = 0, m < 1)$. $\widetilde{B}^s_{q,p}(G) := \{u \mid u \in B^s_{q,p}(\mathbb{R}^r),\ \operatorname{supp}(u) \subset \overline{G}\}$.

(6) $f \in C_\mu^\gamma((0,T];L_p((0,1);L_q(G)))$, *for some* $\gamma \in (0,1]$, $\mu \in [0,1)$;

(7) $u_0 \in \widetilde{W}_p^2((0,1);W_q^2(G;L_0u=0),L_q(G)) := \Big\{u \in W_p^2((0,1);W_q^2(G;L_0u = 0),L_q(G)) \mid L_k u = 0,\ k=1,2,\ y \in G\Big\}$.

Then, problem (2.7)–(2.10) has a unique solution in

$$C\big([0,T];L_p((0,1);L_q(G))\big)\cap$$
$$C^1\big((0,T];\widetilde{W}_p^2((0,1);W_q^2(G;L_0u=0),L_q(G)),L_p((0,1);L_q(G))\big)$$

and, for $t \in (0,T]$, *the following estimates hold:*

$$\begin{aligned} \|u(t,x,y)\|_{L_p((0,1);L_q(G))} &\le C\Big(\|u_0(x,y)\|_{W_p^2((0,1);W_q^2(G),L_q(G))} \\ &+ \|f\|_{C_\mu((0,t];L_p((0,1);L_q(G)))}\Big), \\ \Big\|\frac{\partial u(t,x,y)}{\partial t}\Big\|_{L_p((0,1);L_q(G))} &\le C\Big(\|u_0(x,y)\|_{W_p^2((0,1);W_q^2(G),L_q(G))} \\ &+ t^{-\mu}\|f\|_{C_\mu^\gamma((0,t];L_p((0,1);L_q(G)))}\Big), \end{aligned} \tag{2.14}$$

where C *does not depend on* t.

Proof. Let us denote $E := L_q(G)$ and consider an operator A which is defined by the equalities

$$D(A) := W_q^2(G;L_0u=0), \quad Au := -\sum_{s,j=1}^r a_{sj}(y)D_sD_ju(y) + \lambda_0 u(y),$$

where, by [4, Theorem 8.2], there exists $\lambda_0 > 0$ such that A is an $\mathcal{R}$-sectorial operator in E with the $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} < \pi$. For $x \in [0,1]$, also consider operators

$$B(x)u := b(x,y)u(y) + \int_G c(x,y,z)u(z)dz,$$
$$A_1(x)u := \sum_{j=1}^r b_j(x,y)D_ju(y) + b_0(x,y)u(y)$$
$$+ \sum_{\ell=0}^1\sum_{j=1}^r \int_G c_{\ell j}(x,y,z)D_{z_j}^\ell u(z)dz - \lambda_0 u(y).$$

Then, problem (2.7)–(2.10) can be rewritten in the form

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) + A_1(x)u(t,x) \\ &= f(t,x), \quad (t,x) \in (0,T)\times(0,1), \\ &\alpha_k \frac{\partial^{m_k}u(t,0)}{\partial x^{m_k}} + \beta_k\frac{\partial^{m_k}u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks}u(t,x_{ks}) = 0, \quad t \in (0,T),\ k=1,2, \\ &u(0,x) = u_0(x), \quad x \in (0,1), \end{aligned} \tag{2.15}$$

where $u(t,x) := u(t,x,\cdot)$, $f(t,x) := f(t,x,\cdot)$, and $u_0(x) = u_0(x,\cdot)$ are functions with values in the *UMD* Banach space $E := L_q(G)$, i.e., in the form of problem (2.1)–(2.3). We want now to apply Theorem 2.1 to problem (2.15). Conditions

(7)-(8) of Theorem 2.1 follow from conditions (6)-(7). Conditions (1)-(6) of Theorem 2.1 follow from conditions (1)-(5) and it was shown in the proofs of Theorems 7 and 8 in [8]. □

Examples of T_{ks} (at least for $\partial G \in C^\infty$) satisfying condition (5) of Theorem 2.3 and the corresponding conditions of all further application theorems (for the proof see [8, p. 52]):

1) $(T_{ks}u)(y) := \gamma_{ks}u(y)$, where $\gamma_{ks} \in \mathbb{C}$;

2) $(T_{ks}u)(y) := \int_G \sum_{|\ell|\le 1} T_{ks\ell}(x,y)\frac{\partial^{|\ell|}u(x)}{\partial x_1^{\ell_1}\cdots\partial x_r^{\ell_r}}dx$, where $T_{ks\ell} \in L_{t'}(G\times G)$, $\frac{1}{t'}+\frac{1}{t}=1$, $t=\min(q,q')$, $\frac{1}{q'}+\frac{1}{q}=1$, and $T_{ks\ell}(x,y)$ are continuously differentiable with respect to y_j, $j=1,\dots,r$ and $\frac{\partial}{\partial y_j}T_{ks\ell} \in L_{t'}(G\times G)$. So, we consider, in particular, integro-differential boundary conditions.

