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ON SOLVABILITY CONDITIONS OF A BOUNDARY VALUE PROBLEM WITH AN OPERATOR IN THE BOUNDARY CONDITION FOR A SECOND ORDER ELLIPTIC OPERATOR-DIFFERENTIAL EQUATION

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In memory of M. G. Gasymov on his 75th birthday

Abstract. On a semi-axis we consider a boundary value problem for a class of second order operator-differential equations of second order with a discontinuous coefficient. Note that the principal part of the equation under investigation contains a normal operator, and some abstract operator takes part in the boundary condition. By means of properties of operator coefficients of the boundary value problem we find sufficient conditions providing its regular solvability.

1. Introduction

Let H be a separable Hilbert space, A a normal operator with a completely continuous inverse A^{-1}, whose spectrum is contained in the angular sector $S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\}$, $0 \le \varepsilon < \pi/4$. If $\{e_n\}_{n=1}^{\infty}$ is a orthonormalized basis of eigenvectors of the operator A, i.e. $Ae_n = \lambda_n e_n$, $\lambda_n = \mu_n e^{i\varphi_n}$, $|\varphi_n| \le \varepsilon$, $0 < \mu_1 < \mu_2 < \dots$, $(e_n, e_m) = \delta_{nm}$, where δ_{nm} is the Kronecker symbol, then the operator A $(A(\cdot) = \sum_{n=1}^{\infty} \lambda_n(\cdot\,, e_n)e_n)$ is representable in the form $A = UC$, where $D(A) = D(C)$, C $(C(\cdot) = \sum_{n=1}^{\infty} \mu_n(\cdot\,, e_n)e_n)$ is a self-adjoint positive operator, and U $\left(U(\cdot) = \sum_{n=1}^{\infty} e^{i\varphi_n}(\cdot\,, e_n)e_n\right)$ is a unitary operator in H. The domain of definition of the operator C^γ $(\gamma \ge 0)$ is Hilbert space with respect to the norm $\|x\|_\gamma = \|C^\gamma x\|$. For $\gamma = 0$ we assume that $H_0 = H$.

Denote by $L_2((a,b);H)$, $-\infty \le a < b \le +\infty$ Hilbert space of all vector-functions $f(t)$ determined in (a,b) almost everywhere with the values in H, with the finite norm

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f(t)\|^2\, dt\right)^{1/2}.$$

In the sequel, we denote by $L(X,Y)$ a space of bounded operators acting from the Banach space X to another Banach space Y.

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We introduce the Hilbert space [6]

$$W_2^2((a,b);H) = \{u : C^2u \in L_2((a,b);H),\ \ u'' \in L_2((a,b);H)\}$$

with the norm

$$\|u\|_{W_2^2((a,b);H)} = \left(\left\|C^2u\right\|^2_{L_2((a,b);H)} + \left\|u''\right\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

Here and in the sequel, all derivatives are understood in the sense of distribution theory [6]. For $a = 0$, $b = +\infty$ we assume $L_2((0,\infty);H) = L_2(\mathbb{R}_+;H)$, $W_2^2((0,\infty);H) = W_2^2(\mathbb{R}_+;H)$, $\mathbb{R}_+ = (0,\infty)$.

Let the operator $T \in L(H_{1/2}, H_{3/2})$. Then

$$W_{2,T}^2(\mathbb{R}_+;H) = \{u : u \in W_2^2(\mathbb{R}_+;H),\ u(0) = Tu'(0)\}$$

is a subspace (complete) of the space $W_2^2(\mathbb{R}_+;H)$.

In space H consider the boundary value problem

$$-u''(t) + \rho(t)A^2u(t) + A_1u'(t) + A_2u(t) = f(t), \quad t \in \mathbb{R}_+, \tag{1.1}$$

$$u(0) = Tu'(0), \tag{1.2}$$

where the coefficients satisfy the conditions:
1) A is a normal operator in H with a completely continuous inverse operator whose spectrum is contained in the sector $S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\}$, $0 \le \varepsilon < \pi/4$;
2) $\rho(t) = \alpha^2$ for $t \in (0,1)$ and $\rho(t) = \beta^2$ for $t \in (1,\infty)$, $0 < \alpha \le \beta < \infty$;
3) $T \in L(H_{1/2}, H_{3/2})$;
4) $B_1 = A_1A^{-1}$, $B_2 = A_2A^{-2}$ are bounded operators in H.