3. Completeness of a system of root functions of abstract elliptic boundary value problems and application to elliptic boundary value problems

Consider the corresponding spectral problem to problem (2.1)-(2.2) in a Banach space E, i.e.,

$$L(\lambda)u := \lambda u(x) - u''(x) + B(x)u'(x) + Au(x) + A_1(x)u(x) = 0, \quad x\in(0,1), \tag{3.1}$$

$$L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks}u(x_{ks}) = 0,\ k=1,2, \tag{3.2}$$

where λ is a complex parameter, $m_k \in \{0,1\}$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x\in[0,1]$, and A, T_{ks} are, generally speaking, unbounded operators in E. A number λ_0 is called an **eigenvalue** of problem (3.1)-(3.2) if the problem

$$L(\lambda_0)u = 0, \quad L_k u = 0,\ k=1,2,$$

has a non-trivial solution $u_0(x)$ that belongs to $W_p^2((0,1);E(A),E)$, for some $1<p<\infty$, and $u_0(x)$ is called an **eigenfunction** of problem (3.1)-(3.2) corresponding to the eigenvalue λ_0. A solution $u_m(x)$, for a natural number $m\ge 1$, of the problem

$$L(\lambda_0)u_m + u_{m-1} = 0, \quad L_k u_m = 0,\ k=1,2,$$

belonging to $W_p^2((0,1);E(A),E)$, is called an m**-th associated function** to the eigenfunction $u_0(x)$ of problem (3.1)-(3.2). We combine the eigenfunctions and associated functions of problem (3.1)-(3.2) under the general name of **root functions** of problem (3.1)-(3.2).

Let us, first, formulate and prove a completeness theorem in the framework of Hilbert spaces.

Let an operator C from a Hilbert space H into a Hilbert space H_1 be bounded. Then its **adjoint operator** C^* from H_1 into H is bounded and, for $u\in H$, $u_1\in H_1$, we have

$$(Cu,u_1)_{H_1} = (u, C^*u_1)_H.$$

Since $(C^*C)^* = C^*C^{**} = C^*C$, the operator C^*C in H is selfadjoint. From $(C^*Cu, u)_H = (Cu, Cu)_H \geq 0$ it follows that the operator C^*C in H is non-negative. In turn, it implies that there exists a unique non-negative selfadjoint operator $T := (C^*C)^{\frac{1}{2}}$ in H. If C from a Hilbert space H into a Hilbert space H_1 is compact, then, in addition to the above, the operator $T = (C^*C)^{\frac{1}{2}}$ in H is compact. The eigenvalues of the operator T are called **singular numbers** of the compact operator C and are denoted by $s_j(C; H, H_1)$. Enumerate the singular numbers in decreasing order, taking into account their multiplicities, so that

$$s_j(C; H, H_1) := \lambda_j(T), \quad j = 1, \ldots, \infty.$$

Theorem 3.1. *Let the following conditions be fulfilled:*

(1) α_k, β_k *are complex numbers;* $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$*;* $x_{ks} \in [0, 1]$*;*
(2) *the embedding* $H(A) \subset H$ *is compact and for some* $t > 0$*, for the embedding operator* J*, it holds that* $s_j(J; H(A), H) \leq Cj^{-t}$*,* $j = 1, 2, \ldots$;
(3) *the operator* A *is closed, densely defined in a Hilbert space* H *and, for some* φ *such that* $\frac{2\pi}{2+t} < \varphi < \pi$*,*

$$\|R(\lambda, A)\| \leq C(1 + |\lambda|)^{-1}, \quad |\arg \lambda| \geq \pi - \varphi;$$

(4) *for any* $\varepsilon > 0$ *and for almost all* $x \in [0, 1]$*,*

$$\|B(x)u\| \leq \varepsilon\|A^{\frac{1}{2}}u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}}),$$
$$\|A_1(x)u\| \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A);$$

for $u \in D(A^{\frac{1}{2}})$ *the function* $B(x)u$ *and for* $u \in D(A)$ *the function* $A_1(x)u$ *are measurable on* $[0, 1]$ *in* H;
(5) *if* $m_k = 0$*, then* $T_{ks} = 0$; *if* $m_k = 1$*, then for* $\varepsilon > 0$ *and* $u \in (H(A), H)_{\frac{1}{4},2}$*,*

$$\|T_{ks}u\|_{(H(A),H)_{\frac{3}{4},2}} \leq \varepsilon\|u\|_{(H(A),H)_{\frac{1}{4},2}} + C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\| \leq \varepsilon\|u\|_{(H(A),H)_{\frac{1}{2},2}} + C(\varepsilon)\|u\|.$$

Then, the spectrum of problem (3.1)-(3.2) is discrete and a system of root functions of problem (3.1)-(3.2) is complete in the spaces $W_2^2\big((0, 1); H(A), H; L_k u = 0,\ k = 1, 2\big)$ *and* $L_2((0, 1); H)$*.*

Proof. In the space $\mathbb{H} := L_2((0, 1); H)$, consider an operator $\mathbb{A}$ which is defined by the equalities

$$\begin{aligned} D(\mathbb{A}) &:= W_2^2\big((0, 1); H(A), H; L_k u = 0,\ k = 1, 2\big), \\ \mathbb{A}u &= -u''(x) + B(x)u'(x) + Au(x) + A_1(x)u(x). \end{aligned} \tag{3.3}$$

Apply [13, Theorem 2.2.2/1] to the operator $\mathbb{A}$ in $\mathbb{H}$.

Using the same technique as in the proof of [12, Theorem 4], one can show that $W_2^2\big((0, 1); H(A), H;\ L_k u = 0,\ k = 1, 2\big)$ is dense in $L_2((0, 1); H)$, i.e., the first condition of [13, Theorem 2.2.2/1] is fulfilled.

By [13, Theorem 5.2.1/1], the embedding $W_2^2((0, 1); H(A), H) \subset L_2((0, 1); H) = \mathbb{H}$ is compact, i.e., the embedding $\mathbb{H}(\mathbb{A}) \subset \mathbb{H}$ is also compact. By [13, Lemmas 1.2.10/3 and 1.7.8/6],

$$s_j(J; \mathbb{H}(\mathbb{A}), \mathbb{H}) \leq C\, s_j\left(J; W_2^2((0, 1); H(A), H), L_2((0, 1); H)\right) \leq Cj^{-\frac{2t}{2+t}},$$

i.e., the second condition of [13, Theorem 2.2.2/1] is also fulfilled (with $p = \frac{2t}{2+t}$ in [13, Theorem 2.2.2/1]).

By [8, Theorem 6] (in the framework of Hilbert spaces and for $p = 2$ in [8, Theorem 6]; see Theorem 5.3 in the Appendix), in view of that in Hilbert spaces $\mathcal{R}$-boundedness is just norm-boundedness, for sufficiently large $|\lambda|$ from the angle $|\arg \lambda| \leq \varphi < \pi$, for a solution of the equation

$$\lambda u + \mathbb{A}u = f$$

it holds the estimate

$$|\lambda| \, \|u\|_{L_2((0,1);H)} + \left\|u''\right\|_{L_2((0,1);H)} + \|Au\|_{L_2((0,1);H)} \leq C \, \|f\|_{L_2((0,1);H)} \, .$$

From the last estimate, for sufficiently large $|\lambda|$ from the angle $|\arg \lambda| \leq \varphi < \pi$, we have

$$\|R\,(\lambda, -\mathbb{A})\| \leq C \, |\lambda|^{-1} \, .$$

Consequently,

$$\|R\,(-\lambda, \mathbb{A})\| \leq C \, |\lambda|^{-1} \, , \qquad |\arg(-\lambda)| \geq \pi - \varphi, \quad |\lambda| \to \infty.$$

Since $\varphi > \frac{2\pi}{2+t}$ then $\pi - \varphi < \frac{\pi t}{2+t}$. Therefore, condition (3) of [13, Theorem 2.2.2/1] is also satisfied (with $\eta = 1$ and, previously chosen, $p = \frac{2t}{2+t}$ in [13, Theorem 2.2.2/1]). □

In order to formulate a completeness theorem in the framework of Banach spaces, we need a definition of approximation numbers (of a compact operator) which coincide with (the operator's) singular numbers in the framework of Hilbert spaces (see, e.g., [13, Theorem 1.2.10/2]). Let C be a compact operator from a Banach space E into a Banach space E_1. Then,

$$\tilde{s}_j(C; E, E_1) := \inf_{\substack{\dim R(K) < j \\ K \in B(E, E_1)}} \|C - K\|_{B(E, E_1)}$$

are said to be the **approximation numbers** of C.

Consider now problem (3.1)-(3.2) in a separable, reflexive *UMD* Banach space E and in the space $\mathbb{E} := L_p((0,1); E)$, which is also a separable, reflexive *UMD* Banach space (for reflexivity see [10, Theorem 5.7]), introduce an operator $\mathbb{A}$ defined by the equalities

$$D(\mathbb{A}) := W_p^2\big((0,1); E(A), E; L_k u = 0, \ k = 1, 2\big),$$
$$\mathbb{A}u = -u''(x) + B(x)u'(x) + Au(x) + A_1(x)u(x).$$

Theorem 3.2. *Let the spectrum of the operator* $\mathbb{A}$ *be non empty; for some* $s > 0$,

$$\tilde{s}_j(J; W_p^2((0,1); E(A), E), L_p((0,1); E)) \leq C j^{-s};$$

and let all conditions of [8, Theorem 6] *(see Theorem 5.3 in the Appendix) be satisfied. Moreover, condition (2) of* [8, Theorem 6] *is satisfied for some* $\frac{2-s}{2}\pi < \varphi < \pi$ *if* $0 < s \leq 2$ *and for some* $0 \leq \varphi < \pi$ *if* $s > 2$.

Then, the spectrum of problem (3.1)-(3.2) in E *is discrete and a system of root functions of problem (3.1)-(3.2) is complete in the spaces* $W_p^2\big((0,1); E(A), E; L_k u = 0, \ k = 1, 2\big)$ *and* $L_p\left((0,1); E\right)$.