Definition. If for any $f(t) \in L_2(\mathbb{R}_+;H)$ there exists a vector-function $u(t) \in W_2^2(\mathbb{R}_+;H)$, satisfying equation (1.1) almost everywhere in R_+, boundary condition (1.2) in the sense of $\lim\limits_{t\to+0}\|u(t) - Tu'(t)\|_{3/2} = 0$ and the estimation

$$\|u\|_{W_2^2(\mathbb{R}_+;H)} \le const\,\|\,f\,\|_{L_2(\mathbb{R}_+;H)},$$

we say that $u(t)$ is a *regular solution* of boundary value problem (1.1), (1.2), and the boundary value problem (1.1), (1.2) itself is *regularly solvable.*

Note that for $\varepsilon = 0$, $\rho(t) = 1$, $T = 0$ boundary value problem (1.1), (1.2) was first studied in M. G. Gasymov's papers [3, 4]. The results obtained in these papers were generalized for some $T \in L(H_{1/2}, H_{3/2})$ in the paper of M. G. Gasymov and S. S. Mirzoev [5]. For $\varepsilon = 0$, $T = 0$ and $\rho(t) \ne 1$ satisfying condition 2), problem (1.1), (1.2) was studied in the paper of S. S. Mirzoev and A. R. Aliev [7]. The analog of this case for higher order equations was investigated in the papers [1, 2]. In the sequel, for $\varepsilon = 0$, $\rho(t) \ne 1$, $T \ne 0$ boundary value problem (1.1), (1.2) was considered in the paper [8].

Note that in the present paper $\varepsilon \ne 0$, $\rho(t) \ne 1$ and $T \ne 0$, and this problem was not studied even in the case $\rho(t) = 1$. Therefore, the obtained results are new in the case $\rho(t) = 1$ as well. Notice that we can similarly consider the case $0 < \beta \le \alpha < \infty$.

In the present paper we find conditions in the terms of operator coefficients of boundary value problem (1.1), (1.2), that provide its regular solvability.

2. Main results

Denote by

$$P_0 u = P_0(d/dt)u = -u'' + \rho(t)A^2 u,$$
$$P_1 u = P_1(d/dt)u = A_1 u' + A_2 u, u \in W^2_{2,T}(\mathbb{R}_+; H).$$

It holds

Lemma. *Let conditions* 1)−4) *be fulfilled. Then P_0 and P_1 are bounded operators from the space $W^2_{2,T}(\mathbb{R}_+; H)$ to the space $L_2(\mathbb{R}_+; H)$.*

The **proof** follows from the boundedness condition of the function $\rho(t)$ and the operators B_1 and B_2 with regard to the intermediate derivatives theorem [6].

Theorem 1. *Let conditions* 1) − 3) *be fulfilled, and $Re\, UAT \geq 0$ in $H_{1/2}$. Then for any $u \in W^2_{2,T}(\mathbb{R}_+; H)$ the following inequalities hold*

$$\left\| A^2 u \right\|_{L_2(\mathbb{R}_+;H)} \leq c_0(\varepsilon) \left\| P_0 u \right\|_{L_2(\mathbb{R}_+;H)}, \tag{2.1}$$

$$\left\| A u' \right\|_{L_2(\mathbb{R}_+;H)} \leq c_1(\varepsilon) \left\| P_0 u \right\|_{L_2(\mathbb{R}_+;H)}, \tag{2.2}$$

where $c_0(\varepsilon) = \alpha^{-2}$, $c_1(\varepsilon) = 2^{-1}(\cos 2\varepsilon)^{-1/2}\alpha^{-1}$.

Proof. For $u \in W^2_{2,T}(\mathbb{R}_+; H)$ we have:

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(\mathbb{R}_+;H)} = \left\| \rho^{-1/2} u'' - A^2 u \right\|^2_{L_2(\mathbb{R}_+;H)} = \left\| \rho^{-1/2} u'' \right\|^2_{L_2(\mathbb{R}_+;H)} +$$
$$\left\| \rho^{1/2} A^2 u \right\|^2_{L_2(\mathbb{R}_+;H)} - 2Re(u'', A^2 u)_{L_2(\mathbb{R}_+;H)}. \tag{2.3}$$

Integrating by parts and taking into account spectral expansion of the operator A, we get:

$$-(u'', A^2 u)_{L_2(\mathbb{R}_+;H)} = (C^{1/2} u'(0), C^{3/2} U^2 u(0)) + (A^* u', A u')_{L_2(\mathbb{R}_+;H)} \geq$$
$$(u'(0), UATu'(0))_{1/2} + \cos 2\varepsilon \left\| A u' \right\|^2_{L_2(\mathbb{R}_+;H)}. \tag{2.4}$$