Proof. The proof repeats the steps of the proof of Theorem 3.1, but one has to use, instead of [13, Theorem 2.2.2/1], the Burgoyne's theorem [3, Theorem 4.5], which is also presented in a more convenient form, for using in application, in [14, Theorem 1]. As well, one should use [14, Lemma 2] instead of [13, Lemma 1.2.10/3]. □

Let us show an application of Theorem 3.1. We will use the same differential operators from the application part of section 2 since (almost) all conditions of Theorem 3.1 for them have been already checked there. Our purpose here is just to demonstrate to the reader a possible application of abstract settings to PDEs and do not give various problems of ordinary and partial differential equations to which our abstract methods can be applied.

So, again, let $\Omega := (0,1)\times G$, where $G\subset\mathbb{R}^r$, $r\geq 2$ be a bounded open domain with an $(r-1)$-dimensional boundary ∂G which locally admits rectification, and let us consider in the domain a very nonclassical elliptic boundary value problem (with integro-differential terms in the equation and unbounded operators and the values of the unknown function in intermediate points in boundary conditions)

$$\begin{aligned}\lambda u(x,y) - D_x^2u(x,y) + b(x,y)D_xu(x,y) + \int_G c(x,y,z)D_xu(x,z)dz\\ - \sum_{s,j=1}^r a_{sj}(y)D_sD_ju(x,y) + \sum_{j=1}^r b_j(x,y)D_ju(x,y) + b_0(x,y)u(x,y)\\ + \sum_{\ell=0}^1\sum_{j=1}^r\int_G c_{\ell j}(x,y,z)D_{z_j}^\ell u(x,z)dz = 0, \quad (x,y)\in\Omega, \qquad (3.4)\end{aligned}$$

$$\begin{aligned}L_ku := \alpha_kD_x^{m_k}u(0,y) + \beta_kD_x^{m_k}u(1,y) + \sum_{s=1}^{N_k}T_{ks}u(x_{ks},\cdot) = 0, \quad y\in G,\\ k=1,2, \qquad (3.5)\end{aligned}$$

$$L_0u := \sum_{j=1}^r c_j(y')D_ju(x,y') + c_0(y')u(x,y') = 0, \quad (x,y')\in(0,1)\times\partial G, \qquad (3.6)$$

where $D_x := \frac{\partial}{\partial x}$, $D_{z_j} := \frac{\partial}{\partial z_j}$, $D_j := -i\frac{\partial}{\partial y_j}$, $D_y := (D_1,\ldots,D_r)$, $m_k\in\{0,1\}$, α_k, β_k are complex numbers, $y := (y_1,\ldots,y_r)$, $x_{ks}\in[0,1]$, T_{ks} are, generally speaking, unbounded operators in $L_2(G)$. Let m be the order of the differential boundary operator L_0 in (3.6), i.e., $m=0$ if all $c_j(y')\equiv 0$, $j=1,\ldots,r$ (and then $c_0(y')\neq 0$, $\forall y'\in\partial G$), and $m=1$ if at least one of $c_j(y')$, $j=1,\ldots,r$, is not identically zero.

Theorem 3.3. *Assume that conditions (1)-(5) (with $p=q=2$) of Theorem 2.3 are fulfilled (conditions (2)-(3) with some $0<\beta<\pi-\frac{2\pi r}{2r+2}$).*

Then, the spectrum of problem (3.4)–(3.6) is discrete and a system of root functions of problem (3.4)–(3.6) is complete in the spaces $W_2^2\big((0,1);W_2^2(G;L_0u=0),L_2(G);L_ku=0,\ k=1,2,y\in G\big)$ and $L_2(\Omega)$.

Proof. We are going to use Theorem 3.1. Construction of all operators and all necessary explanations are the same as in the proof of Theorem 2.3. The only additional checking is of condition (2) and the corresponding restriction on φ in condition (3) of Theorem 3.1. From, e.g., [11, formula 4.10.2/(14)], it follows that

$$s_j(J; W_2^2(G; L_0 u = 0), L_2(G)) \leq C j^{-\frac{2}{r}}, \quad j = 1, 2, \ldots,$$

i.e, condition (2) of Theorem 3.1 is fulfilled with $t = \frac{2}{r}$. In turn, this implies the restriction on φ in condition (3) of Theorem 3.1 for $\varphi = \pi - \beta$. Note that in the proof of Theorem 2.3 it is mentioned that for the constructed operator A the corresponding $\mathcal{R}$-boundedness for the resolvent has been proved in [8]. In the framework of Hilbert spaces, this coincides with the just norm-boundedness in condition (3) of Theorem 3.1. □

4. Completeness of elementary solutions of initial abstract parabolic boundary value problems and application to parabolic initial boundary value problems

Consider problem (2.1)–(2.3) with the homogeneous equation (2.1), i.e.,

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) \\ &+ A_1(x)u(t,x) = 0, \quad (t,x) \in (0,T) \times (0,1), \end{aligned} \tag{4.1}$$

$$\alpha_k \frac{\partial^{m_k} u(t,0)}{\partial x^{m_k}} + \beta_k \frac{\partial^{m_k} u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks} u(t, x_{ks}) = 0, \quad t \in (0,T),\ k = 1,2, \tag{4.2}$$

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{4.3}$$

where $m_k \in \{0,1\}$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x \in [0,1]$, and A, T_{ks} are, generally speaking, unbounded operators in E.

Combining Theorem 2.1 and Theorem 3.1 (or Theorem 3.2), we can get the following theorems about an approximation of the unique solution of problem (4.1)–(4.3) by linear combinations of elementary solutions of (4.1)–(4.2). Remind that, e.g., by [13, Lemma 2.1/1], a function of the form

$$u_i(t,x) := e^{\lambda_i t}\left(\frac{t^{k_i}}{k_i!}u_{i0}(x) + \frac{t^{k_i-1}}{(k_i-1)!}u_{i1}(x) + \cdots + u_{ik_i}(x)\right) \tag{4.4}$$

becomes the **elementary solution** of (4.1)–(4.2) if and only if $u_{i0}(x)$, $u_{i1}(x)$,..., $u_{ik_i}(x)$ is a chain of root functions of problem (3.1)–(3.2) corresponding to the eigenvalue λ_i.

First, consider the Hilbert spaces setting, i.e., we will use Theorems 2.1 and 3.1.

Theorem 4.1. *Let the following conditions be fulfilled:*

1. *α_k, β_k are complex numbers; $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$; $x_{ks} \in [0,1]$;*
2. *the embedding $H(A) \subset H$ is compact and for some $t > 0$, for the embedding operator J, it holds that $s_j(J; H(A), H) \leq Cj^{-t}$, $j = 1, 2, ...$;*

(3) *the operator A is closed, densely defined in a Hilbert space H and for some φ, such that* $\max\left\{\frac{2\pi}{2+t},\frac{\pi}{2}\right\}<\varphi<\pi$,

$$\|R(\lambda,A)\|\le C(1+|\lambda|)^{-1},\quad |\arg\lambda|\ge\pi-\varphi;$$

(4) *for any $\varepsilon>0$ and for almost all $x\in[0,1]$,*

$$\|B(x)u\|\le\varepsilon\|A^{\frac12}u\|+C(\varepsilon)\|u\|,\quad u\in D(A^{\frac12}),$$
$$\|A_1(x)u\|\le\varepsilon\|Au\|+C(\varepsilon)\|u\|,\quad u\in D(A);$$

for $u\in D(A^{\frac12})$ the function $B(x)u$ and for $u\in D(A)$ the function $A_1(x)u$ are measurable on $[0,1]$ in H;

(5) *if $m_k=0$, then $T_{ks}=0$; if $m_k=1$, then for $\varepsilon>0$ and $u\in(H(A),H)_{\frac14,2}$,*

$$\|T_{ks}u\|_{(H(A),H)_{\frac34,2}}\le\varepsilon\|u\|_{(H(A),H)_{\frac14,2}}+C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\|\le\varepsilon\|u\|_{(H(A),H)_{\frac12,2}}+C(\varepsilon)\|u\|;$$

(6) $u_0\in W_2^2\big((0,1);H(A),H;L_k=0,\ k=1,2\big)$, *where* $L_ku:=\alpha_ku^{(m_k)}(0)+\beta_ku^{(m_k)}(1)+\sum_{s=1}^{N_k}T_{ks}u(x_{ks}),\ k=1,2$.

Then, problem (4.1)–(4.3) has a unique solution $u(t,x)$ in

$$C\big([0,T];L_2((0,1);H)\big)\cap C^1\big((0,T];W_2^2\big((0,1);H(A),H;$$
$$L_ku=0,\ k=1,2\big),L_2((0,1);H)\big)$$

and there exist numbers C_{in} such that for the solution it holds

$$\begin{aligned}&\lim_{n\to\infty}\max_{t\in[0,T]}\Big\|u(t,\cdot)-\sum_{i=1}^nC_{in}u_i(t,\cdot)\Big\|_{L_2((0,1);H)}=0,\\&\lim_{n\to\infty}\sup_{t\in(0,T]}\Big\|\frac{\partial u(t,\cdot)}{\partial t}-\sum_{i=1}^nC_{in}\frac{\partial u_i(t,\cdot)}{\partial t}\Big\|_{L_2((0,1);H)}=0,\end{aligned}\tag{4.5}$$

where $u_i(t,x)$ are elementary solutions (4.4) of (4.1)–(4.2).