Then taking into attention inequality (2.4) in (2.3), we have

$$\left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(\mathbb{R}_+;H)} \geq \left\| \rho^{-1/2} u'' \right\|^2_{L_2(\mathbb{R}_+;H)} + \left\| \rho^{1/2} A^2 u \right\|^2_{L_2(\mathbb{R}_+;H)} +$$
$$2\cos 2\varepsilon \left\| A u' \right\|^2_{L_2(\mathbb{R}_+;H)}. \tag{2.5}$$

Hence we get:

$$\left\| A^2 u \right\|^2_{L_2(\mathbb{R}_+;H)} \leq \max_t \rho^{-1}(t) \left\| \rho^{1/2} A^2 u \right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{\alpha^2} \left\| \rho^{-1/2} P_0 u \right\|^2_{L_2(\mathbb{R}_+;H)} \leq$$
$$\frac{1}{\alpha^4} \left\| P_0 u \right\|^2_{L_2(\mathbb{R}_+;H)}.$$

Consequently, inequality (2.1) is proved.

Prove inequality (2.2). From inequality (2.4) it follows that

$$\cos 2\varepsilon \left\| A u' \right\|^2_{L_2(\mathbb{R}_+;H)} \leq -Re(u'', A^2 u)_{L_2(\mathbb{R}_+;H)} \leq \left| (\rho^{-1/2} u'', \rho^{1/2} A^2 u)_{L_2(\mathbb{R}_+;H)} \right| \leq$$
$$\frac{1}{2} \left(\left\| \rho^{-1/2} u'' \right\|^2_{L_2(\mathbb{R}_+;H)} + \left\| \rho^{1/2} A^2 u \right\|^2_{L_2(\mathbb{R}_+;H)} \right). \tag{2.6}$$

Taking into account inequality (2.5) in (2.6), we have

$$\cos 2\varepsilon \left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\left(\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(\mathbb{R}_+;H)} - 2\cos 2\varepsilon \left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)}\right).$$

Hence we get:

$$2\cos 2\varepsilon \left\|Au'\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2}\left\|\rho^{-1/2}P_0u\right\|^2_{L_2(\mathbb{R}_+;H)} \leq \frac{1}{2\alpha^2}\left\|P_0u\right\|^2_{L_2(\mathbb{R}_+;H)},$$

i.e.

$$\left\|Au'\right\|_{L_2(\mathbb{R}_+;H)} \leq 2^{-1}\alpha^{-1}(\cos 2\varepsilon)^{-1/2}\left\|P_0u\right\|_{L_2(\mathbb{R}_+;H)}.$$

The theorem is proved.

Now let's solve boundary value problem (1.1), (1.2). At first prove the following theorem.

Theorem 2. *Let conditions* 1) − 3) *be fulfilled, and the operator*

$$Q = \left(E - \frac{\beta-\alpha}{\beta+\alpha}e^{-2\alpha A}\right) + \alpha AT\left(E + \frac{\beta-\alpha}{\beta+\alpha}e^{-2\alpha A}\right)$$

be inversible in $H_{1/2}$*. Then the operator* P_0 *realizes isomorphism from the space* $W^2_{2,T}(\mathbb{R}_+;H)$ *onto the space* $L_2(R_+;H)$*.*

Proof. Show that the kernel of the operator P_0 consists of only a zero element, and the image of the operator P_0 coincides with $L_2(\mathbb{R}_+;H)$. Then the assertion of the theorem follows from the lemma and the Banach theorem on the inverse operator. Obviously, the equation $P_0(d/dt)u(t) = 0$ has a general solution from $W^2_2(\mathbb{R}_+;H)$ in the form

$$u_0(t) = \begin{cases} \xi_1(t) = e^{-\alpha tA}\varphi_1 + e^{-\alpha(1-t)A}\varphi_2, \ 0 < t < 1, \\ \xi_2(t) = e^{-\beta(t-1)A}\varphi_3\,, \quad t > 1\,, \ \varphi_j \in H_{3/2}, \ j = \overline{1,3}\,. \end{cases}$$

Then from the condition $u \in W^2_{2,T}(\mathbb{R}_+;H)$ it follows that $\xi_1(0) = T(\xi_1'(0))$, $\xi_1^{(j)}(1-0) = \xi_1^{(j)}(1+0)$, $j = 0,1$. Consequently, with respect to φ_1, φ_2 and φ_3 we get the following equations:

$$\varphi_1 + e^{-\alpha A}\varphi_2 = T(-\alpha A\varphi_1 + \alpha Ae^{-\alpha A}\varphi_2),$$
$$\varphi_3 = e^{-\alpha A}\varphi_1 + \varphi_2,$$
$$\beta\varphi_3 = \alpha e^{-\alpha A}\varphi_1 - \alpha\varphi_2.$$

Then $\varphi_2 = \frac{\alpha-\beta}{\alpha+\beta}e^{-\alpha A}\varphi_1$ and from the first equation it follows that

$$\left(\left(E - \frac{\beta-\alpha}{\beta+\alpha}e^{-2\alpha A}\right) + \alpha TA\left(E + \frac{\beta-\alpha}{\beta+\alpha}\right)e^{-2\alpha A}\right)\varphi_1 = 0.$$

Assuming $A\varphi_1 = y_1 \in H_{1/2}$ we get

$$\left(\left(E - \frac{\beta-\alpha}{\beta+\alpha}e^{-2\alpha A}\right) + \alpha AT\left(E + \frac{\beta-\alpha}{\alpha+\beta}\right)e^{-2\alpha A}\right)y_1 = 0,$$

i.e. $Qy_1 = 0$. Since from the theorem condition Q is inversible in $H_{1/2}$, then $y_1 = 0$. Hence it follows $\varphi_1 = \varphi_2 = \varphi_3 = 0$, i.e. $u_0(t) = 0$.

Now show that the equation $P_0u = f$ has a solution for any $f \in L_2(\mathbb{R}_+;H)$. Let's consider the vector function

$$u_\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\xi^2E + \alpha^2A^2)^{-1}\left(\int_0^\infty f(s)e^{-is\xi}ds\right)e^{i\xi t}d\xi,\ t \in (0,1),$$

$$u_\beta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\xi^2 E + \beta^2 A^2)^{-1}\left(\int_0^\infty f(s)e^{-is\xi}ds\right)e^{i\xi t}d\xi,\ t\in(1,\infty).$$

It is easy to see that $u_\alpha(t)\in W_2^2((0,1);H)$, $u_\beta(t)\in W_2^2((1,\infty);H)$ and satisfy the equation $P_0(d/dt)u(t) = f(t)$ in $(0,1)$ and in $(1,\infty)$ almost everywhere, respectively. Then we'll look for the solution of the equation $P_0u = f$ in the form

$$u(t) = u_\alpha(t) + e^{-\alpha tA}\varphi_1 + e^{-\alpha(1-t)A}\varphi_2 \text{ for } t\in(0,1)$$

and

$$u(t) = u_\beta(t) + e^{-\beta(t-1)A}\varphi_3 \text{ for } t\in(1,\infty),$$

where φ_1,φ_2 and $\varphi_3\in H_{3/2}$ are unknown vectors. From the inversibility of the operator Q in $H_{1/2}$ and from the condition $u(t)\in W_{2,T}^2(\mathbb{R}_+;H)$, the vectors φ_i, $i=1,2,3$ are uniquely determined. Thus $u(t)\in W_{2,T}^2(\mathbb{R}_+;H)$ and $P_0u = f$. The theorem is proved.

This theorem implies

Corollary 1. *The norms* $\|u\|_{W_2^2(\mathbb{R}_+;H)}$ *and* $\|P_0u\|_{L_2(\mathbb{R}_+;H)}$ *are equivalent in the space* $W_{2,T}^2(\mathbb{R}_+;H)$.

Corollary 2. *The norms*

$$N_{0,T} = \sup_{0\neq u\in W_{2,T}^2(\mathbb{R}_+;H)}\left\|A^2u\right\|_{L_2(\mathbb{R}_+;H)}\|P_0u\|_{L_2(\mathbb{R}_+;H)}^{-1}$$

and

$$N_{1,T} = \sup_{0\neq u\in W_{2,T}^2(\mathbb{R}_+;H)}\left\|Au'\right\|_{L_2(\mathbb{R}_+;H)}\|P_0u\|_{L_2(\mathbb{R}_+;H)}^{-1}$$

are finite and $N_{0,T}\le\alpha^{-2}$, $N_{1,T}\le 2^{-1}(\cos 2\varepsilon)^{-1/2}\alpha^{-1}$.

The proof follows from the intermediate derivatives theorem [6] and from theorem 2.

Now, using the obtained results, we prove the main theorem of the present paper.