Proof. By Theorem 3.1, a system of root functions of problem (3.1)–(3.2) is complete in $W_2^2\big((0,1);\ H(A),H;L_ku=0,\ k=1,2\big)$. Hence, there exist numbers C_{in} such that

$$\lim_{n\to\infty}\Big\|u_0(\cdot)-\sum_{i=1}^nC_{in}u_i(0,\cdot)\Big\|_{W_2^2((0,1);H(A),H)}=0,\tag{4.6}$$

where $u_i(t,x)$ are elementary solutions (4.4) of (4.1)–(4.2). On the other hand, from Theorem 2.1 (remind, in the framework of Hilbert spaces, $\mathcal{R}$-boundedness is just norm-boundedness), we get that problem (4.1)–(4.3) has a unique solution $u(t,x)$ in

$$C\big([0,T];L_2((0,1);H)\big)\cap C^1\big((0,T];W_2^2\big((0,1);H(A),H;$$
$$L_ku=0,\ k=1,2\big),L_2((0,1);H)\big)$$

and for the solution the corresponding estimates in (2.4), with $f=0$, are fulfilled. Then, $u(t,x)-\sum_{i=1}^nC_{in}u_i(t,x)$ is a unique solution of problem (4.1)–(4.3) but with the initial function is equal to $u_0(x)-\sum_{i=1}^n C_{in}u_i(0,x)$ and the corresponding estimates in (2.4) will be

$$\|u(t,\cdot)-\sum_{i=1}^n C_{in}u_i(t,\cdot)\|_{L_2((0,1);H)} \le C\|u_0(\cdot)-\sum_{i=1}^n C_{in}u_i(0,\cdot)\|_{W_2^2((0,1);H(A),H)},$$
$$\Big\|\frac{\partial u(t,\cdot)}{\partial t}-\sum_{i=1}^n C_{in}\frac{\partial u_i(t,\cdot)}{\partial t}\Big\|_{L_2((0,1);H)} \le C\|u_0(\cdot)-\sum_{i=1}^n C_{in}u_i(0,\cdot)\|_{W_2^2((0,1);H(A),H)}, \tag{4.7}$$

where C does not depend on $t\in(0,T]$. From (4.6) and (4.7) we get (4.5). □

Consider now problem (4.1)-(4.3) in a separable, reflexive *UMD* Banach space E and in the space $\mathbb{E}:=L_p((0,1);E)$, which is also separable, reflexive *UMD* Banach space (for reflexivity see [10, Theorem 5.7]), introduce an operator $\mathbb{A}$ defined by the equalities

$$D(\mathbb{A}) := W_p^2\big((0,1);E(A),E;L_ku=0,\ k=1,2\big),$$
$$\mathbb{A}u = -u''(x)+B(x)u'(x)+Au(x)+A_1(x)u(x).$$

Theorem 4.2. *Let the spectrum of the operator $\mathbb{A}$ be non empty; for some $s>0$,*

$$\tilde{s}_j(J;W_p^2((0,1);E(A),E),L_p((0,1);E))\le Cj^{-s};$$

and let all conditions of [8, Theorem 6] *(see Theorem 5.3 in the Appendix) be satisfied. Moreover, condition (2) of* [8, Theorem 6] *is satisfied for some $\frac{2-s}{2}\pi<\varphi<\pi$ if $0<s\le 1$ and for some $\frac{\pi}{2}<\varphi<\pi$ if $s>1$; finally, $u_0\in W_p^2\big((0,1);E(A),E;L_k=0,\ k=1,2\big)$, where $L_ku:=\alpha_k u^{(m_k)}(0)+\beta_k u^{(m_k)}(1)+\sum_{s=1}^{N_k}T_{ks}u(x_{ks}),\ k=1,2.$*

Then, problem (4.1)–(4.3) has a unique solution $u(t,x)$ in

$$C\big([0,T];L_p((0,1);E)\big)\cap C^1\big((0,T];W_p^2\big((0,1);E(A),E;$$
$$L_ku=0,\ k=1,2\big),L_p((0,1);E)\big)$$

and there exist numbers C_{in} such that for the solution it holds

$$\lim_{n\to\infty}\max_{t\in[0,T]}\Big\|u(t,\cdot)-\sum_{i=1}^n C_{in}u_i(t,\cdot)\Big\|_{L_p((0,1);E)}=0,$$
$$\lim_{n\to\infty}\sup_{t\in(0,T]}\Big\|\frac{\partial u(t,\cdot)}{\partial t}-\sum_{i=1}^n C_{in}\frac{\partial u_i(t,\cdot)}{\partial t}\Big\|_{L_p((0,1);E)}=0, \tag{4.8}$$

where $u_i(t,x)$ are elementary solutions (4.4) of (4.1)–(4.2).

Proof. The proof is the same as that of Theorem 4.1. We only use Theorem 3.2 instead of Theorem 3.1. □

Show an application of Theorem 4.1. In fact, all necessary data are given in the application part of section 2. We just take the homogeneous equation (instead of the nonhomogeneous equation (2.7))

$$
\begin{aligned}
&D_t u(t,x,y) - D_x^2 u(t,x,y) + b(x,y) D_x u(t,x,y) + \int_G c(x,y,z) D_x u(t,x,z) dz \\
&- \sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u(t,x,y) + \sum_{j=1}^{r} b_j(x,y) D_j u(t,x,y) + b_0(x,y) u(t,x,y) \\
&+ \sum_{\ell=0}^{1} \sum_{j=1}^{r} \int_G c_{\ell j}(x,y,z) D_{z_j}^{\ell} u(t,x,z) dz = 0, \quad (t,x,y) \in (0,T) \times (0,1) \times G,
\end{aligned} \tag{4.9}
$$

with boundary conditions (2.8)–(2.9) and initial condition (2.10).

Theorem 4.3. *Assume that conditions (1)-(5) and (7) (with $p = q = 2$) of Theorem 2.3 are fulfilled (conditions (2)-(3) with some $0 < \beta < \pi - \frac{2\pi r}{2r+2}$).*

Then, problem (4.9), (2.8)–(2.10) has a unique solution $u(t,x,y)$ in

$$
\begin{aligned}
C\big([0,T];\, L_2((0,1); L_2(G))\big) \cap C^1\big((0,T]; W_2^2\big((0,1); W_2^2(G; L_0 u = 0), L_2(G); \\
L_k u = 0,\ k = 1,2, y \in G\big), L_2((0,1); L_2(G))\big)
\end{aligned}
$$

and there exist numbers C_{in} such that for the solution it holds

$$
\begin{aligned}
&\lim_{n\to\infty} \max_{t\in[0,T]} \Big\| u(t,x,y) - \sum_{i=1}^{n} C_{in} u_i(t,x,y) \Big\|_{L_2((0,1);L_2(G))} = 0, \\
&\lim_{n\to\infty} \sup_{t\in(0,T]} \Big\| \frac{\partial u(t,x,y)}{\partial t} - \sum_{i=1}^{n} C_{in} \frac{\partial u_i(t,x,y)}{\partial t} \Big\|_{L_2((0,1);L_2(G))} = 0,
\end{aligned} \tag{4.10}
$$

where $u_i(t,x,y) := e^{\lambda_i t}\Big(\frac{t^{k_i}}{k_i!} u_{i0}(x,y) + \frac{t^{k_i-1}}{(k_i-1)!} u_{i1}(x,y) + \cdots + u_{ik_i}(x,y)\Big)$ are elementary solutions of a system (4.9), (2.8)–(2.9), i.e., $u_{i0}(x,y)$, $u_{i1}(x,y)$,..., $u_{ik_i}(x,y)$ is a chain of root functions of problem (3.4)–(3.6).

Proof. We use Theorem 4.1. Like to the proof of Theorem 3.3, we note that all operators and all necessary explanations are the same as in the proof of Theorem 2.3. As in the proof of Theorem 3.3, the only thing is to check condition (2) and the corresponding restriction on φ in condition (3) of Theorem 4.1. In our case, they are the same as in Theorem 3.1 (which have been already checked in the proof of Theorem 3.3) since $\max\left\{\frac{2\pi}{2+t}, \frac{\pi}{2}\right\} = \frac{2\pi}{2+t}$ for $t = \frac{2}{r}$ and $r \geq 2$. □

5. Appendix

Consider, in a Banach space X, the Cauchy problem

$$
\begin{aligned}
u'(t) &= Lu(t) + f(t), \\
u(0) &= u_0.
\end{aligned} \tag{5.1}
$$

Theorem 5.1. *(*[13, Theorem 7.2.2/1]*) Let the following conditions be satisfied*

(1) *L is a linear closed operator in X and, for some $\eta \in (0,1]$, $\alpha > 0$,*
$$\|R(\lambda, L)\|_{B(X)} \le M|\lambda|^{-\eta}, \quad |\arg\lambda| \le \frac{\pi}{2} + \alpha,\ |\lambda| \to \infty;$$
(2) *$f \in C^{\gamma}_{\mu}((0,T];X)$, for some $\gamma \in (1-\eta, 1]$, $\mu \in [0,\eta)$;*
(3) *$u_0 \in D(L)$.*

Then, problem (5.1) has a unique solution in $C([0,T];X) \cap C^1((0,T];X(L),X)$, and the solution can be represented in the form
$$u(t) := e^{tL}u_0 + \int_0^t e^{(t-\tau)L} f(\tau)\, d\tau, \tag{5.2}$$
where the semigroup $e^{tL} := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda, L)d\lambda$ and Γ is completely contained in $\rho(L)$ and coincides with the rays $\arg\lambda = \pm(\frac{\pi}{2} + \alpha)$, for large $|\lambda|$. Moreover, for $t \in (0,T]$, the following estimates hold:
$$\begin{aligned} \|u(t)\| &\le C\Big(\|Lu_0\| + \|u_0\| + \|f\|_{C_\mu((0,t];X)}\Big),\\ \|u'(t)\| + \|Lu(t)\| &\le C\Big[t^{\eta-1}\Big(\|Lu_0\| + \|u_0\|\Big) + t^{\eta-\mu-1}\|f\|_{C^{\gamma}_{\mu}((0,t];X)}\Big], \end{aligned} \tag{5.3}$$
where C does not depend on t.

Let us now formulate another theorem for problem (5.1) which is a corollary of [7, Theorem 7.2].