Theorem 3. *Let conditions* 1) – 4) *be fulfilled, the operator* Q *be inversible in* $H_{1/2}$, $Re\,UAT\ge 0$ *in* $H_{1/2}$, *and the operators* B_1 *and* B_2 *be such that*

$$q(\varepsilon) = 2^{-1}(\cos 2\varepsilon)^{-1/2}\alpha^{-1}\|B_1\| + \alpha^{-2}\|B_2\| < 1.$$

Then boundary value problem (1.1), (1.2) *is regularly solvable.*

Proof. From the lemma it follows that we can write boundary value problem (1.1), (1.2) in the form of the operator equation $P_0u + P_1u = f$. Since P_0^{-1} is an isomorphism, then after substitution of $P_0u = v$ we get the equation $v + P_1P_0^{-1}v = f$ in $L_2(\mathbb{R}_+;H)$. On the other hand, for any $v\in L_2(\mathbb{R}_+;H)$

$$\left\|P_1P_0^{-1}v\right\|_{L_2(\mathbb{R}_+;H)} \le \|B_1\|\left\|Au'\right\|_{L_2(\mathbb{R}_+;H)} + \|B_2\|\left\|A^2u\right\|_{L_2(\mathbb{R}_+;H)} \le$$
$$\|B_1\|\cdot c_1(\varepsilon)\|P_0u\|_{L_2(\mathbb{R}_+;H)} + \|B_2\|\cdot c_0(\varepsilon)\|P_0u\|_{L_2(\mathbb{R}_+;H)} =$$
$$q(\varepsilon)\|P_0u\|_{L_2(\mathbb{R}_+;H)} = q(\varepsilon)\|v\|_{L_2(\mathbb{R}_+;H)}.$$

Here we used the inequalities from theorem 1.

Since $q(\varepsilon) < 1$, then $v = (E + P_1P_0^{-1})^{-1}f$, and $u = P_0^{-1}(E + P_1P_0^{-1})^{-1}f$. Hence we have $\|u\|_{W_2^2(\mathbb{R}_+;H)}\le const\,\|f\|_{L_2(\mathbb{R}_+;H)}$. The theorem is proved.

Corollary 3. *Let* A *be a self-adjoint positive operator,* $T = 0$, *and conditions* 2), 4) *be fulfilled. Then subject to the inequality*

$$2^{-1}\alpha^{-1}\|B_1\| + \alpha^{-2}\|B_2\| < 1$$

boundary value problem (1.1), (1.2) *is regularly solvable.*

This result was obtained in the papers [1, 2, 7]. In the case $\rho(t) \equiv 1$ we get the result of M. G. Gasymov's paper [4].

Corollary 4. *Let conditions* 1), 3), 4) *be fulfilled, and* $\rho(t) \equiv 1$, $Re\, UAT \geq 0$ *in* $H_{1/2}$ *and it hold the inequality*

$$2^{-1}(\cos 2\varepsilon)^{-1/2} \|B_1\| + \|B_2\| < 1.$$

Then boundary value problem (1.1), (1.2) *is regularly solvable.*

This result is independent and in fact is new.

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References

[1] A. R. Aliev, Boundary-value problems for a class of operator differential equations of high order with variable coefficients, *Math. Notes*, **74** (2003), no. 5-6, 761–771 (translated from *Mat. Zametki*, **74** (2003), no. 6, 803–814).

[2] A. R. Aliev and S. S. Mirzoev, On boundary value problem solvability theory for a class of high-order operator-differential equations, *Funct. Anal. Appl.*, **44** (2010), no. 3, 209–211 (translated from *Funkc. Anal. Prilozh.*, **44** (2010), no. 3, 63–65).

[3] M. G. Gasymov, On the theory of polynomial operator pencils, *Soviet Math. Dokl.* **12** (1972), 1143–1147 (translated from *Dokl. Akad. Nauk SSSR*, **199** (1971), no.4, 747–750).

[4] M. G. Gasymov, The multiple completeness of part of the eigen- and associated vectors of polynomial operator bundles, *Izv. Akad. Nauk Armjan. SSR Ser. Mat.*, **6** (1971), no. 2-3, 131–147 (in Russian)

[5] M. G. Gasymov and S. S. Mirzoev, Solvability of boundary value problems for second-order operator-differential equations of elliptic type, *Differ. Equ.*, **28** (1992), no. 4, 528–536 (translated from *Differentsial'nye Uravneniya*, **28** (1992), no. 4, 651–661).

[6] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Dunod, Paris, 1968; Mir, Moscow, 1971; Springer-Verlag, Berlin, 1972.

[7] S. S. Mirzoev and A. R. Aliev, On a boundary value problem for second-order operator-differential equations with a discontinuous coefficient, *Proc. Inst. Math. Mech. Acad. Sci. Azerb.*, **6** (1997), 117–121.

[8] S. S. Mirzoev, A. R. Aliev, and L. A. Rustamova, On the boundary value problem with the operator in boundary conditions for the operator-differential equation of second order with discontinuous coefficients, *Journal of Mathematical Physics, Analysis, Geometry*, **9** (2013), no. 2, 207–226.

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