Theorem 5.2. *(a corollary of* [7, Theorem 7.2]*) Let the following conditions be satisfied*

(1) *L is a linear closed operator in X and, for some $\eta \in (0,1]$, $\alpha > 0$,*
$$\|R(\lambda, L)\|_{B(X)} \le M|\lambda|^{-\eta}, \quad |\arg\lambda| \le \frac{\pi}{2} + \alpha,\ |\lambda| \to \infty;$$
(2) *$f \in C^{\gamma}_0([0,T];X)$ with $f(0) + Lu_0 \in W_\gamma$, for some $\gamma \in (1-\eta, 1)$, where $W_\gamma := \{v \in X \mid \exists\lambda_0 > 0$ big enough such that $\sup_{\lambda>\lambda_0}(1+\lambda)^{\gamma}\|L(\lambda I - L)^{-1}v\|_X < \infty\}$;*
(3) *$u_0 \in D(L)$.*

Then, problem (5.1) has a unique strict solution in $C^1([0,T];X(L),X)$ with the regularity $Lu \in C^{\gamma+\eta-1}_0([0,T];X)$ and $u' \in C^{\gamma+\eta-1}_0([0,T];X)$.

Remark 5.1. In fact, if $\eta = 1$ then the theorem implies maximal C^γ-regularity. On the other side, it is well-known that there is no maximal C-regularity for problem (5.1)!

Consider, in a Banach space E, an abstract elliptic boundary value problem with a patameter
$$\begin{aligned} L(\lambda)u &:= \lambda u(x) - u''(x) + B(x)u'(x) + Au(x) + A_1(x)u(x) = f(x),\\ L_k u &:= \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks}u(x_{ks}) = f_k,\ k = 1,2, \end{aligned} \tag{5.4}$$
where λ is a complex parameter, $m_k \in \{0,1\}$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x \in [0,1]$, and A, T_{ks} are, generally speaking, unbounded operators in E.

Theorem 5.3. *([8, Theorem 6]) Let the following conditions be satisfied:*

(1) *an operator A is closed, densely defined in a UMD Banach space E;*
(2) $\mathcal{R}\{\lambda R(\lambda, A) \ : \ |\arg\lambda| \geq \pi - \varphi\} < \infty$ *for some* $0 \leq \varphi < \pi$;[4]
(3) *the embedding $E(A) \subset E$ is compact;*
(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;
(5) *for any $\varepsilon > 0$ and for almost all $x \in [0, 1]$,*

$$\|B(x)u\| \leq \varepsilon\|A^{\frac{1}{2}}u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}}),$$
$$\|A_1(x)u\| \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A);$$

for $u \in D(A^{\frac{1}{2}})$ the function $B(x)u$ and for $u \in D(A)$ the function $A_1(x)u$ are measurable on $[0, 1]$ in E;

(6) *if $m_k = 0$, then $T_{ks} = 0$; if $m_k = 1$, then for $\varepsilon > 0$ and $u \in (E(A), E)_{\frac{1}{2p},p}$, where $p \in (1, \infty)$,*

$$\|T_{ks}u\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \leq \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\| \leq \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2},p}} + C(\varepsilon)\|u\|.$$

Then,

(a) *the operator $\mathbb{L}(\lambda) \ : \ u \to \mathbb{L}(\lambda)u := \Big(L(\lambda)u, L_1u, L_2u\Big)$, for $|\arg\lambda| \leq \varphi$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2((0, 1); E(A), E)$ onto $L_p((0, 1); E) \dot{+} (E(A), E)_{\theta_1,p} \dot{+} (E(A), E)_{\theta_2,p}$, where $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$, and for these λ, the following coercive estimate holds for the solution of problem (5.4)*

$$|\lambda|\|u\|_{L_p((0,1);E)} + \|u''\|_{L_p((0,1);E)} + \|Au\|_{L_p((0,1);E)}$$
$$\leq C\Big[\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{2}\Big(\|f_k\|_{(E(A),E)_{\theta_k,p}} + |\lambda|^{1-\theta_k}\|f_k\|\Big)\Big];$$

(b) *the operator $u \to (L_1u, L_2u)$, for $|\arg\lambda| \leq \varphi$ and $|\lambda|$ sufficiently large, from $W_p^2((0, 1); E(A), E)$ into $(E(A), E)_{\theta_1,p} \dot{+} (E(A), E)_{\theta_2,p}$, has a continuous right-inverse; in other words, there exists such an operator $R(f_1, f_2) = u$ continuous from $(E(A), E)_{\theta_1,p} \dot{+} (E(A), E)_{\theta_2,p}$ into $W_p^2((0, 1); E(A), E)$, where u is a solution of the system $L_ku = f_k$, $k = 1, 2$.*

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[4]In original version it is written, by a misprint, that the $\mathcal{R}$-boundedness is claimed in the angle but for big enough λ. In fact, one can see, from the proof of [8, Theorem 6], that replacing A and $A_1(x)$ by $A + M_0I$ and $A_1(x) - M_0I$, respectively, for some big enough $M_0 > 0$, the latter can be also treated.

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